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The stability of the Rogers–Shephard inequality and of some related inequalities

Abstract

An essentially optimal stability version of the Rogers–Shephard inequality and stability versions of some other related affine invariant inequalities are verified. The characteristic feature of all these inequalities is that simplices provide the equality case.

1 Around the Rogers–Shephard inequality

Let K be a convex body in $\mathbb{R}^n$; namely, a compact convex set with non–empty interior. Since the days of H. Minkowski, affine invariant inequalities for convex bodies are of high interest (see E. Lutwak [15]), and the stability versions of many have been established (see H. Groemer [9]). In this paper we provide stability versions of some affine invariant inequalities where equality holds only for simplices. We recall that the Banach-Mazur distance $d_{BM}(K,T)$ of K from any simplex T is the minimum of all $\lambda \geq 1$ such that for some simplex $S \subset K$, a translate of λS contains K.

If DK denotes the difference body $K-K$ then the celebrated Rogers–Shephard inequality (see C.A. Rogers and G.C. Shephard [18]) states

$$V(DK) \leq \binom{2n}{n} V(K). \tag{1}$$

Theorem 1 *If K is a convex body in $\mathbb{R}^n$ and*

$$V(DK) = (1-\varepsilon)\binom{2n}{n} V(K)$$

then

$$1 + \tfrac{1}{n}\,\varepsilon \leq d_{BM}(K,T) \leq 1 + n^{50n^2}\varepsilon.$$

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[1] Research supported in part by OTKA grants 43520 and 42769

In light of various other stability results (see H. Groemer [9], and Sections 2 and 3) it is somewhat surprising that the error terms in the lower and the upper bound in Theorem 1 are of the same order. A weaker form of Theorem 1 was announced in K. Böröczky, Jr. [2] (without a detailed proof).

What the proof of the Rogers–Shephard inequality (1) yields directly is that equality holds if and only if K is a Choquet–simplex; namely, $K \cap (K + x)$ is homothetic to K for any $x \in \mathrm{int}DK$ (see also Sections 7). In turn this property is equivalent to K being a simplex (see C.A. Rogers and G.C. Shephard [18]). Therefore it is not surprising that the proof of Theorem 1 depends on understanding how close $K \cap (K + x)$ is to be homothetic to K for $x \in \mathrm{int}DK$. Given a convex body C, let $r_K(C)$ and $R_K(C)$ denote the largest number r and the smallest number R such that a translate of rK is contained in C, and a translate of RK contains C, respectively. Hence C is homothetic to K if and only if $r_K(C) = R_K(C)$. We recall that if C is o–symmetric then the corresponding Minkowski–norm is

$$\|x\|_C = \max\{\lambda \geq 0 : \, x \in \lambda C\}.$$

It is not hard to verify (see Proposition 7) that

$$r_K(K \cap (x + K)) = 1 - \|x\|_{DK} \tag{2}$$

for any $x \in \mathrm{int}DK$. A side result of the proof of Theorem 1 is

Theorem 2 *If K is a convex body in $\mathbb{R}^n$ and $d_{BM}(T, K) = 1 + \varepsilon$ then*

(i) $R_K(K \cap (x + K)) \leq 1 - \|x\|_{DK} + \varepsilon$ for any $x \in \mathrm{int}DK$;
(ii) there exists an $x \in DK$ with $\|x\|_{DK} = \frac{1}{2}$ satisfying

$$R_K(K \cap (x + K)) \geq \tfrac{1}{2} + n^{-38}\varepsilon.$$

2 Inequalities for the pth radial mean body

The method of proof of Theorem 1 can be applied to some related results where equality holds for simplices. We exhibit this possibility for the Zhang projection inequality proved in G. Zhang [22] (see (3) below), and for the inclusions (6) for the pth radial mean body, $p \geq -1$, verified by R.J. Gardner and G. Zhang [5]. The latter inclusions provide a common generalization of the Rogers–Shepard and the Zhang projection inequalities. Another possible application might be the inclusion verified by R. Schneider [20] for mixed polytopes.

The Zhang projection inequality (see G. Zhang [22]) is concerned with the *polar projection body* Π^*K of K, and states that

$$V(K)^{n-1}V(\Pi^*K) \geq n^{-n}\binom{2n}{n} \tag{3}$$

where equality holds if and only if K is a simplex. Let us provide a definition of Π^*K, which shows its affine invariance. We say that a non–empty compact set S is a *star-shaped body* if $\lambda x \in \mathrm{int}S$ for any $0 \leq \lambda < 1$ and $x \in S$, and define the radial function of S for $u \neq 0$ by

$$\varrho_S(u) = \max\{\lambda : \lambda u \in S\}.$$

In addition a convex body is called a cylinder if it is the Minkowski sum of an $(n-1)$–dimensional convex body and a segment. For non–zero $u \in \mathbb{R}^n$, let $Z_K(u)$ be a cylinder of minimal volume among the ones that contain K and whose generating segments are parallel to u. We define Π^*K to be the o–symmetric convex body satisfying

$$\varrho_{\Pi^*K}(u) = \tfrac{1}{\|u\|_{DK}V(Z_K(u))} \tag{4}$$

for $u \neq 0$. Actually Π^*K is an o–symmetric convex body because it is the polar of the so–called projection body (see R. Schneider [19] or Section 10).

Theorem 3 *If K is a convex body in $\mathbb{R}^n$ and*

$$V(K)^{n-1}V(\Pi^*K) = (1+\varepsilon)n^{-n}\binom{2n}{n}$$

then

$$1 + n^{-3n^2}\varepsilon \leq d_{BM}(K,T) \leq 1 + n^{88n}\varepsilon^{1/n}.$$

Example 19 shows that the upper bound of Theorem 3 can not be replaced by $1+\varepsilon^{\frac{1}{n-1}}$. On the other hand the error term in the lower bound of Theorem 3 is of optimal order.

Next we discuss a common generalization of the difference body DK and the polar projection body Π^*K. For $x \in K$ and for $u \neq 0$, let

$$\varrho_K(x,u) = \max\{\lambda : x + \lambda u \in K\}.$$

Following R.J. Gardner and G. Zhang [5], the pth radial mean body R_pK of K is the star-shaped body defined by

$$\rho_{R_pK}(u) = \left(\tfrac{1}{V(K)}\int_K \rho_K(x,u)^p dx\right)^{1/p} \tag{5}$$

for non–zero $p > -1$, and if $p = 0$ then let

$$\rho_{R_0 K}(u) = \exp\left(\tfrac{1}{V(K)} \int_K \log \rho_K(x,u)dx\right).$$

Actually $R_p K$ is invariant under linear maps. Using the constants

$$c_{n,p} = \left(n \int_0^1 (1-t)^p t^{n-1} dt\right)^{-1/p}$$

for nonzero $p > -1$, and

$$c_{n,0} = \exp\left(\sum_{k=1}^n \tfrac{1}{k}\right),$$

we define a scaled version $\overline{R}_p K$ of $R_p K$ by

$$\overline{R}_p K = c_{n,p} R_p.$$

Then $\overline{R}_p K$ is convex for $p \geq 0$, and the definition can be extended by continuity to include

$$\overline{R}_{-1} K = nV(K)\Pi^* K \quad \text{and} \quad \overline{R}_\infty K = DK.$$

The normalized bodies $\overline{R}_p K$ satisfy that if $-1 \leq p < q \leq \infty$ then

$$\overline{R}_q K \subset \overline{R}_p K, \tag{6}$$

with equality if and only if K is a simplex. All these properties are proved by R.J. Gardner and G. Zhang [5].

Theorem 4 *Let K be a convex body in $\mathbb{R}^n$ and let $-1 \leq p < q \leq \infty$. There exist a positive constant c depending only on p, q, and n such that if $V(\overline{R}_p K) = (1+\varepsilon)V(\overline{R}_q K)$ then*

$$1 + n^{-3n^2}\varepsilon \leq d_{BM}(K,T) \leq 1 + c\varepsilon^{1/n}.$$

Since R.J. Gardner and G. Zhang [5] verified

$$V(\overline{R}_n K) = \tbinom{2n}{n} V(K),$$

Theorem 4 implies Theorem 3 up to the value of the constants.

We do not know the exact value of the constant in the upper bound of Theorem 4 because it is related to a stability version of the Brunn–Minkowski inequality proved by V.I. Diskant [3] where only the existence of the implied constant is verified. Based on H. Groemer [8], one could get explicit constant in Theorem 4 if the error term is of order $\varepsilon^{\frac{1}{n+1}}$.

3 Some historical remarks and the underlying ideas

Let K be a convex body in $\mathbb{R}^n$. In our context it is natural to consider the *Minkowski measure of symmetry* q_K where q_K is the minimal positive λ such that K is contained in a translate of $-\lambda K$. Readily $q_K \geq 1$, with equality if and only if K is centrally symmetric. On the other hand already H. Minkowski knew that $q_K \leq n$ (say $-K \subset nK$ if the centroid of K is the origin), and $q_K = n$ if K is a simplex. It took surprisingly long until V.L. Klee [13] showed that the property $q_K = n$ characterizes simplices (he used a different notation, see Section 5).

The first fundamental inequality whose stability was discussed already by H. Minkowski is the Brunn–Minkowski inequality (see R. Schneider [19] for thorough discussion). The Brunn–Minkowski inequality yields that

$$V(DK) \geq 2^n V(K) \tag{7}$$

where equality holds if and only if K is centrally symmetric. The known stability versions of (7) (see H. Groemer [9]) can be written in the following form: if $V(DK) = (1+\varepsilon)2^n V(K)$ then

$$1 + c_1\varepsilon \leq q_K \leq 1 + c_2\varepsilon^{1/n}$$

where c_1 and c_2 are positive constants depending on n. Here the lower bound is trivial, and the upper bound follows from V.I. Diskant [3]. In addition the orders of both error terms are optimal.

The proofs of our stability results follow similar paths. Let us describe the common idea in case of the Rogers–Shepard inequality. The lower bound of Theorem 1 is rather simple to verify. The proof of the non–trivial part of Theorem 1 is based on the method of C.A. Rogers and G.C. Shephard [18]. We use the notion of the *covariogram* of K; more precisely, we consider its normalized version

$$g_K(x) = \frac{V(K \cap (x+K))}{V(K)}. \tag{8}$$

It satisfies $g_K(x) \geq (1 - \|x\|_{DK})^n$ for $x \in DK$ according to (2), and the core of our argument is the following statement:

Theorem 5 *If K is a convex body in $\mathbb{R}^n$ with $d_{BM}(K,T) = 1+\varepsilon$ then there exists a convex body $C \subset DK$ such that $V(C) \geq n^{-25n^2}V(DK)$, and any $x \in C$ satisfies*

$$g_K(x) \geq (1 - \|x\|_{DK})^n + n^{-48n}\varepsilon.$$

Next we describe the steps which culminate in the proof of Theorem 5 in Section 6. Using the fundamental properties of the John ellipsoid in Section 4 (see Corollary 11), we obtain an error term of higher order instead of the error term of Theorem 5. If $\varepsilon \geq \frac{1}{4n^2}$ then this result already yields Theorem 5 because

$$d_{BM}(T,K) \leq n+2 \tag{9}$$

according to J. Lagarias and G. Ziegler [14]. If $\varepsilon < \frac{1}{4n^2}$ then we need a different approach. In Section 5 we establish a link between the Banach–Mazur distance and the Minkowski measure of symmetry. Notably if $q_K > n-1$ (which holds when $\varepsilon < \frac{1}{4n^2}$ by Corollary 15) then K can be well–approximated by some simplex in a rather explicit way according to V.L. Klee [13] (see also Lemma 12 and the discussion following it). This approximation provides the structural information that allows us to verify Theorem 5 for $\varepsilon < \frac{1}{4n^2}$. We remark that besides the known methods and results, Theorem 5 is the main ingredient of the proofs of Theorems 3 and 4, as well.

As it was mentioned above, the difficult part of Theorem 1 is the upper bound for $d_{BM}(K,T)$. If the dimension $n = 2,3$ then it is much simpler to derive similar lower bounds using some basic properties of mixed volumes. Actually assuming the celebrated conjecture of Godbersen, one may use analogous ideas in any dimension. We discuss this topic in Section 8.

Inequality (9) can simply be verified by considering any simplex of maximal volume inscribed into K (see J. Lagarias and G. Ziegler [14]). A. Giannopoulos, I. Perissinaki, A. Tsolomitis [6] discuss additional properties of a simplex of maximal volume in K, and related problems. It is not clear what the maximum of $d_{BM}(T,K)$ is over all convex bodies K in $\mathbb{R}^n$.

As it was mentioned above, the proofs of our theorems depend on a close analysis of the fact that simplices and Choquet–simplices coincide. This fact in turn has been generalized in various ways by P.M. Gruber. In [10] he proved that K is already a simplex if K and $K \cap (K+x)$ are homothetic for x in a neighbourhood of the origin. A far reaching generalization appeared in [11]: if $K \cap (K+x)$ is an affine image of K for any $x \in \mathrm{int} DK$ then K is either a simplex, or the direct sum of lower dimensional simplices.

In order to simplify notation, we usually write

$$K_x = K \cap (x+K).$$

It is convenient to assume an auxiliary scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, and let B^n and S^{n-1} denote the unit ball in $\mathbb{R}^n$ centred at the origin and its boundary, respectively. In addition we write $d(\cdot,\cdot)$ to denote the Euclidean distance. We frequently use the inequality

$$\binom{2n}{n} < 4^n, \tag{10}$$

which is a consequence of the Stirling formula. Another useful fact is that if $0 < \alpha \leq \beta \leq 1$ then

$$n\alpha^n(\beta - \alpha) \leq \beta^n - \alpha^n \leq n(\beta - \alpha). \tag{11}$$

The paper is organized as follows: Sections 4 and 5 prepare the proofs of Theorems 2 and 5, which are done in Section 6. Theorem 1 is verified in Section 7, and Section 8 discusses the simpler proof in dimensions $n = 2, 3$. Section 9 provides additional stability results about the covariogram, which lead to the proofs of Theorem 3 in Section 10, and of Theorem 4 in Section 11.

4 Comparing K and $K \cap (K + x)$

In this section we present some simpler estimates about how close K and $K_x = K \cap (K+x)$ are to be homothetic for a convex body K in $\mathbb{R}^n$. In particular we verify a good part of Theorem 2, but the results are also essential towards verifying Theorem 1. The following statement readily holds by definition (see also C.A. Rogers and G.C. Shephard [18]):

Proposition 6 *If $p, q \in K$ and $x = \lambda(p - q)$ for a K is a convex body in $\mathbb{R}^n$ and for $0 < \lambda < 1$ then*

$$(1 - \lambda)K + \lambda p \subset K_x.$$

We observe that any $x \in \mathrm{int}DK$ is of the form $\|x\|_{DK}(p-q)$ for some $p, q \in \partial K$ where there exist parallel supporting hyperplanes to K at p and q. In particular there exist parallel supporting hyperplanes to K_x at p and at $q + x$, which are naturally supporting hyperplanes also to $\|x\|_{DK} \cdot p + (1 - \|x\|_{DK})K$. Therefore Proposition 6 yields

Proposition 7 *If $x \in \mathrm{int}DK$ then $r_K(K_x) = 1 - \|x\|_{DK}$.*

Concerning the simplex T, if $x \in \mathrm{int}DT$ then T_x is the intersection of $(n + 1)$ suitable half spaces, hence T_x is homothetic to T. We conclude that

$$r_T(T_x) = R_T(T_x) = 1 - \|x\|_{DT} \quad \text{and} \quad g_T(x) = (1 - \|x\|_{DT})^n. \tag{12}$$

The simple observations so far already yield Theorem 2 (i):

Lemma 8 *If K is a convex body in $\mathbb{R}^n$ and $d(T,K) = 1+\varepsilon$ then any $x \in \mathrm{int}DK$ satisfies*

$$R_K(K_x) \leq 1 - \|x\|_{DK} + \varepsilon.$$

Proof: We may assume that $K \subset T \subset (1+\varepsilon)K$ holds for the simplex T. It follows by (12) that

$$R_K(K_x) \leq R_K(T_x) = (1 - \|x\|_{DT})R_K(T) \leq \left(1 - \tfrac{1}{1+\varepsilon}\|x\|_{DK}\right) \cdot (1+\varepsilon),$$

which in turn yields Lemma 8. $\square$

Next we sketch the argument leading to the following statement essentially verified by C.A. Rogers and G.C. Shephard [18] and H. Martini [16]:

$$\text{If } K_x \text{ is homothetic to } K \text{ whenever } \|x\|_{DK} = \tfrac{1}{2} \text{ then } K \text{ is a simplex.} \tag{13}$$

We recall that a *cone* is the convex hull of its base and its apex where the base is an $(n-1)$–dimensional compact convex set, and the apex is a point that does not lie in the affine hull of the base. Let $p_0, \ldots, p_n$ be smooth points of ∂K such that the tangent hyperplanes are pairwise different, and for any i, let $q_i \in \partial K$ satisfy $\|p_i - q_i\|_{DK} = 1$. We apply the condition to $x_i = \frac{1}{2}(p_i - q_i)$, hence $K_{x_i} = \frac{1}{2}K + \frac{1}{2}p_i$. Now if L^+ is a half plane bounded by the line p_iq_i then $L^+ \cap K$ is a triangle, which in turn yields by the smoothness of $p_i \in \partial K$ that K is a cone with apex q_i. Thus q_i is contained in the tangent hyperplane to K at any p_j, $j \neq i$, and we conclude that K is a simplex (say by induction on n).

Next we verify a stability version of (13) (see Lemma 10 below) by choosing $p_0, \ldots, p_n$ to be suitable contact points of ∂K and its John ellipsoid. We recall that the John ellipsoid is the unique ellipsoid of maximal volume contained in K. Now if the John ellipsoid of K is B^n and $u \in S^{n-1}$ then there exists a $p \in S^{n-1} \cap \partial K$ satisfying $\langle u, p\rangle \geq \frac{1}{n}$ (see, for example, K.M. Ball [1]). We deduce the following basic properties of the John ellipsoid where (i) is well-known, and (ii) is probably less commonly stated.

Proposition 9 *If B^n is the John ellipsoid of a convex body K in $\mathbb{R}^n$ then*

(i) $B^n \subset K \subset nB^n$;
(ii) there exist contact points $p_0, \ldots, p_n$ of B^n and ∂K whose convex hull contains o, and the angle of p_i and p_j is at least $\arcsin \frac{1}{n}$ *for $i \neq j$.*

Lemma 10 *If K is a convex body in $\mathbb{R}^n$ and $d_{BM}(T,K) = 1+\varepsilon$ then there exsits an x satisfying $\|x\|_{DK} = \frac{1}{2}$ and $R(K_x) \geq \frac{1}{2} + 2^{-22}n^{-12}\varepsilon^2$.*

Proof: What we actually verify is the following equivalent statement: If

$$R_K(K_x) \le \tfrac{1}{2} + \varepsilon \ \text{ holds for any } x \text{ satisfying } \|x\|_{DK} = \tfrac{1}{2} \tag{14}$$

then

$$d_{BM}(T, K) \le 1 + 2^{11} n^6 \sqrt{\varepsilon}. \tag{15}$$

Let us now assume (14). If $\varepsilon \ge 2^{-20} n^{-10}$ then (15) follows by $d_{BM}(T,K) \le 2n$ (see (9)). Therefore let

$$\varepsilon < 2^{-20} n^{-10}.$$

In the calculations below, we frequently make use of this condition without explicitly stating it. We assume that B^n is the John ellipsoid of K, and recall that the Hausdorff distance $d_H(C, M)$ of convex bodies C and M is the minimal r such that

$$C \subset M + rB^n \qquad \text{and} \qquad M \subset C + rB^n.$$

We verify (15) in two steps.

Step I: If $p \in B^n \cap \partial K$ and $q \in \partial K$ satisfies $\|p - q\|_{DK} = 1$ then there exists a cone C whose apex is q and whose base touches B^n at q such that $d_H(C, K) \le 2^5 n^5 \sqrt{\varepsilon}$.

Let q be a point of ∂K where $-p$ is an exterior normal. With the help of Proposition 9 (i), $\langle p, p - q\rangle \ge 2$ yields that B^n, hence also K contains a Euclidean ball $\widetilde{B}$ of radius $\frac{1}{n^2}$ whose centre lies in the segment pq. It follows that $\frac{1}{2}K + \frac{1}{2}p$ contains a Euclidean ball $\widetilde{B}'$ of radius $\frac{1}{2n^2}$ whose centre lies in the segment pq. During the argument we consider the shadow boundary Σ of K with respect to pq; namely, the family of $y \in \partial K$ such that the line parallel to pq and passing through y does not intersect $\mathrm{int}K$.

Let us fix

$$x = \tfrac{1}{2}(p - q),$$

hence combining $\frac{1}{2}K + \frac{1}{2}p \subset K_x$, the condition $R_K(K_x) \le \frac{1}{2} + \varepsilon$ and $DK \subset 2nB^n$ leads to

$$d(w, \tfrac{1}{2}K + \tfrac{1}{2}p) \le 2n\varepsilon \tag{16}$$

holds for any $w \in K_x$. The crucial observation is the claim that if $y \in \Sigma$ and L^+ is a (two–dimensional) half plane that is bounded by the line pq and contains y then any $z \in L^+ \cap K$ satisfies

$$d(z, \operatorname{conv}\{p, q, y\}) \leq 16n^4\varepsilon. \tag{17}$$

Let w be the point of $L^+ \cap \partial K$ such that $w - x \in \partial K$, hence $w \in L^+ \cap K_x$. In addition let l be the line which is parallel to $\operatorname{aff}\{p, q\}$, is contained in L^+ and halves the distance between y and pq. Since l touches $\frac{1}{2}K + \frac{1}{2}p$, there exists a hyperplane H that contains l and separates w and $\frac{1}{2}K + \frac{1}{2}p$. It follows by (16) that $d(w, H) \leq 2n\varepsilon$. On the other hand the centre of $\widetilde{B}'$ is of Euclidean distance at least $\frac{1}{2n^2}$ away from H, hence

$$\frac{d(l,w)}{d(l,y)} = \frac{d(l,w)}{d(l,\operatorname{aff}\{p,q\})} \leq 4n^3\varepsilon.$$

In particular the Euclidean length of $l \cap K$ is at most

$$\frac{1}{1-4n^3\varepsilon} \cdot \frac{1}{2}\, d(p,q) < (1 + 8n^3\varepsilon) \cdot \frac{1}{2}\, d(p,q).$$

Therefore the Euclidean distance of z from the triangle pqy is at most $8n^3\varepsilon\, d(p,q)$, which in turn yields the claim (17) by $d(p,q) < 2n$.

Next we claim that any $y \in \Sigma$ satisfies

$$\langle p, p - y\rangle \leq 16n^4\sqrt{\varepsilon}. \tag{18}$$

Let L^+ be the half plane that is bounded by the line pq and contains y. The existence of $\widetilde{B}$ shows that the radius ϱ of the circular arc $L^+ \cap S^{n-1}$ satisfies $\varrho \geq \frac{1}{n^2}$. Let z be the point of this arc where the tangent line is parallel to yp and the line yp separates q and z. In addition we write Δ to denote the Euclidean distance of z from the segment yp, and α to denote the angle of the half line yp and the tangent half line to $L^+ \cap S^{n-1}$ at p. Since $d(y,p) \leq 2n$, and $\Delta \leq 16n^4\varepsilon$ according to (17), the claim (18) follows by

$$\langle p, p - y\rangle \leq 2n \sin\alpha = 2n\,\frac{\sqrt{2\Delta\varrho - \Delta^2}}{\varrho} \leq 2n^2\sqrt{2\Delta} \leq 16n^4\sqrt{\varepsilon}.$$

Now let M be the family of points $z \in K$ such that $\langle p, p - z\rangle = 16n^4\sqrt{\varepsilon}$, and let $\widetilde{C}$ be the cone with base M and apex q. In particular (17) and (18) yield that any $z \in K$ satisfying $\langle p, p - z\rangle \geq 16n^4\sqrt{\varepsilon}$ is of Euclidean distance at most $16n^4\sqrt{\varepsilon}$ from K. Finally we define the cone C to be the homothetic image of $\widetilde{C}$ with centre q and ratio $\frac{\langle p, p-q\rangle}{\langle p, z-q\rangle}$ for some $z \in M$. Since the Euclidean diameter of $\widetilde{C}$ is at most $2n$, we have completed Step I.

Step II: The proof of (15)

Let $p_0,\dots,p_n$ be the $n+1$ contact points of B^n and ∂K provided by Proposition 9 (ii). We write H_i to denote the common tangent hyperplane to B^n and to K at p_i, and H_i^+ and H_i^- to denote the half spaces containing and not containing K, respectively. In addition let C_i be the cone associated to p_i in Step I, and let q_i be the apex of C_i. We deduce that

$$d_H(C_i,C_j)\leq 2^6n^5\sqrt{\varepsilon} \tag{19}$$

for any i,j. Now $\frac{1}{2}\,B^n\subset C_i$, hence writing r_i to denote the intersection of the line oq_i and H_i, we have

$$(r_i+\tfrac{1}{2}\,B^n)\cap\partial C_i=(r_i+\tfrac{1}{2}\,B^n)\cap H_i. \tag{20}$$

We claim that for any $i\neq j$,

$$d(q_i,H_j)\leq 2^{10}n^6\sqrt{\varepsilon}. \tag{21}$$

Now (19) yields that $d(r_j,y)\leq 2^6n^5\sqrt{\varepsilon}$ for some $y\in\partial C_i$. Since the angle of p_i and p_j is at least $\arcsin\frac{1}{n}$ (see Proposition 9 (ii)), it follows that $y\not\in H_i$, hence y is contained in a segment q_iz for some $z\in\partial C_j\cap H_j$. Let u be a point of the segment q_iz that satisfies $d(u,y)=\frac{1}{4}$. Now combining (19) and (20) leads to $d(u,H_j)\leq 2^6n^5\sqrt{\varepsilon}$. Since the Euclidean length of q_iz is at most $2n$, we conclude claim (21).

Next we define the half spaces

$$\widetilde{H}_i^+=H_i^+-2^{10}n^6\sqrt{\varepsilon}\cdot p_i\qquad\text{and}\qquad\widetilde{H}_i^-=H_i^--2^{10}n^6\sqrt{\varepsilon}\cdot p_i$$

for $i=0,\dots,n$, and in addition

$$\widetilde{S}=\widetilde{H}_0^+\cap\ldots\cap\widetilde{H}_n^+\qquad\text{and}\qquad S=H_0^+\cap\ldots\cap H_n^+.$$

These objects satisfy

$$o\in\operatorname{int}\widetilde{S}\qquad\text{and}\qquad q_i\in\cap_{j\neq i}\widetilde{H}_j^-\ \text{ for any } i. \tag{22}$$

Let us consider a minimal subset $p_{i_0},\dots,p_{i_k}$ of $p_0,\dots,p_n$ with the property that the convex hull of $p_{i_0},\dots,p_{i_k}$ contains the origin. Then

$$\widetilde{H}_{i_0}^-\cap\ldots\cap\widetilde{H}_{i_k}^-=\emptyset,$$

hence (22) yields that $k=n$. In particular S and $\widetilde{S}$ are homothetic simplices. We deduce by (22) that $\widetilde{S}$ is contained in the convex hull of $q_0,\dots,q_n$, thus in K, as well. Since the ratio of homothety between S and $\widetilde{S}$ is

$$\tfrac{1}{1-2^{10}n^6\sqrt{\varepsilon}} < 1 + 2^{11}n^6\sqrt{\varepsilon},$$

we conclude (15). $\square$

Finally we verify a simple consequence of Lemma 10:

Corollary 11 *If K is a convex body in $\mathbb{R}^n$ and $d_{BM}(T,K) = 1+\varepsilon$ then there exists some x satisfying $\|x\|_{DK} = \frac{1}{2}$ such that K_x contains disjoint translates of $\frac{1}{2}K$ and $2^{-25}n^{-14}\varepsilon^2 DK$.*

Proof: We may assume that $DK \subset \operatorname{int} 2nK$. Let x be the one given by Lemma 10, hence K_x contains a translate K' of $\frac{1}{2}K$ and a point $y \in K_x$ such that $y + cDK$ is disjoint from K' for $c = 2^{-23}n^{-13}$. Therefore $K'' = (1-c\varepsilon^2)y + c\varepsilon^2 K'$ is contained in K_x, and does not intersect K'. On the other hand K'' contains a translate of $\frac{c}{4n}\varepsilon^2 DK$. $\square$

5 The Minkowski measure of symmetry

If K is a convex body in $\mathbb{R}^n$ then the *Minkowski measure of symmetry* is

$$q_K = \inf\{\lambda > 0 : \exists x \in K : K \subset x - \lambda(K - x)\}.$$

In this section we relate this notion to the Banach–Mazur distance from simplices.

The Minkowski measure of symmetry (in a different framework) is thoroughly discussed in V.L. Klee [13] (see also B. Grünbaum [12], Section 6.1). V.L. Klee [13] works with a quantity r_K, defined in terms of chords of K, that he calls the critical ratio of K. The relation $q_K = r_K/(1-r_K)$ is a consequence of [13] (4.1) and the ensuing discussion. Below Lemma 12 (i) is [13] (4.6), and (ii) follows from [13] (3.10).

Lemma 12 *Let K be a convex body in $\mathbb{R}^n$. Then*

(i) $q_K \leq n$, with equality if and only if K is a simplex;
(ii) If $q_K > n-1$ then there exists a unique point s_K of K satisfying that $K \subset s_K - q_K(K - s_K)$;
(iii) If $q_K > n-1$ then there exists an n-simplex T_K that contains s_K, and whose vertices are common boundary points of K and $s_K - q_K(K - s_K)$.

Remark: T_K may not be unique but we fix one suitable simplex. It is not hard to see that $n-1$ in (ii) and (iii) can not be replaced by any number smaller than $n-1$.

Let K be a convex body in $\mathbb{R}^n$ with $q_K > n-1$. In the arguments below, we will denote the vertices of T_K by $v_0, \ldots, v_n$, and the facet of T_K opposite to v_i by F_i. We may assume that T_K is a regular simplex whose centroid is the origin, and each v_i is a (Euclidean) unit vector, hence

$$\langle v_i, y\rangle = \tfrac{-1}{n} \text{ for any } y \in F_i.$$

The definition of s_K provides $w_i \in \partial K$, $i = 0, \ldots, n$, such that there exist different parallel supporting hyperplanes at v_i and w_i to K for each i, and

$$s_K = \tfrac{1}{q_K+1}\, v_i + \tfrac{q_K}{q_K+1}\, w_i. \tag{23}$$

In other words, the chords v_iw_i meet in the common point s_K, and $\frac{\|v_i - s_K\|}{\|w_i - s_K\|} = q_K$, $0 \le i \le n$.

Proposition 13 *Let $0 < \varepsilon < \frac{1}{2}$, and let K be a convex body in $\mathbb{R}^n$ with $q_K = n - \varepsilon$. Then using the notation as above,*

$$w_i \in \tfrac{-1}{n}\, v_i - \tfrac{2\varepsilon}{n}\, T_K \textit{ for } i = 0, \ldots, n.$$

Proof: Since $\langle v_i, w_i\rangle \le \frac{-1}{n}$, (23) yields

$$\langle v_i, s_K\rangle \le \left(\tfrac{n}{q_K+1} - \tfrac{q_k}{q_K+1}\right) \cdot \tfrac{1}{n} = \tfrac{n-\varepsilon}{n+1-\varepsilon} \cdot \tfrac{1}{n}.$$

In particular $s_K \in -\frac{\varepsilon}{n+1-\varepsilon}\, T_K$. Since w_i is the image of s_K *via* the homothety of centre v_i and of ratio $\frac{q_K+1}{q_K}$, we deduce

$$w_i \in -\tfrac{\varepsilon}{n-\varepsilon}\, T_K - \tfrac{1}{n-\varepsilon}\, v_i = -\tfrac{\varepsilon}{n-\varepsilon}\, T_K - \tfrac{\varepsilon}{n(n-\varepsilon)}\, v_i - \tfrac{1}{n}\, v_i \in -\tfrac{n+1}{n(n-\varepsilon)}\, T_K - \tfrac{1}{n}\, v_i.$$

In turn we conclude Proposition 13. $\square$

We have arrived at our main estimate linking the Minkowski measure of symmetry to the Banach–Mazur distance.

Lemma 14 *If K is a convex body in $\mathbb{R}^n$ with $q_K = n - \varepsilon$ then*

(i) $d_{BM}(K, T) \ge 1 + \frac{\varepsilon}{n}$;
(ii) $d_{BM}(K, T) < 1 + 4n\varepsilon$ *provided* $0 < \varepsilon < \frac{1}{4n}$.

Remark: Assuming the normalization as after Lemma 12, we actually verify (ii) in the form

$$K \subset (1+4n\varepsilon)T_K. \tag{24}$$

Proof: Let $T \subset K$ be a simplex such that a translate of $d_{BM}(T,K)\,T$ contains K. Since $-T$ is contained in a translate of $q_K K$, it follows that a translate of $q_K d_{BM}(T,K)\,T$ contains $-T$. Since $q_K d_{BM}(T,K) \geq n$ holds by the equality condition of Proposition 12(i), we deduce

$$d_{BM}(T,K) \geq \tfrac{n}{n-\varepsilon} > 1 + \tfrac{\varepsilon}{n}.$$

Next we assume that $0 < \varepsilon < \frac{1}{4n}$, and use the notation for T_K as after Lemma 12. In order to verify (24), let $y \in K$ be different from the origin where we may assume that $y \in \operatorname{pos}\{v_1, \ldots, v_n\}$. We choose $\lambda > 0$ in a way that $\frac{1}{\lambda}\,y \in F_0$. Since $-\frac{1}{n}\,v_0$ is the centroid of F_0, there exist $z \in F_0$ and $\frac{1}{n} \leq \mu \leq \frac{n-1}{n}$ such that

$$\tfrac{\mu}{\lambda}\,y + (1-\mu)z = -\tfrac{1}{n}\,v_0.$$

Now $-\frac{1+2\varepsilon}{n}\,v_0$ is a vertex of $-\frac{1}{n}\,v_i - \frac{2\varepsilon}{n}\,T_K$, and $-\frac{1+2\varepsilon}{n}\,v_0$ is not contained in $\operatorname{int} K$ according to Proposition 13. Since $y, z \in K$ and $o \in \operatorname{int} K$, we deduce

$$\tfrac{(1+2\varepsilon)\mu}{\lambda} + (1+2\varepsilon)\cdot(1-\mu) \geq 1,$$

which in turn yields by $\varepsilon < \frac{1}{4n}$ that

$$\lambda \leq \frac{1+2\varepsilon}{1+2\varepsilon-\frac{2\varepsilon}{\mu}} \leq \frac{1+2\varepsilon}{1+2\varepsilon-2n\varepsilon} < 1+4n\varepsilon.$$

Therefore we conclude (24). $\square$

Corollary 15 *If $d_{BM}(K,T) = 1+\varepsilon$ for a convex body K in $\mathbb{R}^n$ and for $0 < \varepsilon < \frac{1}{4n^2}$ then*

$$n - \tfrac{1}{4n} \leq q_K < n - \tfrac{\varepsilon}{4n}.$$

Proof: Lemma 14 (i) implies that

$$q_K \geq 2n - n d_{BM}(K,T) = n - n\varepsilon > n - \tfrac{1}{4n}.$$

Therefore we may apply Lemma 14 (ii), and deduce that

$$q_K < \tfrac{4n^2+1-d_{BM}(K,T)}{4n} = n - \tfrac{\varepsilon}{4n}.$$

$\square$

6 Proofs of Theorems 2 and 5

The arguments leading to Theorem 2 (ii) and to Theorem 5 follow essentially the same line (only the proof of Theorem 2 (ii) is much simpler):

Proof of Theorem 5: First let $\varepsilon \geq \frac{1}{4n^2}$, in which case the proof of Theorem 5 depends on Corollary 11. In particular there exist an $x_0 \in DK$ with $\|x_0\|_{DK} = \frac{1}{2}$, moreover some y and z such that

$$y + \tfrac{1}{2}K \quad \text{and} \quad z + c\varepsilon^2 DK, \quad c = 2^{-25}n^{-14}$$

are disjoint and contained in K_{x_0}. We define

$$C = z + \tfrac{c^n}{4n}\,\varepsilon^{2n} DK.$$

Let $x \in C$, hence

$$\|x\|_{DK} \geq \tfrac{1}{2} - \tfrac{c^n}{4n}\,\varepsilon^{2n}.$$

Since $\varepsilon \leq 2n$ (see (9)), it follows that K_x contains the disjoint convex bodies

$$y + (1-\lambda)K, \quad \lambda = \tfrac{1}{2} + \tfrac{c^n}{4n}\,\varepsilon^{2n} \qquad \text{and} \qquad Q = z + \tfrac{c}{2}\,\varepsilon^2 DK$$

where the Brunn–Minkowski inequality (7) yields

$$V(Q) \geq c^n \varepsilon^{2n} V(K).$$

Now according to (11), we have

$$(1 - \|x\|_{DK})^n - (1-\lambda)^n \leq n(\lambda - \|x\|_{DK}) \leq \tfrac{c^n}{2}\,\varepsilon^{2n},$$

thus $\varepsilon^{2n} \geq 2^{-4n}n^{-4n}$ and $\frac{1}{2} \geq \frac{1}{n}\varepsilon$ imply

$$g_K(x) \geq (1 - \|x\|_{DK})^n + \tfrac{c^n}{2}\,\varepsilon^{2n} \geq (1 - \|x\|_{DK})^n + 2^{-29n}n^{-19n}\varepsilon.$$

In addition

$$V(C) \geq 2^{-30n}n^{-19n}V(DK),$$

which completes the proof of Theorem 5 if $\varepsilon \geq \frac{1}{4n^2}$.

Therefore let $\varepsilon < \frac{1}{4n^2}$ where we may assume that $\varepsilon > 0$. In the calculations below, we use the condition $\varepsilon < \frac{1}{4n^2}$, and the following observations without explicitly referring to them: Given $0 \leq i \leq n$, we have $-v_i = \sum_{j \neq i} v_j$, and if $y = \sum_{j\neq i} \alpha_j v_j$ for $\alpha_j \geq 0$ then

$$y \in \left(\textstyle\sum_{j\neq i} \alpha_j\right) \cdot T_K.$$

According to Corollary 15, we may use the notation for T_K, *etc*, as after Lemma 12, and the Minkowski measure of symmetry satisfies

$$q_K < n - \tfrac{\varepsilon}{4n}.$$

We deduce by Proposition 13 that

$$w_i \in \tfrac{-1}{n}\, v_i + 2\varepsilon T_K \subset \tfrac{-1}{n}\, v_i + \tfrac{1}{2n^2}\, T_K \quad \text{for } i = 0, \ldots, n. \tag{25}$$

For each w_i, there exists a supporting hyperplane H_i to K at w_i such that the parallel hyperplane through v_i supports K, as well. We write H_i^- to denote the half space determined by H_i and not intersecting $\operatorname{int} K$. In addition let

$$T_i = \operatorname{conv}\{w_i, F_i\}.$$

We may assume that $s_K \in \operatorname{conv}\{o, v_1, \ldots, v_n\}$, hence

$$\tfrac{\langle v_1 - w_0, v_0\rangle}{\langle v_0 - v_1, v_0\rangle} \geq \tfrac{q_K^{-1} - n^{-1}}{1 + n^{-1}} \geq \tfrac{\varepsilon}{6n^3}. \tag{26}$$

We may also assume that v_1 is a vertex of F_0 such that $F_0 - v_1 \subset H_0^- - w_0$, and define

$$T^* = w_0 + \tfrac{1}{16n}\,(T_0 - v_1).$$

It follows by the properties of H_0^-, and by (25) that

$$T^* \cap \operatorname{int} K = \emptyset; \tag{27}$$

$$T^* \subset \tfrac{-1}{n}\, v_0 + \tfrac{1}{3}\, T_K. \tag{28}$$

We deduce by (24) that

$$V(K) \leq (1 + 4n\varepsilon)^n V(T_K) < 3V(T_K). \tag{29}$$

Thus combining (26) and (29) leads to

$$V(T_0) \geq \tfrac{\langle v_1 - w_0, v_0\rangle}{\langle v_0 - v_1, v_0\rangle} \cdot V(T_K) \geq \tfrac{\varepsilon}{6n^3}\, V(T_K) > \tfrac{\varepsilon}{18n^3}\, V(K),$$

which in turn yields

$$V(T^*) > \tfrac{\varepsilon}{2^{10n} n^n} \cdot V(K). \tag{30}$$

First we construct C. Let B^{n-1} be the unit $(n-1)$–dimensional Euclidean ball whose centre is the origin, and whose affine hull is parallel to $\operatorname{aff} F_1$. It follows by (25) that the segment w_1v_1 intersects F_1 in a point r, and $r+\frac{1}{2n}\,B^{n-1}$ is contained in the relative interior of F_1. Let

$$B^* = w_1 + \gamma(w_1 - r) + \tfrac{\gamma}{2n}\,B^{n-1} \quad \text{for} \quad \gamma = \tfrac{1}{2^{13n}n^n},$$

which satisfies by the definition of H_1^- and by (25) that

$$B^* \subset H_1^- \quad \text{and} \quad B^* \subset \tfrac{-1}{n}\,v_1 + \tfrac{1}{6}\,T_K. \tag{31}$$

Now we define

$$C = \{t(y - v_1) : \tfrac{1}{3} \le t \le \tfrac{1}{2} \;\text{ and }\; y \in B^*\}.$$

Since F_1 is contained in a translate of B^{n-1}, and $V(DK) < 3\cdot 4^n V(T_K)$ according to (29) and the Rogers–Shepard inequality (1), we deduce that

$$V(C) > (\tfrac{1}{2^n} - \tfrac{1}{3^n})\tfrac{\gamma^n}{2^n n^n}\,V(T_K) > \tfrac{\gamma^n}{2^{6n}n^n}\,V(DK) > \tfrac{1}{2^{19n^2}n^{2n^2}}\,V(DK).$$

Given $x \in C$, we write p_x to denote the second intersection point of the line $v_1(v_1+x)$ with ∂K, and define $\lambda_x > 0$ by the relation

$$x = \lambda_x(p_x - v_1).$$

Thus the definition of C and (31) yield that

$$\tfrac{1}{3} < \lambda_x < \tfrac{2}{3} \quad \text{and} \quad p_x \in \tfrac{-1}{n}\,v_1 + \tfrac{1}{3}\,T_K. \tag{32}$$

Since $\lambda_x \frac{-1}{n}\,v_1 + (1-\lambda_x)\frac{-1}{n}\,v_0 \in \frac{2}{3}\,T_K$, combining (28) and (32) leads to

$$\lambda_x p_x + (1-\lambda_x)T^* \subset T_K \subset K.$$

On the other hand $\lambda_x p_x + (1-\lambda_x)T^* - x \subset T_0 \subset K$, hence we deduce

$$\lambda_x p_x + (1-\lambda_x)T^* \subset K_x.$$

Now $\lambda_x p_x + (1-\lambda_x)K \subset K_x$, as well, according to Proposition 6 (ii), therefore (27) and (30) imply

$$V(K_x) > (1-\lambda_x)^n(V(K) + V(T^*)) > \left((1-\lambda_x)^n + \tfrac{\varepsilon}{2^{12n}n^n}\right)\cdot V(K). \tag{33}$$

The final part of the proof is to compare λ_x and $\|x\|_{DK}$ for $x \in C$. We write y_x to denote the intersection point of the line v_1p_x and B^*. Since

$$w_1 - \gamma(w_1 - r) + \tfrac{\gamma}{2n}\,B^{n-1}$$

is contained in $T_1 \subset K$, it intersects the segment $v_1 p_x$. It follows by (25) and the definition of B^* that

$$\langle p_x - y_x, v_1 \rangle < \tfrac{2\gamma}{n} \cdot \varepsilon,$$

which in turn yields

$$\|x\|_{DK} \leq \lambda_x < \left(1 + \tfrac{2\gamma}{n}\,\varepsilon\right) \cdot \|x\|_{DK} < \|x\|_{DK} + \tfrac{2\gamma}{n}\,\varepsilon.$$

Since $(1 - \|x\|_{DK})^n - (1 - \lambda_x)^n < 2\gamma\,\varepsilon$, the estimate (33) leads to

$$g_K(x) \geq (1 - \|x\|_{DK})^n + \tfrac{\varepsilon}{2 \cdot 2^{12n} n^n} > (1 - \|x\|_{DK})^n + 2^{-13n} n^{-n} \varepsilon.$$

Therefore we conclude Theorem 5. $\square$

Proof of Theorem 2: Theorem 2 (i) was verified as Lemma 8.

If $\varepsilon \geq \frac{1}{4n^2}$ then Lemma 10 yields that there exists x satisfying $\|x\|_{DK} = \frac{1}{2}$ and $R(K_x) \geq \frac{1}{2} + 2^{-24} n^{-14} \varepsilon$, hence we deduce Theorem 2 (ii) in this case.

Therefore let $0 < \varepsilon < \frac{1}{4n^2}$. In this case the first half of the proof runs word by word as the corresponding argument for Theorem 5 until (28), and we only discuss how to complete the argument. It follows by (24) that $K \subset 2T_K$, hence (26) implies that

$$r_K(T^*) = \tfrac{1}{16n}\, r_K(T_0) \geq \tfrac{1}{16n^2} \cdot \tfrac{\langle v_1 - w_0, v_0 \rangle}{\langle v_0 - v_1, v_0 \rangle}\, r_K(T_K) \geq \tfrac{\varepsilon}{2^8 n^5}. \tag{34}$$

The x in Theorem 2 (ii) will be

$$x = \tfrac{1}{2}\,(w_1 - v_1).$$

Since $\frac{1}{2}\,(\frac{-1}{n}\, v_0 + \frac{-1}{n}\, v_1) \in \frac{1}{2}\, T_K$, (28) yields that $\widetilde{T} = \frac{1}{2}\, T^* + \frac{1}{2}\, w_1$ is contained in T_K. In addition $\widetilde{T} - x = \frac{1}{2}\, T^* + \frac{1}{2}\, v_1$ is contained in K, therefore $\widetilde{T} \subset K_x$. On the other hand $\frac{1}{2}\, K + \frac{1}{2}\, w_1$ is disjoint from $\widetilde{T}$ by (27), and contained in K_x according to Proposition 6. Thus (34) implies that

$$R(K_x) \geq \tfrac{1}{2} + 2^{-9} n^{-5} \varepsilon,$$

which completes the proof of Theorem 2. $\square$

7 The proof of Theorem 1

We start with the easy part of Theorem 1. If $d_{BM}(K,T)=1+\varepsilon$ then we may assume that $T\subset K\subset(1+\varepsilon)T$, hence

$$V(DK)\geq V(DT)=\binom{2n}{n}V(T)\geq\frac{1}{(1+\varepsilon)^n}\binom{2n}{n}V(K)\geq(1-n\varepsilon)\binom{2n}{n}V(K).$$

Therefore we conclude the lower bound in Theorem 1.

In order to verify the non–trivial part of Theorem 1, let us summarize some formulae observed by C.A. Rogers and G.C. Shephard [18]. Writing χ_M to denote the characteristic function of a subset M of $\mathbb{R}^n$, the normalized covariogram $g_K(x)$ (see (8)) satisfies

$$\begin{aligned}\int\limits_{DK} g_K(x)\,dx &= \frac{1}{V(K)}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\chi_K(y)\chi_K(y-x)\,dydx\\ &=\frac{1}{V(K)}\int\limits_{\mathbb{R}^n}\chi_K(y)\left(\int\limits_{\mathbb{R}^n}\chi_K(y-x)\,dx\right)dy=V(K). \qquad (35)\end{aligned}$$

Since Proposition 7 yields

$$g_K(x)\geq(1-\|x\|_{DK})^n\,, \qquad (36)$$

the Roger-Shepard inequality (1) readily follows by

$$\begin{aligned}\int\limits_{DK}(1-\|x\|_{DK})^n\,dx &= n\int\limits_0^1(1-t)^nt^{n-1}dt\cdot V(DK)\\ &=\binom{2n}{n}^{-1}V(DK). \qquad (37)\end{aligned}$$

Now let $d_{BM}(K,T)=1+\varepsilon$. We apply the extra information provided by Theorem 5; namely, even

$$g_K(x)\geq(1-\|x\|_{DK})^n+n^{-48n}\varepsilon$$

if $x\in C$ for some convex body $C\subset DK$ such that $V(C)\geq n^{-25n^2}V(DK)$. Therefore

$$\begin{aligned}V(K)=\int\limits_{DK} g_K(x)\,dx &\geq\left(\binom{2n}{n}^{-1}+n^{-50n^2}\varepsilon\right)V(DK)\\ &\geq\binom{2n}{n}^{-1}\left(1-n^{-50n^2}\varepsilon\right)^{-1}V(DK),\end{aligned}$$

which in turn yields the upper bound of Theorem 1. $\square$

8 The Godbersen conjecture

In this section we use the notion of mixed volumes (see R. Schneider [19] for thorough discussion). We write $V(K,C;i)$ to denote mixed volumes depending on the convex bodies K and C in $\mathbb{R}^n$ where C occurs i times.

We may assume that $-K \subset q_K K$. According to the definition of mixed volumes, we have

$$V(DK) = \sum_{i=0}^{n} \tbinom{n}{i} V(K,-K;i)$$

where $V(K) = V(K,-K;0) = V(K,-K;n)$. We deduce by the monotonicity of the mixed volumes that

$$V(K,-K;1) = V(K,-K;n-1) \leq q_K V(K). \tag{38}$$

Let $d_{BM}(K,T) = 1+\varepsilon$. It follows by Lemma 14 (ii) and by $d_{BM}(K,T) \leq 2n$ that

$$q_K < n - \tfrac{1}{8n^2} \cdot \varepsilon,$$

hence

$$V(K,-K;1) = V(K,-K;n-1) \leq \left(n - \tfrac{1}{8n^2} \cdot \varepsilon\right) V(K). \tag{39}$$

If $n = 2,3$ then we conclude that

$$V(DK) \leq \left(6 - \tfrac{1}{32} \cdot \varepsilon\right) V(K) \quad \text{in } \mathbb{R}^2 \ ;$$
$$V(DK) \leq \left(40 - \tfrac{1}{36} \cdot \varepsilon\right) V(K) \quad \text{in } \mathbb{R}^3.$$

Now the conjecture of C. Godbersen [7] states that

$$V(K,-K;i) \leq \tbinom{n}{i} V(K) \quad \text{for any } i. \tag{40}$$

Here equality holds if K is a simplex, and the conjecture is verified in C. Godbersen [7] if K is a convex body of constant width. If K is any convex body and $i = 1, n-1$ then (39) yields (40). On the other hand for $n \geq 4$ and a convex body K in $\mathbb{R}^n$, we have that if the Godbersen conjecture holds for any i then

$$V(DK) \leq \left(\tbinom{2n}{n} - \tfrac{1}{4n^2} \cdot \varepsilon\right) V(K). \tag{41}$$

9 More about the covariogram

Given a convex body K in $\mathbb{R}^n$, we present some additional properties of the normalized covariogram $g_K(x)$. A simple observation (see also R.J. Gardner and G. Zhang [5]) is that if $x \in \partial DK$ then

$$g_K(tx) \text{ is a strictly monotone decreasing function of } t \in [0,1]. \tag{42}$$

Instead of (7), we will use the general form of the Brunn–Minkowski inequality (see R. Schneider [19]); namely, if M is any convex body in $\mathbb{R}^n$ and $0 \leq \lambda \leq 1$ then

$$V(\lambda K + (1-\lambda)M)^{1/n} \geq \lambda V(K)^{1/n} + (1-\lambda)V(M)^{1/n}. \tag{43}$$

For any $x, y \in DK$, we readily have

$$\lambda K_x + (1-\lambda)K_y \subset K_{\lambda x + (1-\lambda)y}. \tag{44}$$

We conclude by the Brunn–Minkowski inequality that

$$g_K(x)^{1/n} \text{ is concave on } DK \tag{45}$$

(see also R.J. Gardner and G. Zhang [5]).

The goal of this section is to provide stability versions of (45). The natural tool is to use a stability version of the Brunn–Minkowski inequality, say the following one: *There exists a positive constant c_0 depending on n such that if the convex bodies K and M in $\mathbb{R}^n$ satisfy $\frac{1}{7} \leq r_K(M) \leq R_K(M) \leq 1$ and $\frac{R_K(M)}{r_K(M)} \geq 1 + \varepsilon$ then*

$$V(\lambda K + (1-\lambda)M)^{1/n} \geq \lambda V(K)^{1/n} + (1-\lambda)V(M)^{1/n} + c_0 V(K)^{1/n}\varepsilon^n \tag{46}$$

holds for any $\frac{1}{7} \leq \lambda \leq \frac{6}{7}$.
This result can be easily deduced from V.I. Diskant [3] (see also H. Gromer [9]), and the actual value of c_0 is unknown because the corresponding constant in Diskant's result is unknown. Actually H. Groemer [8] verifies a stability version of the Brunn–Minkowski inequality with explicit constant, only the error term is of order ε^{n+1}.

Lemma 16 will be an essential ingredient of the proof of Theorem 4. We say that a closed convex set Σ is an *infinite cone* if $\lambda x \in \Sigma$ for any $x \in \Sigma$ and $\lambda \geq 0$.

Lemma 16 *Given a convex body K in $\mathbb{R}^n$ with $d_{BM}(K,T)=1+\varepsilon$, there exists an infinite cone Σ with $V(\Sigma\cap DK)\geq n^{-25n^2}V(DK)$ such that the following holds: if $x\in\Sigma\cap\partial DK$ and either $\frac{1}{7}\leq s\leq\frac{2}{7}$ and $\frac{3}{7}\leq t\leq\frac{4}{7}$, or $\frac{3}{7}\leq s\leq\frac{4}{7}$ and $\frac{5}{7}\leq t\leq\frac{6}{7}$ then*

$$g(tx)\leq\tfrac{t}{s}\,g(sx)-\tfrac{t-s}{s}-c\varepsilon^n,$$

where c is a positive constant depending only on n.

Proof: Using the convex body C given by Theorem 5, we define

$$\Sigma=\{\lambda y:\lambda\geq 0\quad\text{and}\quad y\in C\},$$

hence readily $V(\Sigma\cap DK)\geq n^{-25n^2}V(DK)$. Let $\frac{2}{7}\leq t\leq\frac{6}{7}$. It follows by Theorem 5 that

$$g_K(tx)\geq\left(1-t+\tfrac{1}{7n}\,n^{-48n}\varepsilon\right)^n.$$

Since $r_K(K_{tx})=1-t$ according to Proposition 7, we deduce that

$$\tfrac{R_K(K_{tx})}{r_K(K_{tx})}\geq 1+\tfrac{1}{7n}\,n^{-48n}\varepsilon\geq 1+n^{-50n}\varepsilon. \tag{47}$$

Now let s satisfy $\frac{1}{7}\leq s\leq\frac{5}{7}$ and $s\leq t-\frac{1}{7}$. We apply (46) to $M=K_{tx}$ and $\lambda=\frac{t-s}{t}$, and obtain that

$$g_K(sx)^{1/n}\geq\tfrac{t-s}{t}+\tfrac{s}{t}\,g_K(tx)^{1/n}+c_0n^{-50n^2}\varepsilon^n.$$

Therefore elementary calculations yield Lemma 16. □

Based on the stability version of the Brunn–Minkowski inequality in H. Groemer [8], one could get an explicit constant in Lemma 16, only the error term is of order ε^{n+1} instead of ε^n.

In order to prove the stability version Theorem 3 of the Zhang projection inequality, it suffices to use a somewhat weaker statement than Lemma 16, namely Lemma 17 (iii) below. We provide a proof independent of (46) because on this way we obtain an error term that is explicit and is of the same order as the one in Lemma 16.

For any non-zero u, we write $u^\perp$ to denote the linear $(n-1)$–plane orthogonal to u, K/u to denote the projection of K into $u^\perp$, and $|K/u|$ to denote the $(n-1)$–dimensional Lebesgue measure of K/u. In addition we breviate the L^2–norm $\|\cdot\|_{B^n}$ as $\|\cdot\|_2$.

Let $x\in\partial DK$. Then $Z_K(x)/x=K/x$, and any secant of $Z_K(x)$ parallel to x is of Euclidean length $\|x\|_2$, hence

$$V(Z_K(x))=|K/x|\cdot\|x\|_2. \tag{48}$$

It is not hard to verify (see, for example, R.J. Gardner and G. Zhang [5]) that $V(K_{tx})$ is a differentiable function of $t \in [0,1)$, and

$$\tfrac{d}{dt} V(K_{tx}) = -|K_{tx}/x| \cdot \|x\|_2 \tag{49}$$

where the derivative at $t = 0$ is one–sided. We note that Lemma 17 (i) is well–known (see, for example, H. Martini [16]), and Lemma 17 (ii) is verified in R.J. Gardner and G. Zhang [5]. For sake of completeness, we provide the short proofs of these statements, as well.

Lemma 17 *If K is a convex body in $\mathbb{R}^n$ then for any $x \in DK$, $x \neq 0$,*

(i) $V(Z_K(x)) \leq nV(K)$ where equality holds when K is a simplex;
(ii) $g_K(x) \leq \left(1 - \frac{V(Z_K(x))}{nV(K)} \cdot \|x\|_{DK}\right)^n$;
(iii) there exists an infinite cone Σ with $V(\Sigma \cap DK) \geq n^{-25n^2} V(DK)$ such that assuming $d_{BM}(K,T) = 1 + \varepsilon$, $x \in \Sigma$ and $\frac{1}{2} \leq \|x\|_{DK} \leq \frac{3}{4}$, we have

$$g_K(x) \leq \left(1 - \tfrac{V(Z_K(x))}{nV(K)} \cdot \|x\|_{DK}\right)^n - n^{-53n^2}\varepsilon^n.$$

Proof: We fix an $x \in \partial DK$, and define $f(t) = g_K(tx)^{1/n}$ for $0 \leq t \leq 1$. Then f is concave, and

$$f'(t) = -\frac{|K_{tx}/x| \cdot \|x\|_2}{nV(K)^{\frac{1}{n}} V(K_{tx})^{\frac{n-1}{n}}} \tag{50}$$

follows by (49). In particular (48) yields

$$f'(0) = -\tfrac{V(Z_K(x))}{nV(K)}, \tag{51}$$

which in turn implies (ii). Combining (ii) and $f(t) \geq 1 - t$ (see (36)) leads to the inequality of (i). Since $f(t) = 1 - t$ when K is a simplex (see (12)), (51) completes the proof of (i).

We define the infinite cone Σ of (iii) to be the positive hull of the convex body C given by Theorem 5; more precisely,

$$\Sigma = \{\lambda y : \lambda \geq 0 \quad \text{and} \quad y \in C\}.$$

Readily $V(\Sigma \cap DK) \geq n^{-25n^2} V(DK)$.

Let $x \in \partial DK \cap \Sigma$. We verify that the function f defined above satisfies

$$f'(\tfrac{1}{4}) \leq f'(0) - n^{-51n^2}\varepsilon^n. \tag{52}$$

According to Theorem 5 and (11), $f(tx) \geq 1 - t + \frac{1}{n} n^{-48n}\varepsilon$ holds for some $0 < t < 1$, hence the concavity of f implies

$$g_K(\tfrac{1}{4}x) \geq \left(\tfrac{3}{4} + \tfrac{1}{4n} n^{-48n}\varepsilon\right)^n. \tag{53}$$

We define $0 < \varrho \leq 1$ by $\varrho^{n-1} = \frac{|K_{x/4}/x|}{|K/x|}$. It follows by (44) and the Brunn–Minkowski inequality that $|K_{tx}/x|^{\frac{1}{n-1}}$ is a concave function of t, hence $\varrho \geq \frac{3}{4}$. In addition $\frac{|K_{x/4}/x|}{|K/x|}$ is at least $(1-4(1-\varrho)t)^{n-1}$ if $t \leq \frac{1}{4}$, and at most $(1-4(1-\varrho)t)^{n-1}$ if $t \geq \frac{1}{4}$, therefore (49) yields

$$g_K(\tfrac{1}{4}x) = \frac{\int_{1/4}^1 -\left(\frac{d}{dt}V(K_{tx})\right)dt}{\int_0^1 -\left(\frac{d}{dt}V(K_{tx})\right)dt} \leq \frac{\int_{1/4}^1 (1-4(1-\varrho)t)^{n-1}dt}{\int_0^1 (1-4(1-\varrho)t)^{n-1}dt}.$$

Evaluating the integrals leads to

$$g_K(\tfrac{1}{4}x) \leq \begin{cases} \frac{\varrho^n - (4\varrho-3)^n}{1-(4\varrho-3)^n} & \text{if} \quad \varrho < 1 \\ \frac{3}{4} & \text{if} \quad \varrho = 1. \end{cases} \tag{54}$$

If $\varrho < 1$ then we combine (50) and (54), and obtain

$$\frac{|f'(\frac{1}{4})|}{|f'(0)|} = \frac{|K_{x/4}/x|}{|K/x| \cdot g_K(\frac{1}{4}x)^{\frac{n-1}{n}}} \geq \left(\frac{\varrho^n - \varrho^n(4\varrho-3)^n}{\varrho^n - (4\varrho-3)^n}\right)^{\frac{n-1}{n}}.$$

Since the right hand side in (54) is at most ϱ^n, we deduce $\varrho \geq \frac{3}{4} + \frac{1}{4n} n^{-48n}\varepsilon$ by (53), hence

$$\tfrac{|f'(\frac{1}{4})|}{|f'(0)|} \geq \left(1 + \tfrac{1-\varrho^n}{\varrho^n - (4\varrho-3)^n} \cdot n^{-49n^2}\varepsilon^n\right)^{\frac{n-1}{n}} \geq 1 + n^{-50n^2}\varepsilon^n$$

follows by (11) and $(1+s)^{\frac{n-1}{n}} \geq 1 + \frac{1}{2}s$. Now a simpler argument (using only (54) and $\varepsilon < 2n$) leads to $\frac{|f'(\frac{1}{4})|}{|f'(0)|} \geq 1 + n^{-50n^2}\varepsilon^n$, also if $\varrho = 1$. Since $|f'(0)| \geq \frac{1}{n}$, we conclude the claim (52).

Finally (52) yields $f(t) \leq 1 + f'(0)t - \frac{1}{4} n^{-51n^2}\varepsilon^n$ for $t \geq \frac{1}{2}$. On the other hand $f(t) \geq \frac{1}{4}$ for $t \leq \frac{3}{4}$, hence (11) completes the proof of Lemma 17. □

Corollary 18 *If $d_{BM}(K, T) = 1 + \varepsilon$ for a convex body K in $\mathbb{R}^n$, then for any $x \in DK$*

$$g_K(x)^{1/n} \leq 1 - \tfrac{1}{(1+\varepsilon)^n} \|x\|_{DK}.$$

Proof: We may assume that $x \neq 0$ and $T \subset K \subset (1+\varepsilon)T$, hence Lemma 17 (i) implies

$$V(Z_K(x)) \geq V(Z_T(x)) = nV(T) \geq \tfrac{n}{(1+\varepsilon)^n} V(K).$$

Therefore Lemma 17 (ii) completes the argument. $\Box$

10 The proof and optimality of Theorem 3

Given a convex body K in $K \in R^n$, the projection body ΠK is the o–symmetric convex body satisfying that the value of the support function at $u \in S^{n-1}$ is $|K/u|$; namely, $|K/u| = \max\{< u, x >: x \in \Pi K\}$ (see R. Schneider [19]). If K is a polytope then ΠK is a zonotope, which is shown by the following geometric interpretation: Let $F_1, \ldots, F_m$ be the facets of K, and let v_i be the exterior normal to F_i whose Euclidean length is $|F_i|$. Writing $[o, v_i]$ to denote the segment ov_i, we have

$$\Pi K = \sum_{i=1}^{m} [o, v_i]. \tag{55}$$

Now $\Pi^* K$ is the polar of ΠK; namely, if $u \in S^{n-1}$ then $\varrho_{\Pi^* K}(u) = \frac{1}{|K/u|}$, which definition coincides with (4) according to (48).

Proof of Theorem 3: Let K be a convex body in $\mathbb{R}^n$ satisfying $d_{BM}(K,T) = 1+\varepsilon$. We may assume that $T \subset K \subset (1+\varepsilon)T$. Since $1+\varepsilon \leq 2n$ (see (9)), and equality holds in the Zhang projection inequality for simplices, we deduce that

$$V(K)^{n-1} V(\Pi^* K) \leq (1+\varepsilon)^{n(n-1)} V(T) V(\Pi^* T) \leq \left(1 + n^3 (2n)^{n^2} \varepsilon\right) n^{-n} \binom{2n}{n}.$$

In turn we conclude the lower bound of Theorem 3.

In case of the upper bound of Theorem 3, we use the proof of the Zhang projection inequality provided in R.J. Gardner and G. Zhang [5]. Let Σ be the cone described in Lemma 17 (iii), and let

$$E(x) = \begin{cases} n^{-53n^2} \varepsilon^n & \text{if } x \in \Sigma \text{ and } \frac{1}{2} \leq \|x\|_{DK} \leq \frac{3}{4}; \\ 0 & \text{otherwise.} \end{cases}$$

Then $E(x)$ satisfies $\int_{DK} E(x)\,dx \geq \frac{1}{2^n} n^{-88n^2}\varepsilon^n V(DK) \geq n^{-88n^2}\varepsilon^n V(K)$. It follows by (35) and Lemma 17 that

$$V(K) = \int\limits_{DK} g_K(x)\,dx \leq \int\limits_{DK\setminus 0} \left(1 - \tfrac{V(Z_K(x))}{nV(K)} \cdot \|x\|_{DK}\right)^n - E(x)\,dx.$$

After rearranging, and writing u_x to denote the Euclidean exterior unit normal at $x \in \partial DK$, we obtain

$$(1 + n^{-88n^2}\varepsilon^n)V(K) \leq n \int\limits_{\partial DK} \int\limits_0^1 \left(1 - \tfrac{V(Z_K(x))}{nV(K)} \cdot t\right)^n t^{n-1} dt\, \langle u_x, x\rangle dx.$$

Now $\frac{V(Z_K(x))}{nV(K)} \leq 1$ according to Lemma 17 (i), hence the substitution $s = \frac{V(Z_K(x))}{nV(K)} \cdot t$ leads to

$$\int\limits_0^1 \left(1 - \tfrac{V(Z_K(x))}{nV(K)} \cdot t\right)^n t^{n-1} dt \leq \tfrac{n^n V(K)^n}{V(Z_K(x))^n} \int\limits_0^1 (1-s)^n s^{n-1} ds.$$

Therefore

$$(1 + n^{-88n^2}\varepsilon^n)\binom{2n}{n} n^{-n} \leq V(K)^{n-1} \int\limits_{\partial DK} \tfrac{1}{V(Z_K(x))^n} \langle u_x, x\rangle dx.$$

Since the last integral is $V(\Pi^* K)$ according to the definition (4) of $\Pi^* K$, we conclude Theorem 3. $\square$

Example 19 *Given any small $\varepsilon > 0$, there exists a convex body K in $\mathbb{R}^n$ such that $d_{BM}(K,T) \geq 1+\varepsilon$ and*

$$V(K)^{n-1} V(\Pi^* K) \leq \left(1 + \varepsilon^{n-1}\right) n^{-n} \tbinom{2n}{n}.$$

Proof: We obtain the convex body K by truncating T; more precisely, we cut off a translate of εT at each vertex of T. Then $q_K = n - (n+1)\varepsilon$, hence $d_{BM}(K,T) > 1+\varepsilon$ according to Lemma 14 (i). On the other hand (55) yields that $\Pi K = (1 - (n-1)\varepsilon^{n-1})\Pi T$, hence

$$V(\Pi^* K) = (1 - (n-1)\varepsilon^{n-1})^{-n} V(\Pi^* T) > (1 + n(n-1)\varepsilon^{n-1}) V(\Pi^* T).$$

Since $V(K) = (1 - (n+1)\varepsilon^n)V(T)$, we have verified Example 19. $\square$

Writing κ_n to denote the volume of the Euclidean unit ball in $\mathbb{R}^n$, C.M. Petty [17]) verified the so–called Petty projection inequality

$$V(K)^{n-1}V(\Pi^*K) \leq \left(\tfrac{\kappa_n}{\kappa_{n-1}}\right)^n \tag{56}$$

where equality holds if and only if K is an ellipsoid. We would like to point out that no stability version of the Petty projection inequality (56) is known.

11 The proof of Theorem 4

Let K be a convex body in $\mathbb{R}^n$. Given $x \in \partial DK$, we define the function $h_x : [0,1] \to [0,1]$ by

$$h_x(t) = s \text{ if and only if } g_K(sx)^{1/n} = t.$$

We note that h_x is well–defined by (42), and satisfies

$$\varrho_{R_pK}(x) = \left(n \int_0^1 h_x(t)^p t^{n-1} dt\right)^{\frac{1}{p}}$$

for any $p > -1$, $p \neq 0$, according to R.J. Gardner and G. Zhang [5], Lemma 5.4.

We summarize the main properties of h_x based on the results in Section 9; namely, (45), Corollary 18 and Lemma 16 yield

Corollary 20 *Given a convex body K in $\mathbb{R}^n$ with $d_{BM}(K,T) = 1+\varepsilon$, there exist an infinite cone Σ with $V(\Sigma \cap DK) \geq n^{-25n^2}V(DK)$, and a positive c depending only on n that satisfy the following properties: If $x \in \partial DK$ then*

(i) $h_x(0) = 1$, $h_x(1) = 0$, and h_x is concave and decreasing;
(ii) $h_x(t) \leq (1+\varepsilon)^n(1-t)$;
(iii) $h_x(t) \leq 2(1-t)h_x(\frac{1}{2}) - c\varepsilon^n$ in case $x \in \Sigma$ and $\frac{4}{21} \leq t \leq \frac{5}{21}$;
(iv) $h_x(t) \geq 2(1-t)h_x(\frac{1}{2}) + c\varepsilon^n$ in case $x \in \Sigma$ and $\frac{16}{21} \leq t \leq \frac{17}{21}$.

For any $p > -1$, $p \neq 0$, and for $x \in \partial DK$, we write

$$F(p,h_x) = \left(\frac{\int_0^1 h_x(t)^p t^{n-1} dt}{\int_0^1 (1-t)^p t^{n-1} dt}\right)^{\frac{1}{p}} = \varrho_{\overline{R}_pK}(x). \tag{57}$$

According to the main result of R.J. Gardner and G. Zhang [5], Corollary 20

(i) implies that if $-1 < p < q$ then

$$F(q, h_x) \leq F(p, h_x), \tag{58}$$

and equality holds if and only if $h_x(t) = 1 - t$.

Theorem 21 *Given $-1 < p < q$, $p, q \neq 0$ and a convex body K in $\mathbb{R}^n$ with $d_{BM}(K, T) = 1 + \varepsilon$, there exists an infinite cone Σ with $V(\Sigma \cap DK) \geq n^{-25n^2} V(DK)$, and there exists a positive c depending only on n such that if $x \in \partial DK$ then*

(i) $F(p, h_x) \leq (1 + \varepsilon)^n F(q, h_x)$;
(ii) $F(p, h_x) \geq (1 + c\varepsilon^n) F(q, h_x)$ if $x \in \Sigma$.

Proof: Corollary 20 (ii) readily yields (i). In order to prove (ii), we may assume that either $-1 < p < q < 0$ or $0 < p < q$. We consider only the case $0 < p < q$ and $\varepsilon > 0$ because the case $-1 < p < q < 0$ is completely analogous. In order to simplify notation, we write h instead of h_x.

Let $\omega = F(p, h)$ that satisfies $1 < \omega < (2n)^n$. We define

$$\varphi(t) = (h(t)^p - (\omega - \omega t)^p) \cdot t^{n-1},$$

which satisfies $\int_0^1 \varphi(t) = 0$. Now $h(t)$ and $\omega - \omega t$ coincide only at $t = 1$ and at some $0 < t_0 < 1$ because h is concave and not linear, hence

$$\begin{aligned} \varphi(t) < 0 \quad &\text{if } \; 0 < t < t_0; \\ \varphi(t) > 0 \quad &\text{if } \; t_0 < t < 1. \end{aligned}$$

Writing $f(t) = t^{\frac{q}{p}}$, we define

$$\psi(t) = \frac{f(h(t)^p) - f((\omega - \omega t)^p)}{h(t)^p - (\omega - \omega t)^p},$$

therefore

$$\int_0^1 \varphi(t)\psi(t)dt = \left(\int_0^1 (1-t)^q t^{n-1} dt \right) (F(q, h)^q - F(p, h)^q). \tag{59}$$

We observe that ψ has one sided derivatives because h is concave. We also note that ψ is monotone decreasing because $h(t)$ is decreasing and $f(t)$ is convex.

In course of the proof, we use various positive constants $\gamma_1, \gamma_2, \ldots$ that depend on p, q and n. Let us consider $t \neq t_0$ satisfying $\frac{4}{21} \leq t \leq \frac{17}{21}$. We introduce the variables $r = h(t)^p$ and $s = (\omega - \omega t)^p$, and hence $r \neq s$ and $\frac{\partial s}{\partial t} < -\gamma_1$.

The convexity of f yields that if $\Delta t > 0$ denotes the variation of t, and $-\Delta s$ denotes the corresponding variation of s then

$$\lim_{\Delta t\to 0^+} \frac{\psi(t+\Delta t)-\psi(t)}{\Delta t} \leq \lim_{\Delta s\to 0^+} \frac{\gamma_1}{\Delta s}\cdot\left(\frac{f(r)-f(s-\Delta s)}{r-s+\Delta s} - \frac{f(r)-f(s)}{r-s}\right).$$

Here $f(s-\Delta s) \geq f(s) - f'(s)\Delta s$, and hence

$$\begin{aligned}\lim_{\Delta t\to 0^+} \frac{\psi(t+\Delta t)-\psi(t)}{\Delta t} &\leq (-\gamma_1)\cdot\frac{f(r)-f(s)-f'(s)(r-s)}{(r-s)^2}\\ &= \frac{-\gamma_1}{2}\cdot f''(\xi) \leq -\gamma_2,\end{aligned} \tag{60}$$

where ξ lies between r and s. Let us define $0 < t_1 < t_0$ and $t_0 < t_2 < 1$ by

$$\psi(t_1) = \frac{\int_0^{t_0}\varphi(t)\psi(t)\,dt}{\int_0^{t_0}\varphi(t)\,dt} \quad\text{and}\quad \psi(t_2) = \frac{\int_{t_0}^{1}\varphi(t)\psi(t)\,dt}{\int_{t_0}^{1}\varphi(t)\,dt}.$$

Therefore $\int_0^1 \varphi(t)\,dt = 0$ implies that

$$\begin{aligned}\int_0^1 \varphi(t)\psi(t)\,dt &= (\psi(t_2)-\psi(t_1))\cdot\int_{t_0}^1 \varphi(t)\,dt\\ &= (\psi(t_1)-\psi(t_2))\cdot\int_0^{t_0}\varphi(t)\,dt.\end{aligned}$$

For the rest of the argument, we assume that $t_0 \leq \frac{1}{2}$, and note that the case $t_0 \geq \frac{1}{2}$ can be handled similarly. We deduce by concavity of h that

$$\int_{\frac{16}{21}}^1 \varphi(t)\,dt > \gamma_3\int_0^1 \varphi(t)\,dt,$$

and by (60) that $\psi(t) < \psi(t_0) - \gamma_4$ if $t \geq \frac{16}{21}$. Therefore

$$\psi(t_2) \leq \frac{\psi(t_0)\int_{t_0}^{\frac{16}{21}}\varphi(t)\,dt + (\psi(t_0)-\gamma_4)\int_{\frac{16}{21}}^1\varphi(t)\,dt}{\int_{t_0}^1\varphi(t)\,dt} \leq \psi(t_0)-\gamma_5 < \psi(t_1)-\gamma_5.$$

Since $\varphi(t) > \gamma_6\varepsilon^n$ holds if $\frac{16}{21} \leq t \leq \frac{17}{21}$, we conclude that

$$\int_0^1 \varphi(t)\psi(t)\,dt \leq -\tfrac{\gamma_5\gamma_6}{21}\,\varepsilon^n.$$

In turn Theorem 21 is a consequence of (59). □

Proof of Theorem 4: Let $-1 < p < q$, $p, q \neq 0$, and let K be a convex body in $\mathbb{R}^n$ with $d_{BM}(K,T) = 1+\varepsilon$. Combining Theorem 21 (i), (57) and $1+\varepsilon \leq 2n$ leads that

$$V(\overline{R}_p K) \leq (1+\varepsilon)^{n^2} V(\overline{R}_q K) \leq (1 + n^{3n^2}\varepsilon) V(\overline{R}_q K),$$

which implies the lower bound of Theorem 4.

Next it follows by (6) that

$$DK \subset \overline{R}_p K \subset nDK$$

hold for any $p > -1$. Thus we deduce by (58) and Theorem 21 (ii) that

$$V(\overline{R}_p K) \geq (1 + c^{-n}\varepsilon^n) V(\overline{R}_q K)$$

for some positive constant c that depends on p, q, n, concluding the proof of Theorem 4. □

Acknowledgment: I would like thank R.J. Gardner for invaluable help in shaping the paper and its methods, and R. Schneider for constant encouragement. In addition thanks are due to P.M. Gruber, M. Reitzner and the referee for various helpful remarks. The work on this paper started during a stay in Moshe Rosenfeld's cabin.

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