# Strengthened volume inequalities for $L_p$ zonoids for even isotropic measures

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**Abstract**

We strengthen the volume inequalities for $L_p$ zonoids of even istropic measures and for their duals due to Ball, Barthe and Lutwak, Yang, Zhang. The $p = \infty$ case yields a stability version of the reverse isoperimetric inequality for centrally symmetric bodies.

## 1 Introduction

According to the isoperimetric inequality, a Euclidean ball has smallest surface area among convex bodies (compact convex sets with non-empty interiors) of given volume in Euclidean space $\mathbb{R}^n$ with scalar product $\langle \cdot\,, \cdot \rangle$ and norm $\| \cdot \|$, and that Euclidean balls are the only minimizers. Let $B^n$ be the Euclidean unit ball centred at the origin. Denoting by $S(K)$ the surface area and by $V(K)$ the volume of a convex body $K$ in $\mathbb{R}^n$, the isoperimetric inequality can be expressed by the inequality

$$\frac{S(B^n)^n}{V(B^n)^{n-1}} \leq \frac{S(K)^n}{V(K)^{n-1}}, \tag{1}$$

where equality holds if and only if $K$ is a Euclidean ball. Since surface area and volume are continuous functionals (with respect to the Hausdorff metric) and the extremal bodies of the inequality (1) are precisely the Euclidean balls, the following question arises naturally. Assume that a convex body $K$ in $\mathbb{R}^n$ satisfies

$$\frac{S(K)^n}{V(K)^{n-1}} \leq (1+\varepsilon)\frac{S(B^n)^n}{V(B^n)^{n-1}}$$

for some $\varepsilon \geq 0$. Does it follow that $K$ is $\varepsilon$-close to a Euclidean ball? An answer to this question requires that the distance $\operatorname{dist}(K)$ of $K$ from a Euclidean ball is measured in a suitable way. For instance, the distance function $\operatorname{dist}(\cdot)$ should have the same scaling and motion invariance as the isoperimetric problem. The problem can also be stated in the following form. Let again $K$ be a convex body in $\mathbb{R}^n$ and assume that $\operatorname{dist}(K) \geq \varepsilon$ for some $\varepsilon \geq 0$. Does it follow that

$$\frac{S(K)^n}{V(K)^{n-1}} \geq (1+f(\varepsilon))\frac{S(B^n)^n}{V(B^n)^{n-1}},$$

where $f : [0, \infty) \to [0, \infty)$ is a continuous and increasing function with $f(0) = 0$? In other words, is it true that

$$\frac{S(K)^n}{V(K)^{n-1}} \geq (1 + f(\mathrm{dist}(K)))\frac{S(B^n)^n}{V(B^n)^{n-1}}$$

with an explicitly given function $f$? Any such inequality provides a strengthening of the classical isoperimetric inequality and is called a stability version of (1).

Although results of this type can be traced back to the work of Minkowski and Bonnesen, a systematic exploration is much more recent. Introductory surveys on geometric stability results were given by H. Groemer [26, 27], an up-to-date coverage of various aspects (including applications) of the topic is provided throughout R. Schneider's book [44]. More specifically, stability results for the isoperimetric problem (based on the Hausdorff distance) have been found, for instance, by Groemer and Schneider [28]. As a recent breakthrough, N. Fusco, F. Maggi, A. Pratelli [23] obtained an optimal stability version of the isoperimetric inequality in terms of the volume difference, and A. Figalli, F. Maggi, A. Pratelli [21, 22] even extended the result to the Brunn-Minkowski inequality.

Since the reverse isoperimetric inequality and a strofnger form of it for general convex bodies are discussed in K.J. Böröczky, D. Hug [13], in this paper we concentrate on centrally symmetric convex bodies. The ratio $S(K)^n/V(K)^{n-1}$ is unbounded from above if $K$ ranges over all centrally symmetric convex bodies. In fact, simple examples show that $K$ can have arbitrarily small volume while its surface area is equal to a prescribed positive value. In order to avoid this type of situation, it is a well known strategy (see, for instance, F. Behrend [10]) is to consider the affine invariant quantity

$$\mathrm{ir}(K) := \inf\left\{\frac{S(\Phi K)^n}{V(\Phi K)^{n-1}} : \Phi \in \mathrm{GL}(n)\right\}$$

where $\mathrm{GL}(n)$ is the group of non-singular linear transformations of $\mathbb{R}^n$. The infimum is attained and the unique minimizer can be characterized, as shown by C. M. Petty [43] (see also A. Giannopoulos, M. Papadimitrakis [24]). In fact, $K$ minimizes the isoperimetric ratio within its affine equivalence class if and only if the suitably normalized area measure of $K$ is isotropic (as defined below). As a simple consequence, cubes minimize the isoperimetric ratio within the class of parallelotopes. Since the new functional 'ir' is affine invariant and upper semi-continuous, it attains its maximum on the space of convex bodies. In the Euclidean plane, the method of F. Behrend [10] yields that $\mathrm{ir}(K) \leq \mathrm{ir}(W^2)$ with equality if and only if $K$ is a parallelogram; here $W^n$ denotes a cube circumscribed about $B^n$. An extension of such a result to higher dimensions turned out to be a formidable problem which resisted its solution until K.M. Ball [1, 2] established reverse forms of the isoperimetric inequality. To state one of his main results, note that $S(W^n) = n2^n = nV(W^n)$.

**Theorem A** (K.M. Ball) *For any centrally symmetric convex body $K$ in $\mathbb{R}^n$, there exists some $\Phi \in \mathrm{GL}(n)$ such that*

$$\frac{S(\Phi K)^n}{V(\Phi K)^{n-1}} \leq \frac{S(W^n)^n}{V(W^n)^{n-1}}.$$

It was proved by F. Barthe [6] that equality holds in Theorem A only if $K$ is a parallelotope.

The first objective of this paper is to establish a stability version of the reverse isoperimetric inequality for centrally symmetric convex bodies. Following [21, 22, 23], we define an affine invariant distance of origin symmetric convex bodies $K$ and $M$ based on the volume difference. For this, let $\alpha = V(K)^{-1/n}$, $\beta = V(M)^{-1/n}$, and then define

$$\delta_{\rm vol}(K,M) := \min\left\{V\left(\Phi(\alpha K)\Delta(\beta M)\right) :\ \Phi\in {\rm SL}(n)\right\}$$

where $A\Delta B$ is the symmetric difference of the sets $A$ and $B$, and ${\rm SL}(n)$ is the group of linear transformations of $\mathbb{R}^n$ of determinant one. We observe that $\delta_{\rm vol}(\cdot,\cdot)$ induces a metric on the linear equivalence classes of convex bodies.

A crucial tool in geometric analysis, and in particular in the proof of the reverse isoperimetric inequality by K.M. Ball, is the John ellipsoid of a convex body $K$ in $\mathbb{R}^n$. This is the unique ellipsoid of maximal volume contained in $K$, which is origin symmetric if $K$ is origin symmetric. Obviously, there is an affine image of $K$, whose John ellipsoid is the Euclidean unit ball $B^n$. Below (see (2)), we list some properties of the John ellipsoid. For thorough discussions of the properties of the John ellipsoid, and of convex bodies in general, see K.M. Ball [4], P.M. Gruber [29] or R. Schneider [44].

**Theorem 1.1** *Let $K$ be an origin symmetric convex body in $\mathbb{R}^n$, $n\geq 3$, whose John ellipsoid is a Euclidean ball, and let $\varepsilon\in[0,1)$. If $\delta_{\rm vol}(K,W^n)\geq\varepsilon$, then*

$$\frac{S(K)^n}{V(K)^{n-1}}\leq (1-\gamma\,\varepsilon^3)\frac{S(W^n)^n}{V(W^n)^{n-1}},$$

*where $\gamma=n^{-cn^3}$ for some absolute constant $c>0$.*

The stability order (the exponent 3 of $\varepsilon$) in Theorem 1.1 close to be optimal, but most probably it is not optimal. Considering a convex body $K$ which is obtained from $W^n$ by cutting off simplices of height $\varepsilon$ at the vertices of $W^n$, one can see that the exponent of $\varepsilon$ must be at least 1 in Theorem 1.1.

Another affine invariant distance between convex bodies is the Banach-Mazur distance $\delta_{\rm BM}(K,M)$ of origin symmetric convex bodies $K$ and $M$, which is defined by

$$\delta_{\rm BM}(K,M) := \ln\min\{\lambda\geq 1 :\ K\subset\Phi(M)\subset\lambda(K) \text{ for } \Phi\in {\rm GL}(n)\}.$$

Again, $\delta_{\rm BM}(\cdot,\cdot)$ induces a metric on the linear equivalence classes of convex bodies. It is not difficult to see that $\delta_{\rm vol}\leq 2n^2\delta_{\rm BM}$ (see say [13]). In the reverse direction, we have $\delta_{\rm BM}\leq\gamma\,\delta_{\rm vol}^{\frac{1}{n}}$, where $\gamma$ depends on the dimension $n$ (see [12, Section 5]), and the exponent $\frac{1}{n}$ cannot be replaced by anything larger than $\frac{2}{n+1}$ as can be seen from the example of a ball from which a cap is cut off.

**Theorem 1.2** *Let $K$ be an origin symmetric convex body in $\mathbb{R}^n$, $n\geq 3$, whose John ellipsoid is an Euclidean ball, and let $\varepsilon\in[0,1)$. If $\delta_{\rm BM}(K,W^n)\geq\varepsilon$, then*

$$\frac{S(K)^n}{V(K)^{n-1}}\leq (1-\gamma\,\varepsilon^n)\frac{S(W^n)^n}{V(W^n)^{n-1}},$$

*where $\gamma=n^{-cn^3}$ for some absolute constant $c>0$.*

The stability order (the exponent $n$ of $\varepsilon$) in Theorem 1.2 is close to be optimal, but most probably it is not optimal. Considering a convex body $K$ which is obtained from $W^n$ by cutting off simplices of height $\varepsilon$ at the vertices of $W^n$, one can see that the exponent of $\varepsilon$ must be at least $n-1$ in Theorem 1.2.

In the planar case case, modifying the argument of F. Behrend [10] leads to stability results of optimal order.

**Theorem 1.3** *Let $K$ be an origin symmetric convex body in $\mathbb{R}^2$ whose John ellipsoid is an Euclidean dics, and let $\varepsilon \in [0,1)$. If $\delta_{\rm vol}(K,W^2) \geq \varepsilon$ or $\delta_{\rm BM}(K,W^2) \geq \varepsilon$, then*

$$\frac{S(K)^2}{V(K)} \leq \left(1 - \frac{\varepsilon}{54}\right) \frac{S(W^2)^2}{V(W^2)}.$$

As mentioned before, the proof of the reverse isoperimetric inequality by K.M. Ball [1, 2] is based on a volume estimate for convex bodies whose John ellipsoid is the unit ball $B^n$. Let $S^{n-1}$ denote the Euclidean unit sphere. According to a classical theorem of F. John [33] (see also K.M. Ball [4]), $B^n$ is the ellipsoid of maximal volume in an origin symmetric convex body $K$ if and only if $B^n \subset K$ and there exist $\pm u_1, \ldots, \pm u_k \in S^{n-1} \cap \partial K$ and $c_1, \ldots, c_k > 0$ such that

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{Id}_n, \tag{2}$$

where $\mathrm{Id}_n$ denotes the $n \times n$ identity matrix and $\partial K$ is the boundary of $K$.

Following A. Giannopoulos, M. Papadimitrakis [24] and E. Lutwak, D. Yang, G. Zhang [42], let us call an even Borel measure $\mu$ on the unit sphere $S^{n-1}$ isotropic if

$$\int_{S^{n-1}} u \otimes u \, d\mu(u) = \mathrm{Id}_n.$$

In this case, equating traces of both sides we obtain that

$$\mu(S^{n-1}) = n. \tag{3}$$

Let us recall that if $v \in \mathbb{R}^n$, then the support function $h_K$ of a convex compact set $K$ in $\mathbb{R}^n$ satisfies

$$h_K(v) = \max\{\langle v, x \rangle : x \in K\}.$$

For any isotropic measure $\mu$ on $S^{n-1}$ and $p \geq 1$, we define the $L_p$-zonoid $Z_p(\mu)$ associated to $\mu$ by

$$h_{Z_p(\mu)}(v)^p = \int_{S^{n-1}} |\langle u, v \rangle|^p \, d\mu(u),$$

which is a zonoid in the classical sense if $p = 1$. In addition, let

$$Z_\infty(\mu) := \lim_{p \to \infty} Z_p(\mu) = \mathrm{conv}\,\mathrm{supp}\,\mu,$$

and for $1 \leq p \leq \infty$, let $Z_p^*(\mu)$ be the polar of $Z_p(\mu)$. In particular,

$$\begin{aligned} Z_p^*(\mu) &= \left\{x \in \mathbb{R}^n : \int_{S^{n-1}} |\langle x, u\rangle|^p \, d\mu(u) \leq 1\right\} \text{ for } p \geq 1, \\ Z_\infty^*(\mu) &= \{x \in \mathbb{R}^n : \langle x, u\rangle \leq 1 \text{ for } u \in \operatorname{supp} \mu\}, \end{aligned}$$

and hence $Z_2(\mu) = B^n$ for any isotropic measure $\mu$.

It follows from D.R. Lewis [37] (see also E. Lutwak, D. Yang and G. Zhang [40, 41]) that any $n$-dimensional subspace of $L_p$ is isometric to $\|\cdot\|_{Z_p^*(\mu)}$ for some isotropic measure $\mu$ on $S^{n-1}$, where

$$\|x\|_{Z_p^*(\mu)} = \left(\int_{S^{n-1}} |\langle x, u\rangle|^p \, d\mu(u)\right)^{\frac{1}{p}}.$$

We call a measure $\nu$ on $S^{n-1}$ a cross measure if there is an orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$ such that

$$\operatorname{supp} \nu = \{\pm u_1, \ldots, \pm u_n\},$$

and $\nu(\{u_i\}) = \nu(\{-u_i\}) = 1/2$ for $i = 1, \ldots, n$, and hence $\nu$ is even and isotropic. We fix a cross measure $\nu_n$ on $S^{n-1}$. We note that if $p \in [1, \infty]$, and $\Gamma(\cdot)$ is Euler's Gamma function, then

$$V(Z_p(\nu_n)) = \begin{cases} \frac{\Gamma(1+\frac{n}{2})\Gamma(1+\frac{p}{2})}{\Gamma(1+\frac{1}{2})\Gamma(1+\frac{n+p}{2})} & \text{if } p \geq 1 \\ \frac{2^n}{n!} & \text{if } p = \infty. \end{cases}$$

In addition,

$$V(Z_p^*(\nu_n)) = \begin{cases} 2^n \frac{\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} & \text{if } p \geq 1 \\ 2^n & \text{if } p = \infty. \end{cases}$$

The crucial statement leading to the reverse isoperimetric inequality is the case about $Z_\infty^*(\mu)$ of the following.

**Theorem B** *If $\mu$ is an even isotropic measure on $S^{n-1}$ and $p \in [1, \infty]$, then*

$$V(Z_p(\mu)) \geq V(Z_p(\nu_n)) \tag{4}$$

$$V(Z_p^*(\mu)) \leq V(Z_p^*(\nu_n)). \tag{5}$$

*Assuming $p \neq 2$, equality holds if and only if $\mu$ is a cross measure.*

Theorem B is the work of K.M. Ball [2] and F. Barthe [6] if $\mu$ is discrete, and their method was extended to any even isotropic $\mu$ by E. Lutwak, D. Yang, and G. Zhang [40]. The measures on $S^{n-1}$ which have an isotropic linear image are characterized by K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang [14], building on work of E.A. Carlen, and D. Cordero-Erausquin [16], J. Bennett, A. Carbery, M. Christ and T. Tao [11] and B. Klartag [36]. We note that isotropic measures on $\mathbb{R}^n$ play a central role in the KLS conjecture by R. Kannan, L. Lovász and M. Simonovits

[34]; see, for instance, F. Barthe and D. Cordero-Erausquin [8], O. Guedon and E. Milman [32] and B. Klartag [35].

To state a stability version of Theorem B, a natural notion of distance between two isotropic measures $\mu$ and $\nu$ is the Wasserstein distance (also called the Kantorovich-Monge-Rubinstein distance) $\delta_W(\mu,\nu)$. To define it, we write $\angle(v,w)$ to denote the angle non-zero vectors $v$ and $w$; that is, the geodesic distance of the unit vectors $\|v\|^{-1}v$ and $\|w\|^{-1}w$ on the unit sphere. Let $\mathrm{Lip}_1(S^{n-1})$ denote the family of Lipschitz functions; namely, $f: S^{n-1}\to\mathbb{R}$ is in $\mathrm{Lip}_1(S^{n-1})$ if $\|f(x)-f(y)\|\leq\angle(x,y)$ for $x,y\in S^{n-1}$. Then

$$\delta_W(\mu,\nu)=\max\left\{\int_{S^{n-1}} f\,d\mu-\int_{S^{n-1}} f\,d\nu:\ f\in\mathrm{Lip}_1(S^{n-1})\right\}.$$

What we actually need in this paper is the Wasserstein distance of an isotropic measure $\mu$ from the closest cross measure. Therefore in the case of two isotropic measures $\mu$ and $\nu$, we define

$$\delta_{\mathrm{WO}}(\mu,\nu)=\min\{\delta_W(\mu,\Phi_*\nu):\ \Phi\in O(n)\}$$

where $\Phi_*\nu$ denotes the pushforward of $\nu$ by $\Phi: S^{n-1}\to S^{n-1}$.

**Theorem 1.4** *Let $\mu$ be an even isotropic measure on $S^{n-1}$, $n\geq 2$, let $\varepsilon\in[0,1)$, and let $p\in[1,\infty]$ with $p\neq 2$. If $\delta_{\mathrm{WO}}(\mu,\nu_n)\geq\varepsilon>0$, then*

$$\begin{aligned} V(Z_p(\mu)) &\geq (1+\gamma\varepsilon^3)V(Z_p(\nu_n))\\ V(Z_p^*(\mu)) &\leq (1-\gamma\varepsilon^3)V(Z_p^*(\nu_n)) \end{aligned}$$

*where $\gamma=n^{-cn^3}\min\{|p-2|^2,1\}$ for an absolute constant $c>0$.*

To state a another stability version of Theorem B in the case of $p=\infty$, we define the "spherical" Hausdorff distance of compact sets $X,Y\subset S^{n-1}$ by the formula

$$\delta_H(X,Y):=\min\left\{\max_{x\in X}\min_{y\in Y}\angle(x,y),\max_{y\in Y}\min_{x\in X}\angle(x,y)\right\}.$$

In addition, let

$$\delta_{HO}(X,Y):=\min\{\delta_H(X,\Phi Y):\ \Phi\in O(n)\}.$$

We note that if $\delta_{HO}(\mathrm{supp}\,\mu,\mathrm{supp}\,\nu_n)\leq 1/(7n^2)$ for an even isotropic measure $\mu$, then $\delta_{WO}(\mu,\nu_n)\leq 2n\delta_H(\mathrm{supp}\,\mu,\mathrm{supp}\,\nu_n)$ according to Corollary 6.2. Therefore Theorem 1.4 yields the following in the case $p=\infty$.

**Corollary 1.5** *If $\mu$ is an even isotropic measure, and $\delta_{HO}(\mathrm{supp}\,\mu,\mathrm{supp}\,\nu_n)\geq\varepsilon>0$, then*

$$\begin{aligned} V(Z_\infty(\mu)) &\geq (1+\gamma\varepsilon^3)V(Z_\infty(\nu_n))\\ V(Z_\infty^*(\mu)) &\leq (1-\gamma\varepsilon^3)V(Z_\infty^*(\nu_n)) \end{aligned}$$

*where $\gamma=n^{-cn^3}$ for an absolute constant $c>0$.*

We note that the proof of Theorem B is based on the rank one case of the geometric Brascamp-Lieb inequality. An essential tool in our approach is the proof provided by F. Barthe [5, 6], which is based on mass transportation. Therefore, it is instructive to review the argument from [5], which is done in Section 2. At the end of that section, we outline the arguments leading to Theorem 1.1, Theorem 1.2 and Theorem 1.4 and roughly describe the structure of the paper. We also indicate in Section 2 what stability result can be expected concerning the Brascamp-Lieb inequality (see Conjecture 2.1). Along the way to prove our main statements, we also establish some some properties of any (not only even) isotropc measures in Section 5 that might be useful in other applications, as well.

Let us point out that the corresponding question in the non-symmetric setting is wide open. For a centered isotropic measure $\mu$ on $S^{n-1}$, and for $p \in [1,\infty)$, we define the non-symmetric $L_p$-zonoid $Z_p(\mu)$ by

$$\begin{aligned} h_{Z_p(\mu)}(v)^p &= \int_{S^{n-1}} \max\{0, \langle v, u\rangle\}^p \, d\mu(u) \\ Z_p^*(\mu) &= \left\{ x \in \mathbb{R}^n : \int_{S^{n-1}} \max\{0, \langle x, u\rangle\}^p \, d\mu(u) \leq 1 \right\}. \end{aligned}$$

This notion (for any discrete measure on $S^{n-1}$, not only isotropic ones), occurs in M. Weberndorfer [45] in connection with reverse versions of the Blaschke-Santaló inequality.

**Conjecture 1.6** *If $\mu$ is a centered isotropic measure on $S^{n-1}$ and $p \in [1,\infty)$, moreover $\nu$ is the isotropic measure on $S^{n-1}$ such that* $\operatorname{supp} \nu$ *consists of the vertices of a regular simplex, then*

$$V(Z_p(\mu)) \geq V(Z_p(\nu)) \tag{6}$$

$$V(Z_p^*(\mu)) \leq V(Z_p^*(\nu)). \tag{7}$$

If $\mu$ is a centered isotropic measure on $S^{n-1}$, then $Z_\infty(\mu) = \operatorname{conv} \operatorname{supp} \mu$. In particular, if $p = \infty$, then (7) was proved by K.M. Ball in [2] for discrete $\mu$, (6) was proved by F. Barthe in [6] again for discrete $\mu$, and the the case of general centered isotropic $\mu$ was handled E. Lutwak, D. Yang and G. Zhang [42].

## 2 A brief review of the Brascamp-Lieb and the reverse Brascamp-Lieb inequality

The rank one geometric Brascamp-Lieb inequality (8), identified by K.M. Ball [1] as an essential case of the rank one Brascamp-Lieb inequality, due to H.J. Brascamp, E.H. Lieb [15], and the reverse form (9), due to F. Barthe [5, 6], read as follows. If $u_1, \ldots, u_k \in S^{n-1}$ are distinct unit vectors and $c_1, \ldots, c_k > 0$ satisfy

$$\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}_n,$$

and $f_1,\dots,f_k$ are non-negative measurable functions on $\mathbb{R}$, then

$$\int_{\mathbb{R}^n}\prod_{i=1}^k f_i(\langle x,u_i\rangle)^{c_i}\,dx \;\leq\; \prod_{i=1}^k\left(\int_{\mathbb{R}} f_i\right)^{c_i}, \text{ and} \tag{8}$$

$$\int_{\mathbb{R}^n}\sup_{x=\sum_{i=1}^k c_i\theta_i u_i}\prod_{i=1}^k f_i(\theta_i)^{c_i}\,dx \;\geq\; \prod_{i=1}^k\left(\int_{\mathbb{R}} f_i\right)^{c_i}. \tag{9}$$

In (9), we always assume that $\theta_1,\dots,\theta_k\in\mathbb{R}$. Here, naturally, $\theta_1,\dots,\theta_k$ is unique if $k=n$ and $u_1,\dots,u_n$ form an orthonormal basis.

According to F. Barthe [6], if equality holds in (8) or in (9) and none of the functions $f_i$ is identically zero or a scaled version of a Gaussian, then there is an origin symmetric regular crosspolytope in $\mathbb{R}^n$ such that $u_1,\dots,u_k$ lie among its vertices. Conversely, equality holds in (8) and (9) if either each $f_i$ is a scaled version of the same centered Gaussian, or if $k=n$ and $u_1,\dots,u_n$ form an orthonormal basis.

A thorough discussion of the rank one Brascamp-Lieb inequality can be found in E. Carlen, D. Cordero-Erausquin [16]. The higher rank case, due to E.H. Lieb [38], is reproved and further explored by F. Barthe [6] (including a discussion of the equality case), and is again carefully analysed by J. Bennett, T. Carbery, M. Christ, T. Tao [11]. In particular, see F. Barthe, D. Cordero-Erausquin, M. Ledoux, B. Maurey [9] for an enlightening review of the relevant literature and an approach via Markov semigroups in a quite general framework.

F. Barthe [5, 6] provides concise proofs of (8) and (9) based on mass transportation (see also K.M. Ball [4] for (8) ). We sketch the main ideas of this approach, since this will be the starting point of subsequent refinements.

We assume that each $f_i$ is a positive continuous probability density both for (8) and (9), and let $g(t)=e^{-\pi t^2}$ be the Gaussian density. For $i=1,\dots,k$, we consider the transportation map $T_i:\mathbb{R}\to\mathbb{R}$ satisfying

$$\int_{-\infty}^{t} f_i(s)\,ds=\int_{-\infty}^{T_i(t)} g(s)\,ds.$$

It is easy to see that $T_i$ is bijective, differentiable and

$$f_i(t)=g(T_i(t))\cdot T_i'(t),\qquad t\in\mathbb{R}. \tag{10}$$

To these transportation maps, we associate the transformation $\Theta:\mathbb{R}^n\to\mathbb{R}^n$ with

$$\Theta(x):=\sum_{i=1}^k c_iT_i(\langle u_i,x\rangle)\,u_i,\qquad x\in\mathbb{R}^n,$$

which satisfies

$$d\Theta(x)=\sum_{i=1}^k c_iT_i'(\langle u_i,x\rangle)\,u_i\otimes u_i.$$

In this case, $d\Theta$ is positive definite and $\Theta:\mathbb{R}^n\to\mathbb{R}^n$ is injective (see [5, 6]). We will need the following two estimates due to K.M. Ball [1] (see also [6] for a simpler proof of (i)).

(i) For any $t_1, \dots, t_k > 0$, we have

$$\det \left( \sum_{i=1}^{k} t_i c_i u_i \otimes u_i \right) \geq \prod_{i=1}^{k} t_i^{c_i}.$$

(ii) If $z = \sum_{i=1}^{k} c_i \theta_i u_i$ for $\theta_1, \dots, \theta_k \in \mathbb{R}$, then

$$\|z\|^2 \leq \sum_{i=1}^{k} c_i \theta_i^2. \tag{11}$$

Therefore, using first (10), then (i) with $t_i = T_i'(\langle u_i, x \rangle)$, and later the definition of $\Theta$ and (ii), the following argument leads to the Brascamp-Lieb inequality (8).

$$\begin{aligned}
\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle u_i, x \rangle)^{c_i}\, dx &\leq \int_{\mathbb{R}^n} \left( \prod_{i=1}^{k} g(T_i(\langle u_i, x \rangle))^{c_i} \right) \left( \prod_{i=1}^{k} T_i'(\langle u_i, x \rangle)^{c_i} \right) dx && (12) \\
&\leq \int_{\mathbb{R}^n} \left( \prod_{i=1}^{k} e^{-\pi c_i T_i(\langle u_i, x \rangle)^2} \right) \det \left( \sum_{i=1}^{k} c_i T_i'(\langle u_i, x \rangle)\, u_i \otimes u_i \right) dx && (13) \\
&\leq \int_{\mathbb{R}^n} e^{-\pi \|\Theta(x)\|^2} \det\left(d\Theta(x)\right)\, dx \\
&\leq \int_{\mathbb{R}^n} e^{-\pi \|y\|^2}\, dy = 1.
\end{aligned}$$

Finally, the Brascamp-Lieb inequality (8) for arbitrary non-negative integrable functions $f_i$ follows by scaling.

For the reverse Brascamp-Lieb inequality (9), we consider the inverse $S_i$ of $T_i$, and hence

$$\int_{-\infty}^{t} g(s)\, ds = \int_{-\infty}^{S_i(t)} f_i(s)\, ds$$

$$g(t) = f_i(S_i(t)) \cdot S_i'(t), \qquad t \in \mathbb{R}. \tag{14}$$

In addition,

$$d\Psi(x) = \sum_{i=1}^{k} c_i S_i'(\langle u_i, x \rangle)\, u_i \otimes u_i$$

holds for the diffeomorphism $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ with

$$\Psi(x) := \sum_{i=1}^{k} c_i S_i(\langle u_i, x \rangle)\, u_i, \qquad x \in \mathbb{R}^n.$$

In particular, $d\Psi$ is positive definite and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is injective (see [5, 6]). Therefore (i) and (14) lead to

$$\begin{aligned}
\int_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i\theta_i u_i} \prod_{i=1}^k f_i(\theta_i)^{c_i}\,dx &\geq \int_{\mathbb{R}^n} \left( \sup_{\Psi(y)=\sum_{i=1}^k c_i\theta_i u_i} \prod_{i=1}^k f_i(\theta_i)^{c_i} \right) \det\left(d\Psi(y)\right)\,dy \\
&\geq \int_{\mathbb{R}^n} \left( \prod_{i=1}^k f_i(S_i(\langle u_i, y\rangle))^{c_i} \right) \det\left( \sum_{i=1}^k c_i S_i'(\langle u_i, y\rangle)\, u_i \otimes u_i \right) dy \qquad (15) \\
&\geq \int_{\mathbb{R}^n} \left( \prod_{i=1}^k f_i(S_i(\langle u_i, y\rangle))^{c_i} \right) \left( \prod_{i=1}^k S_i'(\langle u_i, y\rangle)^{c_i} \right) dy \qquad (16) \\
&= \int_{\mathbb{R}^n} \left( \prod_{i=1}^k g(\langle u_i, y\rangle)^{c_i} \right) dy = \int_{\mathbb{R}^n} e^{-\pi\|y\|^2}\,dy = 1.
\end{aligned}$$

Again, the Reverse Brascamp-Lieb inequality (9) for arbitrary non-negative integrable functions $f_i$ follows by scaling.

We observe that (i) shows that the optimal constant in the geometric Brascamp-Lieb inequality is 1. The stability version of (i) (with $v_i = \sqrt{c_i}u_i$), Lemma 3.1, is an essential tool in proving a stability version of the Brascamp-Lieb inequality leading to Theorem 1.4.

Let us briefly discuss how K.M. Ball [1] and F. Barthe [6] used the Brascamp-Lieb inequality and its reverse form to prove the discrete version of Theorem B. In this section, we write $\mu$ to denote the isotropic measure on $S^{n-1}$ whose support is $\{u_1, \ldots, u_k\}$, and $\mu(u_i) = c_i$, and we assume that $\mu$ is an even measure. For $i = 1, \ldots, k$, we consider the probability densities (see (25)

$$f_i(t) = \frac{1}{2\Gamma(1+\frac{1}{p})}\, e^{-|t|^p}$$

if $p \in [1, \infty)$, and $f_i = \frac{1}{2}\mathbf{1}_{[-1,1]}$ if $p = \infty$, where

$$\mathbf{1}_{[-1,1]}(t) = \begin{cases} 1 & \text{if } t \in [-1,1], \\ 0 & \text{otherwise.} \end{cases}$$

We will frequently use the following observation due to K. Ball [2]. If $K$ is an orgin symmetric convex body in $\mathbb{R}^n$ with associated norm $\|\cdot\|_K$, then

$$V(K) = \frac{1}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} e^{-\|x\|_K^p}\,dx \qquad (17)$$

where if $x \in \mathbb{R}^n$, then

$$\|x\|_K := \min\{\lambda \geq 0 :\, x \in \lambda K\}.$$

In particular, if $p \in [1,\infty)$, then

$$\begin{aligned} V(Z_p^*(\mu)) &= \frac{1}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}\exp\left(-\sum_{i=1}^k c_i|\langle x,u_i\rangle|^p\right)dx \\ &= \frac{2^n\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}\prod_{i=1}^k f_i(\langle x,u_i\rangle)^{c_i}\,dx \qquad (18)\\ &\leq \frac{2^n\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma(1+\frac{n}{p})}\prod_{i=1}^k\left(\int_{\mathbb{R}} f_i\right)^{c_i} = \frac{2^n\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma(1+\frac{n}{p})}, \qquad (19)\end{aligned}$$

and if $p=\infty$, then using $f_i = \mathbf{1}_{[-1,1]}$, we have

$$V(Z_\infty^*(\mu)) = 2^n\int_{\mathbb{R}^n}\prod_{i=1}^k f_i(\langle x,u_i\rangle)^{c_i}\,dx \leq 2^n\prod_{i=1}^k\left(\int_{\mathbb{R}} f_i\right)^{c_i} = 2^n.$$

Equality in (19) leads to equality in the Brascamp-Lieb inequality, and hence $k=2n$ and $u_1,\ldots,u_k$ form the vertices of a regular crosspolytope in $\mathbb{R}^n$.

For the lower bound on the volume of the $L_p$ zonotopes, let us recall that $\frac{1}{p}+\frac{1}{p^*}=1$, and if $p\in[1,\infty)$, then let us consider the auxiliary origin symmetric convex body

$$M_p(\mu) = \left\{\sum_{i=1}^k c_i\theta_i u_i:\ \sum_{i=1}^k c_i|\theta_i|^p \leq 1\right\}.$$

We drop the reference to $\mu$ if it does not cause misunderstanding. In particular,

$$\|x\|_{M_p} = \left(\inf_{x=\sum_{i=1}^k c_i\theta_i u_i}\sum_{i=1}^k c_i|\theta_i|^p\right)^{\frac{1}{p}}.$$

In addition, let

$$M_\infty(\mu) = \left\{\sum_{i=1}^k c_i\theta_i u_i:\ |\theta_i|\leq 1 \text{ for } i=1,\ldots,k\right\}.$$

We claim that if $p\in[1,\infty]$, then

$$M_p(\mu) \subset Z_{p^*}(\mu). \qquad (20)$$

Let $x\in M_p(\mu)$, and hence $x=\sum_{i=1}^k c_i\theta_i u_i$ where $\sum_{i=1}^k c_i|\theta_i|^p\leq 1$ if $p\in[1,\infty)$, and $|\theta_i|\leq 1$ for $i=1,\ldots,k$ if $p=\infty$. If $p\in(1,\infty)$, then it follows from the Hölder inequality that for any $v\in\mathbb{R}^n$, we have

$$\langle x,v\rangle = \sum_{i=1}^k c_i\theta_i\langle u_i,v\rangle \leq \left(\sum_{i=1}^k c_i|\theta_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^k c_i|\langle u_i,v\rangle|^{p^*}\right)^{\frac{1}{p^*}} \leq h_{Z_{p^*}}(v).$$

If $p=1$, then

$$\langle x,v\rangle=\sum_{i=1}^k c_i\theta_i\langle u_i,v\rangle\leq\max_{i=1,\ldots,k}|\langle u_i,v\rangle|=h_{Z_\infty}(v).$$

In addition, if $p=\infty$, then

$$\langle x,v\rangle=\sum_{i=1}^k c_i\theta_i\langle u_i,v\rangle\leq\sum_{i=1}^k c_i|\langle u_i,v\rangle|=h_{Z_1}(v).$$

Now if $p\in[1,\infty)$, then we deduce from (20) and the reverse Brascamp-Lieb inequality (9) that

$$\begin{aligned}
V(Z_{p^*}(\mu)) &\geq V(M_p(\mu))=\frac{1}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}\exp\left(-\|x\|_{M_p}^p\right)dx\\
&= \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}\sup_{x=\sum_{i=1}^k c_i\theta_iu_i}\prod_{i=1}^k f_i(\theta_i)^{c_i}\,dx \qquad (21)\\
&\geq \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\prod_{i=1}^k\left(\int_{\mathbb{R}}f_i\right)^{c_i}=\frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}. \qquad (22)
\end{aligned}$$

Finally, if $p=\infty$, then $f_i=\frac{1}{2}\mathbf{1}_{[-1,1]}$, and

$$V(Z_1(\mu))\geq V(M_\infty(\mu))=2^n\int_{\mathbb{R}^n}\sup_{x=\sum_{i=1}^k c_i\theta_iu_i}\prod_{i=1}^k f_i(\theta_i)^{c_i}\,dx\geq 2^n\prod_{i=1}^k\left(\int_{\mathbb{R}}f_i\right)^{c_i}=2^n.$$

Equality in (22) leads to equality in the Reverse Brascamp-Lieb inequality, and hence $k=2n$ and $u_1,\ldots,u_k$ form the vertices of a regular crosspolytope in $\mathbb{R}^n$.

The main idea to have a stability version of (19) and (22) is to have a stronger version of (13) and (16), respectively, based on the stronger version Lemma 3.1 of (i). In order to apply the estimate of Lemma 3.1, we need some basic bounds on the derivatives of the transportation maps involved proved in Section 4. The technical Sections 5 and 6 also serve as preparation for the proof of the core statement Proposition 7.2 providing the stabiliy version of (13). The argument for the estimate strenghtening (16) is similar, and is reviewed in Section 9. The stability versions of the Reverse Isoperimetric Inequality in the origin symmetric case are proved in Section 8.

The methods of this paper are very specific for our particular choice of the functions $f_i$, and no method is known to the authors that could lead to a stability version of the Brascamp-Lieb inequality (8) or of its reverse form (9) in general. However, the proof of Theorem 1.4 indicates the following conjecture.

**Conjecture 2.1** *If $f$ is an even probability density function on $\mathbb{R}$ with variance 1, $g(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ is the standard normal distribution, and $\mu$ is an even isotropic measure on $S^{n-1}$*

*supported at* $u_1,\dots,u_k\in S^{n-1}$ *with* $\mu(\{u_i\})=c_i$*, then*

$$\int_{\mathbb{R}^n}\prod_{i=1}^k f(\langle x,u_i\rangle)^{c_i}\,dx \;\leq\; \exp\left(-\gamma\min\{1,\|f-g\|_1\}^\alpha\cdot\delta_{\rm WO}(\mu,\nu_n)^\alpha\right), \tag{23}$$

$$\int_{\mathbb{R}^n}\sup_{x=\sum_{i=1}^k c_i\theta_i u_i}\prod_{i=1}^k f(\theta_i)^{c_i}\,dx \;\geq\; \exp\left(\gamma\min\{1,\|f-g\|_1\}^\alpha\cdot\delta_{\rm WO}(\mu,\nu_n)^\alpha\right) \tag{24}$$

*where* $\gamma>0$ *depends on* $n$*, and* $\alpha>0$ *is an absolute constant.*

# 3 Stability versions of some observations

To obtain a stability version of Theorem B, we need a stability version of the Brascamp-Lieb inequality and its reverse form in the special cases we use. For example, we estimate the derivatives of the corresponding transportation map in Section 4, we will use the following strengthened form of (i) from [13].

**Lemma 3.1** *Let* $k\geq n+1$*,* $t_1,\dots,t_k>0$*, and let* $v_1,\dots,v_k\in\mathbb{R}^n$ *satisfy* $\sum_{i=1}^k v_i\otimes v_i={\rm Id}_n$*. Then*

$$\det\left(\sum_{i=1}^k t_i v_i\otimes v_i\right)\geq\theta^*\prod_{i=1}^k t_i^{\langle v_i,v_i\rangle},$$

*where*

$$\theta^*=1+\frac{1}{2}\sum_{1\leq i_1<\ldots<i_n\leq k}\det[v_{i_1},\dots,v_{i_n}]^2\left(\frac{\sqrt{t_{i_1}\cdots t_{i_n}}}{t_0}-1\right)^2,$$

$$t_0=\sqrt{\sum_{1\leq i_1<\ldots<i_n\leq k}t_{i_1}\cdots t_{i_n}\det[v_{i_1},\dots,v_{i_n}]^2}.$$

In order to estimate $\theta^*$ from below, we use the following observation from [13].

**Claim 3.2** *If* $a,b,x>0$*, then*

$$(xa-1)^2+(xb-1)^2\geq\frac{(a^2-b^2)^2}{2(a^2+b^2)^2}$$

# 4 The transportation maps

We note that for $p\geq 1$, we have

$$\int_{\mathbb{R}}e^{-|t|^p}\,dt=\frac{2}{p}\int_0^\infty e^{-s}s^{\frac{1}{p}-1}\,ds=2\Gamma(1+\tfrac{1}{p}). \tag{25}$$

Thus for $p \in [1, \infty]$, we consider the density functions

$$\varrho_p(x) = \begin{cases} \frac{1}{2\Gamma(1+\frac{1}{p})}\, e^{-|s|^p} & \text{if } p \in [1, \infty); \\ \frac{1}{2}\mathbf{1}_{[-1,1]} & \text{if } p = \infty. \end{cases}$$

In particular, $\varrho_2$ is the Gaussian density function $\pi^{-1/2}e^{-s^2}$. In addition, we define the transportation maps $\varphi_p, \psi_p : \mathbb{R} \to \mathbb{R}$ for $p \in [1, \infty)$, $\varphi_\infty : (-1, 1) \to \mathbb{R}$ and $\psi_\infty : \mathbb{R} \to (-1, 1)$ by

$$\int_{-\infty}^{t} \varrho_p(s)\, ds = \int_{-\infty}^{\varphi_p(t)} \varrho_2(s)\, ds, \tag{26}$$

$$\int_{-\infty}^{\psi_p(t)} \varrho_p(s)\, ds = \int_{-\infty}^{t} \varrho_2(s)\, ds. \tag{27}$$

Here $\varphi_p$ and $\psi_p$ are odd and inverses of each other.

In the following, we use that

$$s - s^2 \leq \log(1 + s) \leq s \quad \text{if } s \geq -\tfrac{1}{2}, \tag{28}$$

and the following properties of the $\Gamma$ function.

(i) $\log \Gamma(t)$ is strictly convex for $t > 0$;

(ii) $\Gamma(1) = \Gamma(2) = 1$;

(iii) $\Gamma(1 + \frac{1}{2.3}) < \Gamma(1 + \frac{1}{2}) = \sqrt{\pi}/2$;

(iv) $\Gamma(t) > 0.8856$ for $t > 0$.

We deduce from (i)-(iv) that the density functions involved satisfy

$$\frac{1}{2e} \leq \varrho_p(s) < \frac{1}{2 \cdot 0.8856} \quad \text{for } p \in [1, \infty] \text{ and } s \in [0, 1]. \tag{29}$$

We note that $e/0.8856 < 3.1$. Therefore (29) and (10) yield that

$$\frac{1}{3.1} < \varphi_p'(s), \psi_p'(s) < 3.1 \quad \text{for } p \in [1, \infty] \text{ and } s \in [0, \tfrac{1}{3.1}]. \tag{30}$$

The following simple estimate will have crucial role in the proofs of Lemma 4.2 and Lemma 4.3.

**Lemma 4.1** *For* $p \in (1, 3)\backslash\{2\}$ *and* $\nu > 0$, *let* $f(t) = \nu t - pt^{p-1}$.

*(a) If* $p \in (1, 2)$, $f(\tau) \leq 0$ *for* $\tau \in (0, 1]$ *and* $t \in (0, \tau/2]$, *then*

$$f(t) < -\frac{p(p-1)(2-p)}{2^{4-p}} \cdot t^{p-1}.$$

*(b) If $p \in (2,3)$, $f(\tau) \geq 0$ for $\tau \in (0,1]$ and $t \in (0, \tau/2]$, then*

$$f(t) > \frac{p(p-1)(p-2)}{2^{4-p}} \cdot t^{p-1}.$$

**Remark** Naturally, the bound could be linear in $t$ with a factor depending on $\nu$, but this way the only influence of $\nu$ is on the value of $\tau$. We only use Lemma 4.1 when $1.5 \leq p \leq 2.3$ and $t > c$ for a positive absolute constant $c$ anyway.

*Proof:* Let $p \in (1,2)$. Since $f(t) = \nu t - pt^{p-1}$ is convex on $[0, \tau]$ and non-positive at $t = 0$ and $t = \tau$, the Taylor formula yields that if $t \in (0, \tau/2]$, then there exist $\tau_1 \in (0,t)$ and $\tau_2 \in (t, 2t)$ such that

$$\begin{aligned} 0 &\geq \frac{1}{2}\left(f(0) + f(2t)\right) = \frac{1}{2}\left(f(t) - f'(t)t + \frac{1}{2} f"(\tau_1)t^2 + f(t) + f'(t)t + \frac{1}{2} f"(\tau_2)t^2\right) \\ &= f(t) + \frac{1}{2}\, \frac{f"(\tau_1) + f"(\tau_2)}{2}\, t^2. \end{aligned}$$

We deduce from $f"(\tau_i) = -p(p-1)(p-2)\tau_i^{p-3} > p(p-1)(2-p)(2t)^{p-3}$, $i = 1, 2$, the estimate

$$f(t) < -\frac{1}{2} p(p-1)(2-p)(2t)^{p-3} \cdot t^2 = -\frac{p(p-1)(2-p)}{2^{4-p}} \cdot t^{p-1}.$$

If $p \in (2,3)$, then $f(t) = \nu t - pt^{p-1}$ is concave on $[0, \tau]$, and the analogous argument yields (b). $\square$

**Lemma 4.2** *Let $p \in [1, \infty]\backslash\{2\}$. For $t \in (0, \frac{1}{8})$, we have*

$$\begin{aligned} \varphi_p"(t) &< -\frac{2-p}{48} \cdot t \quad \text{if } p \in [1,2); \\ \varphi_p"(t) &> \frac{p-2}{5} \cdot t^{1.3} \quad \text{if } p \in (2,3]; \\ \varphi_p"(t) &> 0.2 \cdot t^{1.3} \quad \text{if } p \in (3, \infty] \end{aligned}$$

*Proof:* Let $\varphi = \varphi_p$. We have $\varphi(0) = 0$ as $\varphi$ is odd. Therefore $\varphi(t) > 0$ if $t > 0$.

Let $p \in [1, \infty)\backslash\{2\}$. For $t > 0$, differentiating (26) yields the formula

$$\frac{e^{-t^p}}{2\Gamma(1 + \frac{1}{p})} = \frac{e^{-\varphi(t)^2}\varphi'(t)}{2\Gamma(1+\frac{1}{2})},$$

and hence

$$\frac{-p\Gamma(1+\frac{1}{2})}{\Gamma(1+\frac{1}{p})} \cdot e^{-t^p} t^{p-1} = -2e^{-\varphi(t)^2}\varphi(t)\varphi'(t)^2 + e^{-\varphi(t)^2}\varphi"(t).$$

In particular,

$$\begin{aligned}\varphi'(t) &= \frac{\Gamma(1+\frac{1}{2})}{\Gamma(1+\frac{1}{p})}\, e^{\varphi(t)^2-t^p} &(31)\\ \varphi"(t) &= (2\varphi(t)\varphi'(t)-pt^{p-1})\varphi'(t). &(32)\end{aligned}$$

Case 1 and Case 2 below use the value

$$t_p=(p/2)^{\frac{1}{p-2}} \quad \text{for } p\in[1,\infty)\backslash\{2\}. \tag{33}$$

To estimate $t_p$ for for $p\in(2,4]$, we deduce from the repeated application of (28) that

$$t_p=\exp\left(\frac{\log(2/p)}{p-2}\right)\geq\exp\left(\frac{\frac{2-p}{p}-\frac{(2-p)^2}{p^2}}{p-2}\right)=\exp\left(\frac{2-2p}{p^2}\right)\geq 1+\frac{2-2p}{p^2}.$$

Here differentiation shows that $\frac{2-2p}{p^2}$ has its minimum at $p=2$, thus evaluating at $p=2$ yields

$$t_p\geq\frac{1}{2} \quad \text{for } p\in(2,4]. \tag{34}$$

If $p\in[1,2)$, then $p-2<0$, and hence (28) yields that

$$t_p=\exp\left(\frac{\log(2/p)}{p-2}\right)\geq\exp\left(\frac{\frac{2-p}{p}}{p-2}\right)=\exp\left(\frac{-1}{p}\right)\geq\frac{1}{e} \quad \text{for } p\in[1,2). \tag{35}$$

Moreover Case 1 and Case 3 apply the fact that

$$\text{for given } t\in(0,1/e),\ pt^{p-1} \text{ is a decreasing function of } p\geq 1. \tag{36}$$

**Case 1** If $1\leq p<2$ and $t\in(0,\frac{1}{2e})$, then $\varphi"(t)<-\frac{2-p}{48}\cdot t$
In this case, $\varphi'(0)<1$ by (31), (i) and (ii). Since $\varphi'$ is continuous, there exists largest $s_p\in(0,\infty]$ such that $\varphi'(t)<1$ if $0<t<s_p$. Thus if $t\in(0,s_p)$, then $\varphi(t)<t$, and in turn (32) yields that

$$\varphi"(t)=(2\varphi(t)\varphi'(t)-pt^{p-1})\varphi'(t)<(2t-pt^{p-1})\varphi'(t).$$

It follows from $1\leq p<2$ that $t^{p-1}$ is concave, and hence $2t-pt^{p-1}\leq 0$ if $t\in[0,t_p]$ (see (33)). In particular, $\varphi'(t)$ is monotone decreasing on $(0,\min\{s_p,t_p\})$, which in turn implies that $s_p\geq t_p$. We deduce from (30) and (35) that

$$\varphi"(t)<\frac{2t-pt^{p-1}}{3.1} \qquad \text{if } t\in(0,\tfrac{1}{e}). \tag{37}$$

The rest of the argument in Case 1 splits into two subcases. First let $1.5\leq p<2$. We deduce from (37) and Lemma 4.1 (a) that

$$\varphi"(t)<-\frac{p(p-1)(2-p)}{3.1\cdot 2^{4-p}}\cdot t^{p-1}<-\frac{\frac{3}{4}(2-p)}{3.1\cdot 2^{2.5}}\cdot t<-\frac{2-p}{24}\cdot t \qquad \text{if } t\in(0,\tfrac{1}{2e}). \tag{38}$$

If $1 \leq p \leq 1.5$, then when estimating the right hand side of (37) for given $t \in (0, \frac{1}{2e})$, we may assume that $p = 1.5$ according to (36). In other words, (38) yields that if $1 \leq p \leq 1.5$ and $t \in (0, \frac{1}{2e})$, then

$$\varphi''(t) < \frac{2t - pt^{p-1}}{3.1} \leq \frac{2t - 1.5t^{0.5}}{3.1} \leq -\frac{2 - 1.5}{24} \cdot t \leq -\frac{2 - p}{48} \cdot t. \tag{39}$$

**Case 2** If $2 < p \leq 2.3$ and $t \in (0, \frac{1}{4})$, then $\varphi''(t) > \frac{p-2}{2} \cdot t^{1.3}$
In this case, $\varphi'(0) > 1$ by (31), (i) and (iii). Since $\varphi'$ is continuous, there exists largest $s_p \in (0, \infty]$ such that $\varphi'(t) > 1$ if $0 < t < s_p$. Thus if $t \in (0, s_p)$, then $\varphi(t) > t$, and in turn (32) yields that

$$\varphi''(t) = (2\varphi(t)\varphi'(t) - pt^{p-1})\varphi'(t) > (2t - pt^{p-1})\varphi'(t).$$

It follows from $p > 2$ that $t^{p-1}$ is convex, and hence $2t - pt^{p-1} \geq 0$ if $t \in [0, t_p]$ (see (33)). In particular, $\varphi'(t)$ is monotone increasing on $(0, \min\{s_p, t_p\})$, which in turn implies that $s_p \geq t_p$. We deduce from (34) that

$$\varphi''(t) > 2t - pt^{p-1} \qquad \text{if } t \in (0, \tfrac{1}{2}). \tag{40}$$

We deduce from (40) and Lemma 4.1 (b) that

$$\varphi''(t) > \frac{p(p-1)(p-2)}{2^{4-p}} \cdot t^{p-1} > \frac{2(2-p)}{2^2} \cdot t^{1.3} = \frac{2-p}{2} \cdot t^{1.3} \qquad \text{if } t \in (0, \tfrac{1}{4}). \tag{41}$$

**Case 3** If $p \geq 2.3$ and $t \in (0, \frac{1}{8})$, then $\varphi''(t) > 0.2 \cdot t^{1.3}$
In this case, $\varphi'(0) > \sqrt{\pi}/2$ by (31), (i), (ii) and (iii). Since $\varphi'$ is continuous, there exists largest $s_p \in (0, \frac{1}{4}]$ such that $\varphi'(t) > \sqrt{\pi}/2$ if $0 < t < s_p$. Thus if $t \in (0, s_p]$, then $\varphi(t) > (\sqrt{\pi}/2) \cdot t$, and in turn (32) yields that

$$\varphi''(t) = (2\varphi(t)\varphi'(t) - pt^{p-1})\varphi'(t) > \left(\frac{\pi}{2}\, t - pt^{p-1}\right) \cdot \frac{\sqrt{\pi}}{2}.$$

It follows from (36) that $pt^{p-1} \leq 2.3t^{1.3}$ if $0 < t \leq s_p (\leq \frac{1}{4})$, and hence

$$\varphi''(t) > \left(\frac{\pi}{2}\, t - 2.3t^{1.3}\right) \cdot \frac{\sqrt{\pi}}{2} \qquad \text{if } t \in (0, s_p]. \tag{42}$$

Here $f(\frac{1}{4}) < 0$ for $f(t) = \frac{\pi}{2}\, t - 2.3t^{1.3}$, thus $s_p = \frac{1}{4}$, and hence Lemma 4.1 (b) yields that

$$\varphi''(t) > \frac{(\sqrt{\pi}/2) \cdot 2.3 \cdot 1.3 \cdot 0.3}{2^{1.7}} \cdot t^{1.3} > 0.2 \cdot t^{1.3} \qquad \text{if } t \in (0, \tfrac{1}{8}). \tag{43}$$

**Case 4** If $p = \infty$ and $t > 0$, then $\varphi''(t) > t$
We deduce from differentiating (26) that

$$\varphi'(t) = \Gamma\left(1 + \frac{3}{2}\right) e^{\varphi(t)^2} = \frac{\sqrt{\pi}}{2}\, e^{\varphi(t)^2}; \tag{44}$$

$$\varphi''(t) = 2\varphi(t)\varphi'(t)^2. \tag{45}$$

As $\varphi(t) > 0$ for $t > 0$, we have $\varphi''(t) \geq 0$ by (45), and hence $\varphi'(t)$ is monotone increasing for $t \geq 0$. Therefore $\varphi'(t) \geq \varphi'(0) = \sqrt{\pi}/2$ by (44), which in turn yields by again (45) that

$$\varphi''(t) \geq 2\left(\frac{\sqrt{\pi}}{2}\right)^3 t > t \quad \text{for } t > 0.$$

Combining the estimates of Cases 1-4 yields the estimates of Lemma 4.2 for $\varphi''$. $\square$

**Lemma 4.3** *Let* $p \in [1, \infty]\backslash\{2\}$. *For* $t \in (0, \frac{1}{10})$, *we have*

$$\begin{aligned} \psi_p''(t) &> \frac{2-p}{16} \cdot t \quad \text{\it if } p \in [1,2); \\ \psi_p''(t) &< -\frac{p-2}{11} \cdot t^{1.3} \quad \text{\it if } p \in (2,3]; \\ \psi_p''(t) &< -\frac{t^{1.3}}{11} \quad \text{\it if } p \in (3,\infty] \end{aligned}$$

*Proof:* Let $\psi = \psi_p$. We have $\psi(0) = 0$ as $\psi$ is odd. Therefore $\psi(t) > 0$ if $t > 0$. Turning to $\psi''$, we only sketch the main steps. In this case, differentiating (27) yields the formulas

$$\begin{aligned} \psi'(t) &= \frac{\Gamma(1+\frac{1}{p})}{\Gamma(1+\frac{1}{2})} e^{\psi(t)^p - t^2} && (46) \\ \psi''(t) &= (p\psi(t)^{p-1}\psi'(t) - 2t)\psi'(t). && (47) \end{aligned}$$

**Case 1** If $1 \leq p < 2$ and $t \in (0, \frac{1}{2e})$, then $\psi''(t) > \frac{p-2}{16} \cdot t$
If $p \in [1,2)$, then $\psi'(0) > 1$ by (i), (ii) and (iii), and argument similar to the one in Case 1 in Lemma 4.2 yields

$$\psi''(t) > (p\psi(t)^{p-1}\psi'(t) - 2t)\psi'(t) > pt^{p-1} - 2t \qquad \text{if } t \in (0, \tfrac{1}{e}). \tag{48}$$

If moreover $1.5 \leq p < 2$, then we deduce from (48) and Lemma 4.1 (a) that

$$\varphi''(t) > \frac{p(p-1)(2-p)}{2^{4-p}} \cdot t^{p-1} > \frac{\frac{3}{4}(2-p)}{2^{2.5}} \cdot t > \frac{2-p}{8} \cdot t \qquad \text{if } t \in (0, \tfrac{1}{2e}). \tag{49}$$

If $1 \leq p \leq 1.5$, then when estimating the right hand side of (48) for given $t \in (0, \frac{1}{2e})$, we may assume that $p = 1.5$ according to (36). In other words, (49) yields that if $1 \leq p \leq 1.5$ and $t \in (0, \frac{1}{2e})$, then

$$\varphi''(t) > 2t - pt^{p-1} \geq 2t - 1.5t^{0.5} \geq \frac{2-1.5}{8} \cdot t \geq \frac{2-p}{16} \cdot t. \tag{50}$$

**Case 2** If $2 < p \leq 2.3$ and $t \in (0, \frac{1}{4})$, then $\psi''(t) < -\frac{p-2}{3} \cdot t^{1.3}$
If $p \in (2, 2.3]$, then $\psi'(0) < 1$ by (i), (ii) and (iii), and argument similar to the one in Case 2 in Lemma 4.2 yields

$$\psi''(t) = (p\psi(t)^{p-1}\psi'(t) - 2t)\psi'(t) < -(pt^{p-1} - 2t)\psi'(t) < -\frac{pt^{p-1} - 2t}{3.1} < 0 \qquad \text{if } t \in (0, \tfrac{1}{2}).$$

We deduce from Lemma 4.1 (b) that

$$\varphi"(t) < -\frac{p(p-1)(p-2)}{3.1\cdot 2^{4-p}}\cdot t^{p-1} < -\frac{2(p-2)}{3.1\cdot 2^2}\cdot t^{1.3} < -\frac{p-2}{7}\cdot t^{1.3} \qquad \text{if } t\in(0,\tfrac{1}{4}). \tag{51}$$

**Case 3** If $p\geq 2.3$ and $t\in(0,\frac{1}{10})$, then $\psi"(t) < -t^{1.3}/11$
In this case, $\psi'(0) < 2/\sqrt{\pi}$ by (i), (ii) and (iii). There exists a maximal $s_p\in(0,\frac{1}{5}]$ such that if $t\in(0,s_p)$, then $\psi'(t) < 2/\sqrt{\pi}$. Thus if $t\in(0,s_p]$, then $\psi(t) < (2/\sqrt{\pi})\cdot t$, and in turn (47) yields that

$$\psi"(t) = (p\psi(t)^{p-1}\psi'(t)-2t)\psi'(t) < \left(\left(\frac{2}{\sqrt{\pi}}\right)^p pt^{p-1}-2t\right)\psi'(t). \tag{52}$$

Given $t\in(0,\frac{1}{2}]$,

$$\frac{d}{dp}\log\left[\left(\frac{2}{\sqrt{\pi}}\right)^p pt^{p-1}\right] = \frac{1}{p}+\log\frac{2t}{\sqrt{\pi}} < 0 \quad \text{for } p\in(2,\infty)),$$

and hence (52) yields that if $t\in(0,s_p]$, then

$$\psi"(t) = (p\psi(t)^{p-1}\psi'(t)-2t)\psi'(t) < \left(\left(\frac{2}{\sqrt{\pi}}\right)^{2.3} 2.3t^{1.3}-2t\right)\psi'(t) = f(t)\left(\frac{2}{\sqrt{\pi}}\right)^{2.3}\psi'(t) \tag{53}$$

where

$$f(t) = 2.3t^{1.3} - 2\left(\frac{\sqrt{\pi}}{2}\right)^{2.3} t.$$

Here $f(\frac{1}{5}) < 0$, thus $s_p=\frac{1}{5}$, and Lemma 4.1 (b) yields that

$$f(t) < -\frac{2.3\cdot 1.3\cdot 0.3}{2^{1.7}}\cdot t^{1.3} < -0.27\cdot t^{1.3} \qquad \text{if } t\in(0,\tfrac{1}{10}).$$

We conclude from (30) and (53) that

$$\psi"(t) < -\frac{(\frac{2}{\sqrt{\pi}})^{2.3}\cdot 0.27\cdot t^{1.3}}{3.1} < -\frac{t^{1.3}}{11} \qquad \text{if } t\in(0,\tfrac{1}{10}).$$

**Case 4** If $p=\infty$ and $t\in(0,\frac{1}{3.1})$, then $\psi"(t) < -\frac{2}{3.1}\cdot t$
We deduce from differentiating (27) that if $t>0$, then

$$\psi'(t) = \frac{1}{\Gamma\left(1+\frac{3}{2}\right)}e^{-t^2} = \frac{2}{\sqrt{\pi}}e^{-t^2}; \tag{54}$$

$$\psi"(t) = -2t\psi'(t). \tag{55}$$

We conclude from (30) that $\psi"(t) < -\frac{2t}{3.1}$ for $t\in(0,\frac{1}{3.1})$.

Combining the estimates of Cases 1-4 yields the estimates of Lemma 4.3 for $\psi"$. □

# 5 Basic estimates on isotropic measures

The main result of this section is Lemma 5.4, which shows that for any istropic measure $\mu$ on $S^{n-1}$, there exist $X_1, \ldots, X_n \subset S^{n-1}$ such that $\mu(X_i) > \gamma_1$, $i = 1, \ldots, n$ and $|\det[w_1, \ldots, w_n]| > \gamma_2$ for positive $\gamma_1, \gamma_2$ depending only on $n$.

For some $\alpha \in (0, \frac{\pi}{2}]$ and $v \in S^{n-1}$, we consider the closed and open spherical caps

$$\begin{aligned} \Omega(v,\alpha) &= \{u \in S^{n-1} : \langle u, v\rangle \geq \cos\alpha\} \\ \widetilde{\Omega}(v,\alpha) &= \{u \in S^{n-1} : \langle u, v\rangle > \cos\alpha\}. \end{aligned}$$

**Claim 5.1** *If $\mu$ is an isotropic measure on $S^{n-1}$, $v \in S^{n-1}$, and $\alpha \in (0, \frac{\pi}{2})$ satisfies $\cos\alpha < 1/\sqrt{n}$, then*

$$\mu\left(\widetilde{\Omega}(v,\alpha)\right) + \mu\left(\widetilde{\Omega}(-v,\alpha)\right) \geq 1 - n\cos^2\alpha.$$

*Proof:* Let $X = \{u \in S^{n-1} : |\langle u, v\rangle| \leq \cos\alpha\}$. Since $\mu$ is isotropic, we have $\mu(X) \leq n$, and

$$\begin{aligned} 1 &= \langle v, v\rangle = \int_{S^{n-1}} \langle u, v\rangle^2 \, d\mu(u) = \int_{\widetilde{\Omega}(v,\alpha)\cup\widetilde{\Omega}(-v,\alpha)} \langle u, v\rangle^2 \, d\mu(u) + \int_X \langle u, v\rangle^2 \, d\mu(u) \\ &\leq \mu\left(\widetilde{\Omega}(v,\alpha) \cup \widetilde{\Omega}(-v,\alpha)\right) + n\cos^2\alpha. \quad \square \end{aligned}$$

The next claim follows from a standard argument but we are not aware of any reference. The trivial Borel measure is the one where each set is of measure zero.

**Claim 5.2** *If $\mu$ is a non-trivial measure on $S^{n-1}$, $v \in S^{n-1}$, and $0 < \beta < \alpha < \frac{\pi}{2}$, then there exists a $w \in \Omega(v,\alpha)$, such that*

$$\mu\left(\Omega(v,\alpha) \cap \Omega(w,\beta)\right) > \mu(\Omega(v,\alpha)) \cdot \frac{\sin^{n-1}\beta}{\sqrt{2\pi n}}.$$

*Proof:* We define the measure $\bar{\mu}(X) = \mu(X \cap \Omega(v,\alpha))$ for Borel sets $X \subset S^{n-1}$. Let $\nu$ be the Haar probability measure on $SO(n)$, and hence if $X \subset S^{n-1}$ is a Borel set and $u \in S^{n-1}$, then

$$\nu(\{g \in SO(n) : gu \in X\} = \frac{\mathcal{H}_{n-1}(X)}{\mathcal{H}_{n-1}(S^{n-1})}.$$

We deduce that

$$\begin{aligned}
\mu\left(\Omega(v,\alpha)\right)\cdot\frac{\mathcal{H}_{n-1}(\Omega(v,\beta))}{\mathcal{H}_{n-1}(S^{n-1})} &= \bar{\mu}\left(S^{n-1}\right)\cdot\frac{\mathcal{H}_{n-1}(\Omega(v,\beta))}{\mathcal{H}_{n-1}(S^{n-1})}\\
&= \int_{S^{n-1}}\int_{SO(n)} \mathbf{1}_{\Omega(v,\beta)}(gu)\,d\nu(g)\,d\bar{\mu}(u)\\
&= \int_{SO(n)}\int_{S^{n-1}} \mathbf{1}_{\Omega(v,\beta)}(gu)\,d\bar{\mu}(u)\,d\nu(g)\\
&= \int_{SO(n)} \bar{\mu}(\Omega(g^{-1}v,\beta))\,d\nu(g)\\
&= \int_{SO(n)} \mu(\Omega(v,\alpha)\cap\Omega(g^{-1}v,\beta))\,d\nu(g).
\end{aligned}$$

Therefore there exists some $w_0 \in S^{n-1}$ such that

$$\mu\left(\Omega(v,\alpha)\cap\Omega(w_0,\beta)\right) = \bar{\mu}\left(\Omega(w_0,\beta)\right) \geq \mu\left(\Omega(v,\alpha)\right)\cdot\frac{\mathcal{H}_{n-1}(\Omega(v,\beta))}{\mathcal{H}_{n-1}(S^{n-1})}.$$

Finally, let $w \in \Omega(v,\alpha)$ be the closest point to $w_0$, and hence

$$\Omega(v,\alpha)\cap\Omega(w_0,\beta) \subset \Omega(v,\alpha)\cap\Omega(w,\beta).$$

To conclude the proof, we use that $\mathcal{H}_{n-1}(\Omega(v,\beta)) > \kappa_{n-1}\sin^{n-1}\beta$, $\mathcal{H}_{n-1}(S^{n-1}) = n\kappa_n$, and $\frac{\kappa_{n-1}}{n\kappa_n} > \frac{1}{\sqrt{2\pi n}}$. □

**Claim 5.3** *If* $b_1,\ldots,b_n \in S^{n-1}$*, and* $\|s_i\| \leq |\det[b_1,\ldots,b_n]|/4n$ *for* $s_i \in \mathbb{R}^n$ *and* $i = 1,\ldots,n$*, then*

$$|\det[b_1+s_1,\ldots,b_n+s_n]| \geq |\det[b_1,\ldots,b_n]|/2.$$

*Proof:* Let $D = |\det[b_1,\ldots,b_n]|/4n$. Since any $r_1,\ldots,r_n \in \mathbb{R}^n$ satisfies

$$|\det[r_1,\ldots,r_n]| \leq \|r_1\|\cdot\ldots\cdot\|r_n\|,$$

we deduce from the linearity of the determinant and $e^t < 1+2t$ for $t \in (0,1)$ that

$$\begin{aligned}
|\det[b_1+s_1,\ldots,b_n+s_n]| &\geq |\det[b_1,\ldots,b_n]| - \sum_{i=1}^{n}\binom{n}{i}D^i\\
&= 4nD - (1+D)^n + 1\\
&\geq 4nD - e^{nD} + 1\\
&\geq 4nD - 2nD \geq 2nD = |\det[b_1,\ldots,b_n]|/2. \quad \square
\end{aligned}$$

**Lemma 5.4** *If $\mu$ is an isotropic measure on $S^{n-1}$, then there exist $v_1, \dots, v_n \in S^{n-1}$ such that $\mu(\Omega(v_i, \beta)) \geq \beta^n$ for $i = 1, \dots, n$, and if $w_i \in \Omega(v_i, \beta)$ for $i = 1, \dots, n$, then $|\det[w_1, \dots, w_n]| \geq 2n\beta$, where $\beta = 2^{-(n+1)}n^{-(n+1)/2}$.*

*Proof:* Let $\alpha_n \in (0, \frac{\pi}{2})$ satisfy $\cos\alpha_n = \frac{1}{2\sqrt{n}}$. First, we construct $v_i, p_i \in S^n$ by induction on $i = 1, \dots, n$ in a way such that

$$\begin{aligned} \mu(\Omega(v_i, \beta)) &\geq \beta^n; & (56)\\ \mu(\Omega(p_i, \alpha_n)) &\geq 3/8; & (57)\\ v_i &\in \Omega(p_i, \alpha_n); & (58)\\ \langle p_i, v_j \rangle &= 0 \text{ for } 1 \leq j < i \leq n. & (59) \end{aligned}$$

Let $p \in S^{n-1}$. According to Claim 5.1, we can choose $p_1 \in \{p, -p\}$ such that

$$\mu(\Omega(p_1, \alpha_n)) \geq \frac{1 - n\cos^2\alpha_n}{2} = \frac{3}{8}.$$

Thus Claim 5.2 yields the existence of a $v_1 \in \Omega(p_1, \alpha_n)$ satisfying (56).

If $i \geq 2$, and $v_j, p_j$ are known for $j = 1, \dots, i-1$, then we choose $p_i' \in S^{n-1}$ satisfying (59). Again, Claim 5.1 provides $p_i \in \{p_i', -p_i'\}$ satisfying (57). In addition, the $v_i \in \Omega(p_i, \alpha_n)$ satisfying (56) is provided by Claim 5.2.

We deduce from (58) that if $i = 2, \dots, n$, then $\langle p_i, v_i \rangle \geq \frac{1}{2\sqrt{n}}$, which fact, combined with (59), yields that

$$\text{dist}\,(v_i, \text{aff}\,\{v_1, \dots, v_{i-1}\}) \geq \frac{1}{2\sqrt{n}}.$$

In particular,

$$|\det[v_1, \dots, v_n]| \geq 2^{-(n-1)}n^{-(n-1)/2} = 4n\beta. \tag{60}$$

Next let $w_i \in \Omega(v_i, \beta)$ for $i = 1, \dots, n$, and hence $\|s_i\| < \beta$ for $s_i = w_i - v_i$ and $i = 1, \dots, n$. Therefore Claim 5.3 implies the lemma. □

The following Lemma 5.5 uses the notation of Lemma 5.4.

**Lemma 5.5** *For an isotropic measure $\mu$ on $S^{n-1}$, let $v_1, \dots, v_n \in S^{n-1}$ be as in Lemma 5.4. For $i = 1, \dots, n$ and $\eta \in (0, \beta)$, we have*

*(i) either there exists $q_i \in \Omega(v_i, \beta)$ such that*

$$\mu(\Omega(v_i, \beta) \cap \Omega(q_i, \eta)) \geq \frac{\beta^n}{4n};$$

*(ii) or there exist $\Psi_1, \Psi_2 \subset \Omega(v_i, \beta)$ such that*

$$\begin{aligned} \mu(\Psi_j) &\geq \frac{\beta^n}{4n} \text{ for } j = 1, 2\\ \|a_1 - a_2\| &\geq \frac{\eta}{\sqrt{n}} \text{ for } a_1 \in \Psi_1 \text{ and } a_2 \in \Psi_2. \end{aligned}$$

*Proof:* If there exists $q_i \in \Omega(v_i,\beta)$ such that $\mu(\{q_i\}) \geq \frac{\beta^n}{4n}$, then (i) is satisfied. Therefore we assume that

$$\mu(\{q\}) < \frac{\beta^n}{4n} \text{ for all } q \in \Omega(v_i,\beta). \tag{61}$$

We choose on orthonormal basis $w_1,\dots,w_{n-1}$ for $v_i^\perp$. It follows from (61) that there exist $-1 < s_j \leq t_j < 1$ for $j = 1,\dots,n-1$ such that

$$\mu\left(\{x \in \Omega(v_i,\beta) : \langle w,x\rangle < s_j\}\right) \leq \frac{\beta^n}{4n} \leq \mu\left(\{x \in \Omega(v_i,\beta) : \langle w,x\rangle \leq s_j\}\right)$$

$$\mu\left(\{x \in \Omega(v_i,\beta) : \langle w,x\rangle > t_j\}\right) \leq \frac{\beta^n}{4n} \leq \mu\left(\{x \in \Omega(v_i,\beta) : \langle w,x\rangle \geq t_j\}\right).$$

If there exists $j \in \{1,\dots,n-1\}$ such that $t_j - s_j \geq \eta/\sqrt{n}$, then we define $\Psi_1 = \{x \in \Omega(v_i,\beta) : \langle w,x\rangle \leq s_j\}$ and $\Psi_2 = \{x \in \Omega(v_i,\beta) : \langle w,x\rangle \geq t_j\}$, which satisfy (ii).

Finally, we assume that $t_j - s_j < \eta/\sqrt{n}$ for $j = 1,\dots,n-1$. Let $q_i \in \Omega(v_i,\beta)$ be such that $\langle q_i, w_j\rangle = (s_j+t_j)/2$ for $j = 1,\dots,n-1$, and let

$$\Psi = \{x \in \Omega(v_i,\beta) : s_j \leq \langle v_j,x\rangle \leq t_j,\ j = 1,\dots,n-1\}.$$

On the one hand,

$$\mu(\Psi) \geq \mu(\Omega(v_i,\beta)) - 2n\cdot\frac{\beta^n}{4n} \geq \frac{\beta^n}{2}.$$

On the other hand, $\|x - (q_i|v_i^\perp)\| \leq \eta/2$ for $x \in \Psi|v_i^\perp$. Since $\langle u,v_i\rangle > 1/2$ for $u \in \Omega(v_i,\beta)$, we deduce that $\Psi \subset \Omega(q_i,\eta)$. In turn, we conlude (i). □

# 6 Even isotropic measures and the cross measure

As a consequence of Claim 5.1, we estimate the Wasserstein distance.

**Lemma 6.1** *If $\mu$ is an even isotropic measure, and $\nu$ is a cross measure on $S^{n-1}$ with* $\operatorname{supp}\nu = \{\pm w_1,\dots,\pm w_n\}$*, and for some $\delta \in [0,\frac{1}{7n^2}]$ and $\omega \in [0,1)$, we have*

$$\mu\left(S^{n-1}\setminus\bigcup_{i=1}^{n}(\Omega(w_i,\delta)+\Omega(-w_i,\delta))\right) \leq \omega,$$

*then*

$$\delta_W(\mu,\nu) \leq 2n\delta + 2\pi n^2\omega.$$

*Proof:* We write $-w_i = w_{i+n}$ for $i = 1,\dots,n$. Since $\widetilde{\Omega}\left(w_i,\frac{\pi}{2}-\delta\right)$ is disjoint from $\Omega(w_j,\delta)$ for $i \neq j$, it follows from Claim 5.1 that for each $i = 1,\dots,n$, we have

$$\begin{aligned}\mu\left(\Omega(w_i,\delta)\cup\Omega(-w_i,\delta)\right) &\geq \mu\left(\widetilde{\Omega}\left(w_i,\frac{\pi}{2}-\delta\right)\cup\widetilde{\Omega}\left(-w_i,\frac{\pi}{2}-\delta\right)\right) - \omega \\ &> 1 - n\sin^2\delta - \omega > 1 - n\delta^2 - \omega.\end{aligned}$$

Since $\mu(S^{n-1})=n$ and $\mu$ is even, if $i=1,\ldots,2n$, then

$$\mu(\Omega(w_i,\delta))\leq\frac{1}{2}\left(n-(n-1)(1-n\delta^2-\omega)\right)\leq\frac{1+n^2\delta^2+n\omega}{2}.$$

For $f\in\mathrm{Lip}_1(S^{n-1})$, we may assume that $f(w_1)=0$, and hence $|f(u)|\leq\pi$ for $u\in S^{n-1}$. Therefore

$$\begin{aligned}\int_{S^{n-1}}f\,d\mu-\int_{S^{n-1}}f\,d\nu &< \sum_{i=1}^{2n}\left(\int_{\Omega(w_i,\delta)}(f(u)-f(w_i))\,d\mu(u)+\int_{\Omega(w_i,\delta)}f(w_i)\,d\mu(u)-\frac{f(w_i)}{2}\right)\\ &\quad+\int_{S^{n-1}\setminus(\cup_{i=1}^{2n}\Omega(w_i,\delta))}f(u)\,d\mu(u)\\ &\leq 2n\left(\delta\cdot\frac{1+n^2\delta^2+n\omega}{2}+\pi\cdot\frac{n^2\delta^2+n\omega}{2}\right)+\pi\omega\leq 2n\delta+2\pi n^2\omega.\end{aligned}$$

□

We deduce the following estimate for the Wasserstein distance.

**Corollary 6.2** *If $\mu$ is an even isotropic measure, and $\nu$ is a cross measure on $S^{n-1}$, and $\delta_H(\operatorname{supp}\mu,\operatorname{supp}\nu)\leq 1/(7n^2)$, then*

$$\delta_W(\mu,\nu)\leq 2n\delta_H(\operatorname{supp}\mu,\operatorname{supp}\nu).$$

Finally, we consider the stability of the optimal symmetric coverings by $2n$ congruent spherical caps where symmetric covering stands for an arrangement invariant under the antipodal map. It is a well-known conjecture that in an optimal covering of $S^{n-1}$ by $2n$ congruent spherical caps, the spherical centers of the caps are vertices of a regular crosspolytope (see say L. Fejes Tóth [20]). This conjecture has been verified by L. Fejes Tóth [20] if $n\leq 3$, and by L. Dalla, D. G Larman, P. Mani-Levitska, C. Zong [17] if $n=4$. However, the case when the $2n$ congruent spherical caps are symmetric (see Lemma 6.3 (i)) should be known, but we could not find any reference if $n\geq 5$.

**Lemma 6.3** *Let $n\geq 2$, let $t\in(0,\frac{1}{4^n\sqrt{n!}})$, and let $u_1,\ldots,u_n\in S^{n-1}$.*

*(i) If there exist $i<j$ such that $|\langle u_i,u_j\rangle|\geq\sin t$, then there exists $u\in S^{n-1}$ such that*

$$|\langle u_i,u\rangle|\leq\frac{1}{\sqrt{n}}-\frac{t}{4n^{3/2}}\ \text{ for } i=1,\ldots,n.$$

*(ii) If $|\langle u_i,u_j\rangle|\leq\sin t$ for all $i<j$, then there exists a cross measure $\nu$ such that*

$$\delta_H(\operatorname{supp}\nu,\{\pm u_1,\ldots,\pm u_n\})\leq 4^{n-2}\sqrt{(n-1)!}\cdot t.$$

*Proof:* First we assume that $|\langle u_1, u_2\rangle| \geq \sin t$. We construct sequences $a_2, \ldots, a_n > 0$ and $w_1, \ldots, w_n \in S^{n-1}$ such that $w_i \in \mathrm{lin}\{u_1, \ldots, u_i\}$, and possibly after exchanging some of $u_i$ by $-u_i$, we have

$$\langle w_i, u_j\rangle = a_i \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, i.$$

More precisely, let $w_1 = u_1$, and if $i = 2, \ldots, n$ and $w_{i-1}$ is known, then we choose the direction of $u_i$ in a way such that $\langle u_i, w_{i-1}\rangle \leq 0$. This algorithm determines $a_2, \ldots, a_n > 0$ and $w_1, \ldots, w_n \in S^{n-1}$, and we prove that

$$\langle w_i, u_j\rangle = a_i \leq \frac{1}{\sqrt{i}} - \frac{t}{4i^{3/2}}. \text{ for } i = 2, \ldots, n \text{ and } j = 1, \ldots, i. \tag{62}$$

To verify (62), we use the elementary fact that if $o$ is a vertex of a triangle, two sides meeting at $o$ are of length $a$ and $b$, and enclose an angle $\gamma$, then the distance of $o$ from the line of the third side is

$$h = \frac{ab \sin\gamma}{\sqrt{a^2 + b^2 - 2ab\cos\gamma}}. \tag{63}$$

In addition, we use that if $f(a) = \frac{a}{\sqrt{1+a^2}}$ for $a \in (0, s)$ and $s > 0$, then

$$f'(a) = \frac{1}{(1+a^2)^{3/2}} > \frac{1}{(1+s^2)^{3/2}}. \tag{64}$$

We start with case $i = 2$. We have $\angle(u_1, u_2) \geq \frac{\pi}{2} + t$, and let $w_2 = (u_1 + u_2)/\|u_1 + u_2\|$. Therefore (63) yields that

$$\langle w_2, u_1\rangle = \langle w_2, u_2\rangle \leq \frac{\cos t}{\sqrt{2 + 2\sin t}} < \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1 + \sin t}} < \frac{1}{\sqrt{2}} \cdot \left(1 - \frac{\sin t}{4}\right) < \frac{1}{\sqrt{2}} - \frac{t}{8\sqrt{2}}.$$

Next assume that $2 \leq i < n$, and (62) holds. We apply (63) to the triangle $[o, a_i w_i, u_{i+1}]$. Since $\langle u_{i+1}, w_i\rangle \leq 0$, we deduce from (64) with $s = \frac{1}{\sqrt{i}}$ that

$$a_{i+1} \leq \frac{a_i}{\sqrt{1 + a_i^2}} = f(a_i) < f(s) - \frac{t}{4i^{3/2}(1+s^2)^{3/2}} = \frac{1}{\sqrt{i+1}} - \frac{t}{4(i+1)^{3/2}}.$$

Finally, (62) yields (i) with $u = w_n$.

For (ii), let $v_1, \ldots, v_n$ be an orthonormal base of $\mathbb{R}^n$ such that $v_i \in \mathrm{lin}\{u_1, \ldots, u_i\}$ and $\langle v_i, u_j\rangle \geq 0$ for $i = 1, \ldots, n$, and hence $v_1 = u_1$. We verify that

$$\angle(v_i, u_i) \leq 4^{i-2}\sqrt{(i-1)!} \cdot t \text{ for } i = 2, \ldots, n \tag{65}$$

by induction on $i = 2, \ldots, n$.

If $i = 2$, then readily $\angle(v_2, u_2) \leq t$. If (65) holds for some $i$ with $2 \leq i < n$, then

$$\left|\angle(u_{i+1}, v_j) - \frac{\pi}{2}\right| \leq \left|\angle(u_{i+1}, u_j) - \frac{\pi}{2}\right| + \angle(u_j, v_j) < 2 \cdot 4^{i-2}\sqrt{(i-1)!} \cdot t$$

for $j = 1, \ldots, i$. In other words, $\langle u_{i+1}, v_j \rangle < 2 \cdot 4^{i-2}\sqrt{(i-1)!} \cdot t$ for $j = 1, \ldots, i$, which in turn yields that

$$\sin \angle(u_{i+1}, v_{i+1}) = \|u_{i+1}|v_{i+1}^{\perp}\| < 2 \cdot 4^{i-2}\sqrt{i!} \cdot t.$$

In turn, we conclude $\angle(u_{i+1}, v_{i+1}) < 4^{i-1}\sqrt{i!} \cdot t$. $\square$

Lemma 6.3 yields the following statement with factor $4n^{3/2} \cdot 4^{n-2}\sqrt{(n-1)!} < 4^n n!$.

**Corollary 6.4** *Let $n \geq 2$, let $t \in (0, \frac{1}{4^n n!})$, and let $u_1, \ldots, u_n \in S^{n-1}$. If*

$$\Omega\left(u, \arccos\left(\frac{1}{\sqrt{n}} - t\right)\right) \cap \{\pm u_1, \ldots, \pm u_n\} \neq \emptyset$$

*for any $u \in S^{n-1}$, then then there exists a cross measure $\nu$ such that*

$$\delta_H(\operatorname{supp} \nu, \{\pm u_1, \ldots, \pm u_n\}) \leq 4^n n! \cdot t.$$

**Remark** The condition is equivalent saying that $\Omega\left(\pm u_i, \arccos\left(\frac{1}{\sqrt{n}} - t\right)\right)$, $i = 1, \ldots, n$ cover $S^{n-1}$.

# 7 The volume of $Z_p^*$

**Claim 7.1** *For $u, u_0 \in S^{n-1}$ with $\langle u, u_0 \rangle \geq 0$, we have $V(\Xi_{u,u_0}) \geq \kappa_n / 240^n$ where*

$$\Xi_{u,u_0} = \left\{ y \in 0.1\, B^n : \langle y, u \rangle \geq \frac{1}{30} \text{ and } \langle y, u_0 \rangle \geq \frac{1}{30} \text{ and } \langle y, u - u_0 \rangle \geq \frac{\|u - u_0\|}{120} \right\}.$$

*Proof:* Let $\gamma$ be the half of the angle of $u$ and $u_0$, and hence $\gamma \in (0, \frac{\pi}{4}]$. The set

$$\Xi_0 = \left\{ y \in 0.1\, B^n : \langle y, u \rangle \geq \frac{1}{30},\ \langle y, u_0 \rangle \geq \frac{1}{30} \right\}$$

contains a ball of radius $r$ with center $\frac{0.1 - r}{\|u + u_0\|}(u + u_0)$ provided that

$$(0.1 - r)\cos\gamma \geq \tfrac{1}{30} + r.$$

Since $\cos\gamma \geq 1/\sqrt{2}$, we may choose

$$r = \frac{0.1 - (\sqrt{2}/30)}{\sqrt{2} + 1} > \frac{1}{60}.$$

Therefore $\Xi_{u,u_0}$ contains a ball of radius $r/4 > 1/240$. $\square$

**Proposition 7.2** *If $p \in [1, \infty)\backslash\{2\}$, $\mu$ is an even discrete istropic measure on $S^{n-1}$, and*

$$V(Z_p^*(\mu)) \geq (1-\varepsilon)V(Z_p^*(\nu_n))$$

*for $\varepsilon \in (0,1)$, then there exists a cross measure $\nu$ on $S^{n-1}$ such that*

$$\delta_W(\mu, \nu) \leq n^{cn^3} \max\{|p-2|^{\frac{-2}{3}}, 1\} \cdot \varepsilon^{1/3}$$

*for some absolute constant $c > 0$.*

*Proof:* What we actually prove is that if $\eta > 0$ is small enough, then either

$$V(Z_p^*(\mu)) < (1 - n^{-cn^3} \min\{(p-2)^2, 1\} \cdot \eta^3)V(Z_p^*(\nu_n)), \tag{66}$$

or there exists a cross measure $\nu$ satisfying

$$\delta_W(\mu, \nu) \leq n^{cn}\eta \tag{67}$$

for some absolute constant $c > 0$.

Let $\operatorname{supp}\mu = \{\bar{u}_1, \ldots, \bar{u}_{\bar{k}}\}$, and let $\bar{c}_i = \mu(\{\bar{u}_i\})$. For $c_0 = \min\{\bar{c}_i : i = 1, \ldots, \bar{k}\}$ and $i = 1, \ldots, \bar{k}$, we define $\bar{m}_i = \min\{m \in \mathbb{Z} : m \geq 1 \text{ and } c_i/m \leq c_0\}$, and let $k = \sum_{i=1}^{\bar{k}} \bar{m}_i$. We consider $\xi : \{1, \ldots, k\} \to \{1, \ldots, \bar{k}\}$ such that $\#\xi^{-1}(i) = \bar{m}_i$ for $i = 1, \ldots, \bar{k}$, and define

$$u_i = \bar{u}_{\xi(i)} \text{ and } c_i = \bar{c}_{\xi(i)}/\bar{m}_{\xi(i)}$$

for $i = 1, \ldots, k$. The system $(u_1, \ldots, u_k, c_1, \ldots, c_k)$ is even (i.e. origin symmetric) in the following sense: Any $u \in S^{n-1}$ occurs as $u_i$ exactly as many times as $-u$, and if $u_i = -u_j$, then $c_i = c_j$.

In particular, $\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}_n$ and $\sum_{i=1}^k c_i = n$, and for any Borel $X \subset S^{n-1}$, we have

$$\mu(X) = \sum_{u_i \in X} c_i.$$

The reason for the renormalization is that

$$c_0/2 < c_i \leq c_0 \text{ for } i = 1, \ldots, k. \tag{68}$$

In addition, let $\varphi = \varphi_p$ as defined in (26), let $g(t) = e^{-\pi t^2}$, and for $i = 1, \ldots, k$, let

$$f_i(t) = \frac{1}{2\Gamma(1+\frac{1}{p})} e^{-|t|^p}.$$

We define the map $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Theta(y) = \sum_{i=1}^k c_i \varphi(\langle y, u_i \rangle) \cdot u_i,$$

and hence the differential of $\Theta$ is

$$d\Theta(y) = \sum_{i=1}^{k} c_i \varphi'(\langle y, u_i \rangle) \cdot u_i \otimes u_i \tag{69}$$

where $d\Theta(y)$ is positive definite, and $\Theta$ is injective from $\mathbb{R}^n$ to $\mathbb{R}^n$. It follows from first applying (18), then (12) that

$$\begin{aligned} V(Z_p^*(\mu)) &\leq \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} g(\varphi(\langle u_i, x \rangle))^{c_i}\right) \left(\prod_{i=1}^{k} \varphi'(\langle u_i, x \rangle)^{c_i}\right) dx \\ &= \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \exp\left(-\pi \sum_{i=1}^{k} c_i \varphi(\langle u_i, x \rangle)^2\right) \left(\prod_{i=1}^{k} \varphi'(\langle u_i, x \rangle)^{c_i}\right) dx. \end{aligned} \tag{70}$$

For each fixed $y \in \mathbb{R}^n$, we estimate the product of the two terms in (70) after the integral sign. To estimate the first term in (70), we apply (11) with $\theta_i = \varphi(\langle y, u_i \rangle)$, $i = 1, \ldots, k$, and hence the definition of $\Theta$ yields

$$\exp\left(-\sum_{i=1}^{k} c_i \varphi(\langle y, u_i \rangle)^2\right) \leq e^{-\|\Theta(y)\|^2}. \tag{71}$$

To estimate the second term, we apply Lemma 3.1 with $v_i = \sqrt{c_i} \cdot u_i$ and $t_i = \varphi'(\langle y, u_i \rangle)$ at each $y \in \mathbb{R}^n$, and write $\theta^*(y)$ and $t_0(y)$ to denote the corresponding $\theta^* \geq 1$ and $t_0$. In particular, if $\{i_1, \ldots, i_n\} \subset \{1, \ldots, k\}$ and $y \in \mathbb{R}^n$, then we set

$$\aleph(i_1, \ldots, i_n; y) = c_{i_1} \cdot \ldots \cdot c_{i_n} \det[u_{i_1}, \ldots, u_{i_n}]^2 \left(\frac{\sqrt{\varphi'(\langle y, u_{i_1} \rangle) \cdot \ldots \cdot \varphi'(\langle y, u_{i_n} \rangle)}}{t_0(y)} - 1\right)^2.$$

Therefore for

$$\theta^*(y) = 1 + \frac{1}{2} \sum_{1 \leq i_1 < \ldots < i_n \leq k} \aleph(i_1, \ldots, i_n; y), \tag{72}$$

Lemma 3.1 says

$$\prod_{i=1}^{k} \varphi'(\langle y, u_i \rangle)^{c_i} \, dy \leq \theta^*(y)^{-1} \det d\Theta(y). \tag{73}$$

We conclude from (71) and (73) that

$$V(Z_p^*(\mu)) \leq \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \theta^*(y)^{-1} e^{-\pi\|\Theta(y)\|^2} \det d\Theta(y) \, dy. \tag{74}$$

To provide a lower bound for $\theta^*(y)$, we use the estimates of (30) yielding

$$\frac{1}{3.1} < \varphi'(s) < 3.1 \text{ and } \varphi(s) < 1 \text{ for } p \in [1, \infty] \text{ and } s \in [0, \tfrac{1}{3.1}]. \tag{75}$$

We consider the $v_1,\dots,v_n\in S^n$ provided by Lemma 5.4 such that

$$\begin{aligned}\mu(\Omega(v_i,\beta)) &> \beta^n \quad \text{for } i=1,\dots,n; && (76)\\ |\det[w_1,\dots,w_n]| &\geq 2n\beta \quad \text{for } w_i\in\Omega(v_i,\beta) \text{ and } i=1,\dots,n; && (77)\\ \beta &= 2^{-(n+1)}n^{-(n+1)/2}. && (78)\end{aligned}$$

The remaining discussion is split into three cases, where the first two correspond to the two cases in Lemma 5.5.

**Case 1** There exist $l\in\{1,\dots,n\}$ and $\Psi_1,\Psi_2\subset\Omega(v_l,\beta)$ such that

$$\begin{aligned}\mu(\Psi_j) &\geq \frac{\beta^n}{4n} \quad \text{for } j=1,2\\ \|a_1-a_2\| &\geq \frac{\eta}{\sqrt{n}} \quad \text{for } a_1\in\Psi_1 \text{ and } a_2\in\Psi_2.\end{aligned}$$

In this case, we prove

$$V(Z_p^*(\mu)) < \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}(1-n^{-cn^3}\min\{(p-2)^2,1\}\cdot\eta^2) \tag{79}$$

for some absolute constant $c>0$.

We may assume that $l=n$. For $j=1,2$, let

$$\Pi_j=\{i\in\{1,\dots,k\}:\ u_i\in\Psi_j\}$$

Possibly after interchanging the roles of $\Psi_1$ and $\Psi_2$, we may assume that $\#\Pi_1\leq\#\Pi_2$, and let

$$\tau:\Pi_1\to\Pi_2 \text{ be injective.} \tag{80}$$

Given $u_{i_j}\in\Omega(v_j,\beta)$ for $j=1,\dots,n-1$ and $u_{i_n}\in\Psi_1$, we have have $u_{\tau(i_n)}\in\Psi_2$, and (68) and (77) yield

$$\left.\begin{array}{r} c_{i_1}\cdot\ldots\cdot c_{i_{n-1}}\cdot c_{i_n}\det[u_{i_1},\dots,u_{i_n}]^2\\ c_{i_1}\cdot\ldots\cdot c_{i_{n-1}}\cdot c_{\tau(i_n)}\det[u_{i_1},\dots,u_{i_{n-1}},u_{\tau(i_n)}]^2\end{array}\right\}\geq 4n^2\beta^2c_{i_1}\cdot\ldots\cdot c_{i_{n-1}}\cdot(c_{i_n}/2). \tag{81}$$

In addition, $\beta<\pi/4$ yields that if $u_{i_n}\in\Psi_1$, then

$$\langle u_{i_n},u_{\tau(i_n)}\rangle>0.$$

According to Claim 7.1, we have $V(\Xi_{u,u_0})\geq\kappa_n/240^n$ for $u,u_0\in S^{n-1}$ with $\langle u,u_0\rangle\geq 0$ where

$$\Xi_{u,u_0}=\left\{y\in 0.1\,B^n:\ \langle y,u\rangle\geq\frac{1}{30} \text{ and } \langle y,u_0\rangle\geq\frac{1}{30} \text{ and } \langle y,u-u_0\rangle\geq\frac{\|u-u_0\|}{120}\right\}.$$

In particular, if $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$\begin{aligned} \langle y, u_{i_n}\rangle, \langle y, u_{\tau(i_n)}\rangle &< \frac{1}{8}; \\ \langle y, u_{i_n}\rangle - \langle y, u_{\tau(i_n)}\rangle &= \langle y, u_{i_n} - u_{\tau(i_n)}\rangle \geq \frac{\eta}{120\sqrt{n}}. \end{aligned}$$

Next $\varphi''$ is continuous, and Lemma 4.2 says that if $t \in [\frac{1}{30}, 0.1]$, then

$$|\varphi"(t)| \geq \left\{ \begin{array}{lll} \frac{|p-2|}{48}\left(\frac{1}{30}\right)^{1.3} & > \frac{|p-2|}{2^{12}} & \text{if } p \in [1,3]\backslash 2; \\ 0.2\left(\frac{1}{30}\right)^{1.3} & > 2^{-9} & \text{if } p > 3 \end{array} \right. \tag{82}$$

Therefore

$$|\varphi'(\langle y, u_{i_n}\rangle) - \varphi'(\langle y, u_{\tau(i_n)}\rangle)| > \left\{ \begin{array}{lll} \frac{|p-2|}{2^{12}120\sqrt{n}}\,\eta > & \frac{|p-2|}{2^{19}\sqrt{n}}\,\eta & \text{if } p \in [1,3]\backslash 2; \\ \frac{1}{2^{9}120\sqrt{n}}\,\eta > & \frac{1}{2^{19}\sqrt{n}}\,\eta & \text{if } p > 3. \end{array} \right.$$

It follows from Claim 3.2 and $\varphi'(t) \leq 3.1$ for $p \in [1,\infty)\backslash 2$ and $t \in (0, 0.1]$ (cf. (75) that

$$\begin{aligned} &\left( \frac{\sqrt{\varphi'(\langle y, u_{i_1}\rangle) \cdot \ldots \cdot \varphi'(\langle y, u_{i_{n-1}}\rangle \cdot \varphi'(\langle y, u_{i_n}\rangle)}}{t_0(y)} - 1 \right)^2 \\ &+ \left( \frac{\sqrt{\varphi'(\langle y, u_{i_1}\rangle) \cdot \ldots \cdot \varphi'(\langle y, u_{i_{n-1}}\rangle \cdot \varphi'(\langle y, u_{\tau(i_n)}\rangle)}}{t_0(y)} - 1 \right)^2 \geq \\ &\geq \frac{(\varphi'(\langle y, u_{i_n}\rangle) - \varphi'(\langle y, u_{\tau(i_n)}\rangle))^2}{2(\varphi'(\langle y, u_{i_n}\rangle) + \varphi'(\langle y, u_{\tau(i_n)}\rangle))^2} > \\ &> \frac{\min\{1, (p-2)^2\}}{2^{45}n}\,\eta^2 \end{aligned}$$

Combining this estimate with (81) implies that if $p \in [1,\infty)\backslash 2$ and $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \ldots, n-1$, $u_{i_n} \in \Psi_1$ and $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$\begin{aligned} &\aleph(i_1, \ldots, i_{n-1}, i_n; y) + \aleph(i_1, \ldots, i_{n-1}, \tau(i_n); y) \geq \\ &4n^2\beta^2 c_{i_1} \cdot \ldots \cdot c_{i_{n-1}} \cdot (c_{i_n}/2) \frac{\min\{1, (p-2)^2\}}{2^{45}n}\,\eta^2 \end{aligned}$$

If $u_{i_n} \in \Psi_1$ and $y \in \mathbb{R}^n$, then we define

$$\varrho(i_n; y) = \left\{ \begin{array}{ll} 0 & \text{if } y \notin \Xi_{i_n, \tau(i_n)} \\ \frac{\beta^2 n (p-2)^2}{2^{44}}\,\eta^2 & \text{if } y \in \Xi_{i_n, \tau(i_n)} \text{ and } p \in [1,3]\backslash 2; \\ \frac{\beta^2 n}{2^{44}}\,\eta^2 & \text{if } y \in \Xi_{i_n, \tau(i_n)} \text{ and } p > 3. \end{array} \right.$$

In particular, if $u_{i_j}\in\Omega(v_j,\beta)$ for $j=1,\ldots,n-1$, $u_{i_n}\in\Psi_1$ and $y\in\mathbb{R}^n$, then

$$\aleph(i_1,\ldots,i_{n-1},i_n;y)+\aleph(i_1,\ldots,i_{n-1},\tau(i_n),y)\geq c_{i_1}\cdot\ldots\cdot c_{i_n}\varrho(i_n;y). \tag{83}$$

Substituting (83) into (72) and later using (76) show that if $y\in\mathbb{R}^n$, then

$$\begin{aligned}\theta^*(y) &\geq 1+\frac{1}{2}\sum_{\substack{u_{i_j}\in\Omega(v_j,\beta),\,j=1,\ldots,n-1\\ u_{i_n}\in\Psi_1}} c_{i_1}\cdot\ldots\cdot c_{i_{n-1}}\cdot c_{i_n}\varrho(i_n;y)\\ &= 1+\frac{1}{2}\left(\prod_{j=1}^{n-1}\mu(\Omega(v_j,\beta))\right)\sum_{u_{i_n}\in\Psi_1}c_{i_n}\varrho(i_n;y)\geq 1+\frac{\beta^{n(n-1)}}{2}\sum_{u_{i_n}\in\Psi_1}c_{i_n}\varrho(i_n;y).\end{aligned}$$

Here

$$\frac{\beta^{n(n-1)}}{2}\sum_{u_{i_n}\in\Psi_1}c_{i_n}\varrho(i_n;y)\leq\frac{\beta^{n(n-1)}}{2}\,\mu(\Psi_1)\cdot\frac{\beta^2 n}{2^{44}}\,\eta^2<1,$$

and hence if $y\in\mathbb{R}^n$, then

$$\theta^*(y)^{-1}\leq 1-\frac{\beta^{n(n-1)}}{4}\sum_{u_{i_n}\in\Psi_1}c_{i_n}\varrho(i_n;y). \tag{84}$$

We deduce from (74) and (84) that

$$\begin{aligned}V(Z_p^*(\mu)) &\leq \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}\left(1-\frac{\beta^{n(n-1)}}{4}\sum_{u_{i_n}\in\Psi_1}c_{i_n}\varrho(i_n;y)\right)e^{-\pi\|\Theta(y)\|^2}\det d\Theta(y)\,dy\\ &= \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n}e^{-\pi\|\Theta(y)\|^2}\det d\Theta(y)\,dy-\\ &\quad -\frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\cdot\frac{\beta^{n(n-1)}}{4}\sum_{u_{i_n}\in\Psi_1}c_{i_n}\int_{\mathbb{R}^n}\varrho(i_n;y)e^{-\pi\|\Theta(y)\|^2}\det d\Theta(y)\,dy.\end{aligned}$$

Here

$$\int_{\mathbb{R}^n}e^{-\pi\|\Theta(y)\|^2}\det d\Theta(y)\,dy=\int_{\mathbb{R}^n}e^{-\pi\|z\|^2}\,dz=1.$$

If $y\in\Xi_{i_n,\tau(i_n)}$, then (71), (73) and (75) yield that

$$e^{-\|\Theta(y)\|^2} \geq \exp\left(-\sum_{i=1}^k c_i\varphi(\langle y,u_i\rangle)^2\right)>\exp\left(-\sum_{i=1}^k c_i\right)=e^{-n}, \tag{85}$$

$$\det d\Theta(y) \geq \prod_{i=1}^k\varphi'(\langle y,u_i\rangle)^{c_i}\,dy\geq\prod_{i=1}^k 3.1^{-c_i}=3.1^{-n}. \tag{86}$$

Therefore

$$V(Z_p^*(\mu)) \leq \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\left(1-\sum_{u_{i_n}\in\Psi_1} c_{i_n}\frac{\beta^{n(n-1)}}{4}\cdot\frac{V(\Xi_{i_n,\tau(i_n)})}{(3.1e)^n}\cdot\frac{\beta^2 n\min\{(p-2)^2,1\}}{2^{44}}\cdot\eta^2\right)$$

Since $\sum_{u_{i_n}\in\Psi_1} c_{i_n} = \mu(\Psi_1) > \frac{\beta^n}{4n}$, and $V(\Xi_{i_n,\tau(i_n)}) \geq \kappa_n/240^n$ if $u_{i_n} \in \Psi_1$ according to Claim 7.1, we conclude (79).

**Case 2** There exists $q_i \in \Omega(v_i,\beta)$ for $i = 1,\dots,n$ such that

$$\mu(\Omega(q_i,\eta)) \geq \frac{\beta^n}{4n} \text{ for } i = 1,\dots,n; \tag{87}$$

$$\mu\left(\bigcup_{i=1}^n(\Omega(q_i,2\eta)\cup\Omega(-q_i,2\eta))\right) \leq n-\eta. \tag{88}$$

In this case, we prove

$$V(Z_p^*(\mu)) < \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}(1-n^{-cn^3}\min\{(p-2)^2,1\}\cdot\eta^3) \tag{89}$$

for some absolute constant $c > 0$. The argument is very similar to the one in Case 1.

Any $x \in \widetilde{\Psi}$ for

$$\widetilde{\Psi} = S^{n-1}\backslash\left(\bigcup_{i=1}^n \Omega(q_i,2\eta)\cup\Omega(-q_i,2\eta))\right)$$

can be written in the form

$$x = \sum_{i=1}^n \lambda_i(x)q_i.$$

Here the triangle inequality ensures that there exists some $i \in \{1,\dots,n\}$ satisfying $|\lambda_i(x)| \geq 1/n$. Thus we may reindex $q_1,\dots,q_n$ in a way such that

$$\mu(\Psi) > \frac{\eta}{n} \text{ for } \Psi = \{x\in\widetilde{\Psi} :\ |\lambda_n(x)| \geq 1/n\}. \tag{90}$$

We deduce from (77) that if $x \in \Psi$, then

$$|\det[q_1,\dots,q_{n-1},x]| \geq |\det[q_1,\dots,q_{n-1},q_n]/n \geq 2\beta.$$

Therefore Claim 5.3 implies that if $u_{i_j} \in \Omega(q_j,\eta)$ for $j = 1,\dots,n-1$, and $x \in \Psi$, then

$$|\det[q_1,\dots,q_{n-1},x]| \geq |\det[q_1,\dots,q_{n-1},q_n]/n \geq \beta. \tag{91}$$

We observe that $\Psi = -\Psi$. Thus for

$$\Pi_2 = \{i\in\{1,\dots,k\} : u_i \in \Psi\},$$

there exists $\Pi' \subset \Pi_2$ with $\#\Pi' = \frac{1}{2}\#\Pi_2$, and a bijection $\tilde{\tau} : \Pi' \to \Pi_2 \backslash \Pi'$ such that if $i \in \Pi'$ then

$$u_{\tilde{\tau}(i)} = -u_i.$$

Now let

$$\Pi_1 \subset \{i \in \{1, \ldots, k\} : u_i \in \Omega(q_n, \eta)\}$$

be a maximal subset such that $\#\Pi_1 \leq \#\Pi'$. Therefore (68) and (90) yield that

$$\sum_{i \in \Pi_1} c_i > \frac{\eta}{4n}, \tag{92}$$

and there exists an injective injective $\tau : \Pi_1 \to \Pi_2$ such that if $i \in \Pi_1$, then

$$\langle u_i, u_{\tau(i)} \rangle \geq 0. \tag{93}$$

In addition, if $i \in \Pi_1$, then $u_i \in \Omega(q_n, \eta)$ and $u_{\tau(i)} \not\in \Omega(q_n, 2\eta)$, therefore

$$\|u_i - u_{\tau(i)}\| \geq \eta/2. \tag{94}$$

Given $u_{i_j} \in \Omega(q_j, \eta)$ for $j = 1, \ldots, n-1$ and $i_n \in \Pi_1$, we have have $\tau(i_n) \in \Pi_2$, and (68), (77) and (91) yield

$$\left.\begin{array}{r} c_{i_1} \cdot \ldots \cdot c_{i_{n-1}} \cdot c_{i_n} \det[u_{i_1}, \ldots, u_{i_n}]^2 \\ c_{i_1} \cdot \ldots \cdot c_{i_{n-1}} \cdot c_{\tau(i_n)} \det[u_{i_1}, \ldots, u_{i_{n-1}}, u_{\tau(i_n)}]^2 \end{array}\right\} \geq \beta^2 c_{i_1} \cdot \ldots \cdot c_{i_{n-1}} \cdot (c_{i_n}/2). \tag{95}$$

We deduce from (93) that Claim 7.1 applies to $\Xi_{u_{i_n}, u_{\tau(i_n)}}$. In particular, $V(\Xi_{u_{i_n}, u_{\tau(i_n)}}) \geq \kappa_n/240^n$, and if $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$\begin{aligned} \langle y, u_{i_n} \rangle, \langle y, u_{\tau(i_n)} \rangle &< \frac{1}{8}; \\ \langle y, u_{i_n} \rangle - \langle y, u_{\tau(i_n)} \rangle &= \langle y, u_{i_n} - u_{\tau(i_n)} \rangle \geq \frac{\eta}{240} > \frac{\eta}{2^8}. \end{aligned}$$

It follows from (82) that

$$|\varphi'(\langle y, u_{i_n} \rangle) - \varphi'(\langle y, u_{\tau(i_n)} \rangle)| > \frac{\min\{|p-2|, 1\}}{2^{20}} \cdot \eta.$$

Since $\varphi'(t) \leq 3.1 < 2^2$ for $t \in (0, 0.1]$, if $i_n \in \Pi_1$, then

$$\frac{(\varphi'(\langle y, u_{i_n} \rangle) - \varphi'(\langle y, u_{\tau(i_n)} \rangle))^2}{2(\varphi'(\langle y, u_{i_n} \rangle) + \varphi'(\langle y, u_{\tau(i_n)} \rangle))^2} > \frac{\min\{(p-2)^2, 1\}}{2^{47}} \cdot \eta^2.$$

Thus combining Claim 3.2 and (95) implies that if $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \ldots, n-1$, $i_n \in \Pi_1$ and $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$\aleph(i_1, \ldots, i_{n-1}, i_n; y) + \aleph(i_1, \ldots, i_{n-1}, \tau(i_n); y) \geq \frac{\beta^2 c_{i_1} \cdot \ldots \cdot c_{i_n}}{2} \cdot \frac{\min\{(p-2)^2, 1\}}{2^{47}} \cdot \eta^2.$$

If $i_n \in \Pi_1$ and $y \in \mathbb{R}^n$, then we define

$$\varrho(i_n; y) = \begin{cases} 0 & \text{if } y \notin \Xi_{i_n,\tau(i_n)} \\ \frac{\beta^2 \min\{(p-2)^2,1\}}{2^{48}} \cdot \eta^2 & \text{if } y \in \Xi_{i_n,\tau(i_n)}. \end{cases}$$

In particular, if $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \ldots, n-1$, $i_n \in \Pi_1$ and $y \in \mathbb{R}^n$, then

$$\aleph(i_1, \ldots, i_{n-1}, i_n; y) + \aleph(i_1, \ldots, i_{n-1}, \tau(i_n), y) \geq c_{i_1} \cdot \ldots \cdot c_{i_n} \varrho(i_n; y). \tag{96}$$

Substituting (96) into (72) and later using (76) show that if $y \in \mathbb{R}^n$, then

$$\begin{aligned} \theta^*(y) &\geq 1 + \frac{1}{2} \sum_{\substack{u_{i_j} \in \Omega(v_j,\beta),\ j=1,\ldots,n-1 \\ i_n \in \Pi_1}} c_{i_1} \cdot \ldots \cdot c_{i_{n-1}} \cdot c_{i_n} \varrho(i_n; y) \\ &= 1 + \frac{1}{2} \left( \prod_{j=1}^{n-1} \mu(\Omega(v_j, \beta)) \right) \sum_{i_n \in \Pi_1} c_{i_n} \varrho(i_n; y) \geq 1 + \frac{\beta^{n(n-1)}}{2} \sum_{i_n \in \Pi_1} c_{i_n} \varrho(i_n; y). \end{aligned}$$

Here

$$\frac{\beta^{n(n-1)}}{2} \sum_{i_n \in \Pi_1} c_{i_n} \varrho(i_n; y) \leq \frac{\beta^{n(n-1)}}{2} \mu(\Psi_1) \cdot \frac{\beta^2 n}{2^{48}} \cdot \eta^2 < 1,$$

and hence if $y \in \mathbb{R}^n$, then

$$\theta^*(y)^{-1} \leq 1 - \frac{\beta^{n(n-1)}}{4} \sum_{i_n \in \Pi_1} c_{i_n} \varrho(i_n; y). \tag{97}$$

We deduce from (74) and (97) that

$$\begin{aligned} V(Z_p^*(\mu)) &\leq \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} e^{-\pi \|\Theta(y)\|^2} \det d\Theta(y)\, dy - \\ &\quad - \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \cdot \frac{\beta^{n(n-1)}}{4} \sum_{i_n \in \Pi_1} c_{i_n} \int_{\mathbb{R}^n} \varrho(i_n; y) e^{-\pi \|\Theta(y)\|^2} \det d\Theta(y)\, dy. \end{aligned}$$

Here

$$\int_{\mathbb{R}^n} e^{-\pi \|\Theta(y)\|^2} \det d\Theta(y)\, dy = \int_{\mathbb{R}^n} e^{-\pi \|z\|^2}\, dz = 1,$$

and still have the estimates (85) and (86) if $y \in \Xi_{i_n,\tau(i_n)}$. Therefore

$$V(Z_p^*(\mu)) \leq \frac{2^n \Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \left( 1 - \sum_{i_n \in \Pi_1} c_{i_n} \frac{\beta^{n(n-1)}}{4} \cdot \frac{V(\Xi_{i_n,\tau(i_n)})}{(3.1e)^n} \cdot \frac{\beta^2 \min\{(p-2)^2, 1\}}{2^{48}} \cdot \eta^2 \right)$$

Since $\sum_{i_n \in \Pi_1} c_{i_n} > \frac{\eta}{4n}$ by (92), and $V(\Xi_{i_n,\tau(i_n)}) \geq \kappa_n / 240^n$ if $i_n \in \Pi_1$ according to Claim 7.1, we conclude (89).

**Case 3** There exists $q_i \in \Omega(v_i, \beta)$ for $i = 1, \ldots, n$ such that

$$\mu\left(\bigcup_{i=1}^{n}(\Omega(q_i, 2\eta) \cup \Omega(-q_i, 2\eta))\right) > n - \eta.$$

In this case, we prove that there exists a cross measure $\nu$ such that

$$\delta_W(\nu, \mu) \leq n^{cn}\eta \tag{98}$$

for some absolute constant $c > 0$.

We observe that $\frac{1}{2}(1 - n(\frac{1}{\sqrt{n}} - t)^2) > \eta$ for $t = 2\eta$. Thus Claim 5.1 yields that $\Omega(u, \arccos(\frac{1}{\sqrt{n}} - 2\eta))$ intersects $\cup_{i=1}^{n}\Omega(\pm q_i, 2\eta)$ for any $u \in S^{n-1}$. In turn, we deduce that

$$\Omega\left(u, \arccos\left(\frac{1}{\sqrt{n}} - 6\eta\right)\right) \cap \{\pm q_1, \ldots, \pm q_n\} \neq \emptyset$$

for any $u \in S^{n-1}$, therefore Corollary 6.4 implies that there exists a cross measure $\nu$ such that

$$\delta_H(\text{supp}\nu, \{\pm q_1, \ldots, \pm q_n\}) \leq 4^n n! \cdot 6\eta.$$

In particular, (98) follows from Lemma 6.1.

Finally, Cases 1, 2 and 3 cover all possible even isotropic measure $\mu$ according to Lemma 5.5. We have proved (66) in Cases 1 and 2, and (67) in Case 3. □

**Proof of Theorem 1.4 in the case of $Z_p^*(\mu)$:** If $p \in (1, \infty)\backslash 2$ and $\mu$ is an even discrete isotropic meaure on $S^{n-1}$, and $\delta_{\text{WO}}(\mu, \nu_n) \geq \varepsilon > 0$ for $\varepsilon \in (0, 1)$, then Proposition 7.2 yields that

$$V(Z_p^*(\mu)) \leq (1 - \gamma\varepsilon^3)V(Z_p^*(\nu_n)) \tag{99}$$

where $\gamma = n^{-cn^3}\min\{|p-2|^2, 1\}$ for an absolute constant $c > 0$. Proposition 7.2 yields the estimate for $V(Z_p^*(\mu))$. However, any even isotropic measure can be weakly approximated by even discrete isotropic measures according to F. Barthe [7], and we conclude (99), and in turn Theorem 1.4 in the case of $Z_p^*(\mu)$ for any even isotropic meaure $\mu$ on $S^{n-1}$ and $p \in [1, \infty)\backslash 2$.

Since for any isotropic measure $\mu$, we have

$$\lim_{p\to\infty} Z_p^*(\mu) = Z_\infty^*(\mu),$$

and the factor $\gamma$ in (99) is independent of $p$ for $p \in (2, \infty)$, we deduce the case $p = \infty$, as well. □

# 8 The stability of the Reverse Isoperimetric Inequality in the origin symmetric case

We may assume that the facets of $W^n$ touch $B^d$ in the support of the reference cross measure $\nu_n$.

**Lemma 8.1** *If $\mu$ is an even discrete istropic measure on $S^{n-1}$ such that $\delta_H(\operatorname{supp}\mu, \operatorname{supp}\nu_n) < \alpha$ for $\alpha \in (0, \frac{1}{3n})$, then $e^{-n\alpha}W^n \subset Z^*_\infty(\mu) \subset e^{2n\alpha}W^n$.*

*Proof:* Let $\operatorname{supp}\nu_n = \{\pm e_1, \ldots, \pm e_n\}$. If $v \in S^{n-1}$ satisfies $\angle(e_1, v) \leq \alpha$, and hence $\sum_{i=2}^n \langle e_i, v\rangle^2 \leq \sin^2\alpha$, then we deduce from $\angle(e_1, v) \leq \alpha$ and the Cauchy-Schwarz inequality that

$$\langle e_1, v\rangle \geq \cos\alpha \text{ and } \sum_{i=2}^n |\langle e_i, v\rangle| \leq \sqrt{n-1}\sin\alpha.$$

If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $x_1 = \max\{|x_1|, \ldots, |x_n|\} \geq \lambda$ for $\lambda = (\cos\alpha - \sqrt{n-1}\sin\alpha)^{-1} < e^{2n\alpha}$, then

$$\langle x, v\rangle \geq x_1\langle e_1, v\rangle - x_1\sum_{i=2}^n |\langle e_i, v\rangle| \geq 1.$$

We deduce that $Z^*_\infty((\mu) \subset \lambda W^n$. However, $p = \sum_{i=1}^n \varrho w_i$ for $\varrho = (1 + \sqrt{n-1}\sin\alpha)^{-1} > e^{-n\alpha}$ satisfies

$$\langle v, p\rangle \leq \varrho + \varrho\sqrt{n-1}\sin\alpha \leq 1,$$

and hence $\varrho W^n \subset Z^*_\infty((\mu)$. $\square$

For the proof of Theorem 1.2 (the case of the Banach-Mazur distance), we also need the following statement:

**Lemma 8.2** *If $\tau \in (0, 1/4)$, and te o-symmetric convex bodies $K \subset Z$ satisfy that $(1-\tau)W^n \subset Z$, $(1-2\tau)W^n \not\subset K$ and $V(Z) \leq V(W^n)$, then $V(K) \leq (1-\tau^n)W^n$.*

*Proof:* Let $e_1, \ldots, e_n$ be the orthonormal basis of $\mathbb{R}^n$ such that the facets of $W_n$ touch $S^{n-1}$ at $\{\pm e_1, \ldots, \pm e_n\}$. Possibly reindexing $e_1, \ldots, e_n$, we may assume for some $t > 0$, we have

$$\begin{aligned} t\sum_{i=1}^n e_i &\in \partial K \\ t\sum_{i=1}^n \eta_i e_i &\in K \text{ if } \eta_i \in \{-1, 1\},\ i = 1, \ldots, n, \text{ and some } \eta_i \neq 1. \end{aligned}$$

Since $(1-2\tau)W^n \not\subset K$, we have $t \leq 1 - 2\tau$. It follows that

$$\begin{aligned} (\operatorname{int} K) \cap \left(\tau[0,1]^n + t\sum_{i=1}^n e_i\right) &= \emptyset \\ \tau[0,1]^n + t\sum_{i=1}^n e_i &\subset (1-\tau)W^n \subset Z. \end{aligned}$$

Therefore

$$V(K) \leq V(Z) - \tau^n \leq \left(1 - \frac{\tau^n}{2^n}\right) V(W^n). \quad \Box$$

**Proofs of Theorems 1.1 and 1.2:** Let $K$ be an origin symmetric convex body such that $B^n$ is the maximal volume ellipsoid contained in $K$, and

$$\frac{S(K)^n}{V(K)^{n-1}} \geq (1-\varepsilon)\frac{S(W^n)^n}{V(W^n)^{n-1}} \tag{100}$$

for small $\varepsilon > 0$. If $C$ is a compact convex set with $B^n \subset C$, and $S_C$ is the surface area measure, then

$$V(C) = \int_{S^{n-1}} \frac{h_C(u)}{n}\, dS_C(u) \geq \int_{S^{n-1}} \frac{1}{n}\, dS_C(u) = \frac{S(C)}{n},$$

with equality if $h_C(u) = 1$ for each $u \in \operatorname{supp} S_C$. Therefore $V(W^n) = S(W^n)/n$ and $V(K) \geq S(K)/n$, and hence (100) implies

$$V(K) \geq (1-\varepsilon)V(W^n). \tag{101}$$

Let $\mu$ be an even discrete isotropic measure satisfying $\operatorname{supp}\mu \subset S^{n-1} \cap \partial K$ provided by John's Theorem. In particular,

$$K \subset Z_\infty^*(\mu) \text{ and } V(Z_\infty^*(\mu)) \geq V(K) \geq (1-\varepsilon)V(W^n). \tag{102}$$

We deduce from Corollary 1.5 that, possibly after a suitable rotation, we may assume

$$\delta_H(\operatorname{supp}\mu, \operatorname{supp}\nu_n) \leq n^{c_1 n^3}\varepsilon^{\frac{1}{3}}$$

for an absolute constant $c_1 > 0$. Applying now Lemma 8.1, we have

$$e^{-\omega\varepsilon^{\frac{1}{3}}} W^n \subset Z_\infty^*(\mu) \subset e^{\omega\varepsilon^{\frac{1}{3}}} W^n \tag{103}$$

for $\omega = n^{c_2 n^3}$ and an absolute constant $c_2 > 0$.

To verify the estimate of Theorems 1.1 for $\delta_{\rm vol}$, let us write $\delta_{\rm sym}(C,M) = V(C\Delta M)$ to denote the distance of two convex compact sets according to the symmetric difference metric. For example, (103) yields

$$\delta_{\rm sym}(Z_\infty^*(\mu), W^n) \leq \left(e^{n\omega\varepsilon^{\frac{1}{3}}} - e^{-n\omega\varepsilon^{\frac{1}{3}}}\right) 2^n \leq n^{c_3 n^3}\varepsilon^{\frac{1}{3}} \cdot 2^n$$

for an absolute constant $c_3 > 0$. We note that $V(K) \leq V(Z_\infty^*(\mu)) \leq 2^n$ by K.M. Ball's Theorem B. Let $\lambda \geq 1$ be such that $V(\lambda K) = 2^n$, and hence $V(\lambda K) - V(K) \leq \varepsilon \cdot 2^n$ according to (102). We conlude that

$$\begin{aligned}
\delta_{\rm vol}(K, W^n) &\leq 2^{-n}\delta_{\rm sym}(\lambda K, W^n) \\
&\leq 2^{-n}(\delta_{\rm sym}(\lambda K, K) + \delta_{\rm sym}(K, Z_\infty^*(\mu)) + \delta_{\rm sym}(Z_\infty^*(\mu), W^n)) \\
&\leq n^{c_4 n^3}\varepsilon^{\frac{1}{3}}
\end{aligned}$$

for an absolute constant $c_4 > 0$.

Let us turn to the estimate of Theorem 1.2 for $\delta_{\rm BM}$. Let $\delta_{\rm BM}(K, W^n) \geq \alpha$ for $\alpha \in (0, 1)$. If

$$e^{-\frac{\alpha}{5}} W^n \subset Z_\infty^*(\mu) \subset e^{\frac{\alpha}{5}} W^n, \tag{104}$$

then $\delta_{\rm BM}(K, W^n) \geq \alpha$ yields that $e^{-\frac{4\alpha}{5}} W^n \not\subset K$, and hence $(1 - \frac{2\alpha}{5}) W^n \not\subset K$. On the other hand, $(1 - \frac{\alpha}{5}) W^n \subset Z_\infty^*(\mu)$, thus Lemma 8.2 yields

$$V(K) \leq \left(1 - \frac{\alpha^n}{10^n}\right) V(W^n). \tag{105}$$

Finally we assume that (104) does not hold. Since (101) leads to (103), we have $V(K) < (1 - \varepsilon) V(W^n)$ provided $\frac{\alpha}{5} = \omega \varepsilon^{\frac{1}{3}}$. In other words,

$$V(K) \leq \left(1 - \frac{\alpha^3}{125\omega^3}\right) V(W^n) \tag{106}$$

where $\frac{1}{125\omega^3} \geq n^{-c_5 n^3}$ for an absolute constant $c_5 > 0$. Combining (105) and (106) proves Theorem 1.2. $\square$

# 9 The case of the $L_p$ zonoids in Theorem 1.4

The proof of Theorem 1.4 for $V(Z_p(\mu))$ is analogous to the argument for $V(Z_p^*(\mu))$. In particular, we may assume again that $\mu$ is an even discrete istropic measure, and $p \in (1, \infty) \backslash 2$. Let $p^* \in (1, \infty)$ be defined by $\frac{1}{p} + \frac{1}{p^*} = 1$. We prove that if $\eta \in (0, 1)$, then either

$$V(Z_{p^*}(\mu)) > (1 - n^{-cn^3} \min\{(p-2)^2, 1\} \cdot \eta^3) V(Z_{p^*}(\nu_n)), \tag{107}$$

or there exists a cross measure $\nu$ satisfying

$$\delta_W(\mu, \nu) \leq n^{cn} \eta \tag{108}$$

for some absolute constant $c > 0$. Since if $p \in [\frac{3}{2}, 3]$, then $p^* \in [\frac{3}{2}, 3]$ and $|p-2|/2 \leq |p^* - 2| \leq 2|p-2|$, (107) and (108) yield Theorem 1.4 for $V(Z_p(\mu))$ .

Again, let $\operatorname{supp} \mu = \{\bar{u}_1, \ldots, \bar{u}_{\bar{k}}\}$, and let $\bar{c}_i = \mu(\{\bar{u}_i\})$. For $c_0 = \min\{\bar{c}_i : i = 1, \ldots, \bar{k}\}$ and $i = 1, \ldots, \bar{k}$, we define $\bar{m}_i = \min\{m \in \mathbb{Z} : m \geq 1 \text{ and } c_i/m \leq c_0\}$, and let $k = \sum_{i=1}^{\bar{k}} \bar{m}_i$. We consider $\xi : \{1, \ldots, k\} \to \{1, \ldots, \bar{k}\}$ such that $\#\xi^{-1}(i) = \bar{m}_i$ for $i = 1, \ldots, \bar{k}$, and define

$$u_i = \bar{u}_{\xi(i)} \text{ and } c_i = \bar{c}_{\xi(i)}/\bar{m}_{\xi(i)}$$

for $i = 1, \ldots, k$.

In particular, $\sum_{i=1}^k c_i u_i \otimes u_i = {\rm Id}_n$ and $\sum_{i=1}^k c_i = n$, and for any Borel $X \subset S^{n-1}$, we have

$$\mu(X) = \sum_{u_i \in X} c_i.$$

Again, we have

$$c_0/2 < c_i \leq c_0 \ \text{ for } i = 1, \ldots, k.$$

In addition, let $\psi = \psi_p$ as defined in (27), let $g(t) = e^{-\pi t^2}$, and for $i = 1, \ldots, k$, let

$$f_i(t) = \frac{1}{2\Gamma(1+\frac{1}{p})}\, e^{-|t|^p}.$$

We define the map $\Psi :\, \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Psi(y) = \sum_{i=1}^{k} c_i \psi(\langle y, u_i \rangle) \cdot u_i,$$

and hence the differential of $\Psi$ is

$$d\Psi(y) = \sum_{i=1}^{k} c_i \psi'(\langle y, u_i \rangle) \cdot u_i \otimes u_i$$

where $d\Psi(y)$ is positive definite, and $\Psi$ is injective from $\mathbb{R}^n$ to $\mathbb{R}^n$.

It follows from first applying (21), then (15) that

$$\begin{aligned} V(Z_{p^*}(\mu)) \ &\geq \ V(M_p(\mu)) = \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i\theta_i u_i} \prod_{i=1}^{k} f_i(\theta_i)^{c_i}\, dx \\ &\geq \ \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \left( \prod_{i=1}^{k} f_i(\psi(\langle u_i, y\rangle))^{c_i} \right) \det\left( \sum_{i=1}^{k} c_i \psi'(\langle u_i, y \rangle)\, u_i \otimes u_i \right) dy. \end{aligned}$$

To estimate the second term, we apply Lemma 3.1 with $v_i = \sqrt{c_i} \cdot u_i$ and $t_i = \psi'(\langle y, u_i\rangle)$ at each $y \in \mathbb{R}^n$, and write $\theta^*(y)$ and $t_0(y)$ to denote the corresponding $\theta^* \geq 1$ and $t_0$. In particular, if $\{i_1, \ldots, i_n\} \subset \{1, \ldots, k\}$ and $y \in \mathbb{R}^n$, then we set

$$\aleph(i_1, \ldots, i_n; y) = c_{i_1} \cdot \ldots \cdot c_{i_n} \det[u_{i_1}, \ldots, u_{i_n}]^2 \left( \frac{\sqrt{\psi'(\langle y, u_{i_1}\rangle) \cdot \ldots \cdot \psi'(\langle y, u_{i_n}\rangle)}}{t_0(y)} - 1 \right)^2.$$

Therefore for

$$\theta^*(y) = 1 + \frac{1}{2} \sum_{1 \leq i_1 < \ldots < i_n \leq k} \aleph(i_1, \ldots, i_n; y),$$

Lemma 3.1 and (14) lead to

$$\begin{aligned} V(Z_{p^*}(\mu)) \ &\geq \ \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \theta^*(y) \left( \prod_{i=1}^{k} f_i(\psi(\langle u_i, y\rangle))^{c_i} \right) \left( \prod_{i=1}^{k} \psi'(\langle u_i, y\rangle)^{c_i} \right) dy \\ &= \ \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \theta^*(y) \left( \prod_{i=1}^{k} g(\langle u_i, y\rangle)^{c_i} \right) dy \\ &= \ \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})} \int_{\mathbb{R}^n} \theta^*(y) e^{-\pi\|y\|^2}\, dy. \end{aligned}$$

Now (107) and (108), and hence Theorem 1.4 for $V(Z_p(\mu))$, can be proved as (66) and (67) in Proposition 7.2 were proved following (73).

# 10 Proof of Theorem 1.3

In this section, we prove Theorem 1.3, which is the 2-dimensional (sharper) version of Theorems 1.1 and 1.2. The idea of our proof is essentially the one given by F. Behrend [10]. We begin with recalling some of the details of Behrend's argument. We write $[x_1, \dots, x_k]$ to denote the convex hull of the points $x_1, \dots, x_k \in \mathbb{R}^2$. For the origin symmetric convex body $K \subset \mathbb{R}^2$ and $u \in \mathbb{R}^2 \backslash \{o\}$, we write $H(K,u)$ to denote the supporting line with exterior normal $u$, and $H(K,u)^-$ to denote the corresponding half plane containing $K$.

For $\varepsilon \in [0, \frac{1}{2})$, and for the planar origin symmetric convex body $K$ satisfying

$$\mathrm{ir}(K) \geq (1-\varepsilon)\mathrm{ir}(W^2) = \frac{(1-\varepsilon)S(W^2)^2}{V(W^2)}, \tag{109}$$

we prove that

$$\begin{aligned} \delta_{\mathrm{vol}}(K, W^2) &\leq 54\varepsilon && (110) \\ \delta_{\mathrm{BM}}(K, W^2) &\leq 18\varepsilon. && (111) \end{aligned}$$

Let $u_1, u_2$ form the orthonomal basis of $\mathbb{R}^2$. We may assume that $W^2 = [-1,1]^2$ is a parallelogram of largest area contained in $K$, and hence $p_i \in \partial K \cap H(K, p_i)$ holds for the vertices $p_1 = u_2 + u_1$ and $p_2 = u_2 - u_1$ of $W^2$. It also follows that

$$K \subset \bigcap_{i=1}^{2} H(K, \pm p_i)^- = [\pm 2u_1, \pm 2u_2]. \tag{112}$$

Let $q_i \in \partial K \cap H(K, u_i)$ for $i = 1, 2$. In particular, (112) yields

$$\begin{aligned} q_1 &= (1+t_1, s_1) \text{ where } t_1 \in [0,1] \text{ and } |s_1| \leq 1 - t_1, \\ q_2 &= (s_2, 1+t_2) \text{ where } t_2 \in [0,1] \text{ and } |s_2| \leq 1 - t_2. \end{aligned}$$

Since $K$ contains the parallelogram $P = [\pm q_1, \pm q_2]$, we have

$$\begin{aligned} V(W^2) &\geq V(P) = 2|\det[q_1, q_2]| = 2[(1+t_1)(1+t_2) - s_1 s_2] \\ &\geq 2[(1+t_1)(1+t_2) - (1-t_1)(1-t_2)] = 4(t_1 + t_2), \end{aligned}$$

and hence

$$t = \frac{t_1 + t_2}{2} \leq \frac{1}{2}.$$

We approximate $K$ by suitable polygons to obtain

$$W^2 \subset Q \subset K \subset M \subset (1+t)W^2, \tag{113}$$

where

$$\begin{aligned} M &= \left(\bigcap_{i=1}^{2} H(K,\pm u_i)^-\right)\bigcap\left(\bigcap_{i=1}^{2} H(K,\pm p_i)^-\right) \text{ with } S(M)=(1+(\sqrt{2}-1)t)S(W^2) \\ Q &= [\pm p_1,\pm p_2,\pm q_1,\pm q_2] \text{ with } V(Q)=(1+t)V(W^2). \end{aligned}$$

We deduce from (109) and (113) that

$$(1-\varepsilon)\frac{S(W^2)^2}{V(W^2)} \leq \frac{S(K)^2}{V(K)} \leq \frac{S(M)^2}{V(Q)} = \frac{(1+(\sqrt{2}-1)t)^2 S(W^2)^2}{(1+t)V(W^2)}.$$

Since $\frac{1-t}{1+t} \geq \frac{1}{3}$ by $t \leq \frac{1}{2}$, we have

$$\varepsilon \geq 1-\frac{(1+(\sqrt{2}-1)t)^2}{1+t} = \frac{(3-2\sqrt{2})t(1-t)}{1+t} \geq \frac{(3-2\sqrt{2})t}{3} \geq \frac{t}{18}. \tag{114}$$

Therefore combining (113) and (114) leads to

$$\delta_{\mathrm{BM}}(K,W^2) \leq \ln(1+t) \leq t \leq 18\varepsilon,$$

and combining (113) and (114) with an elementary argument leads to

$$\delta_{\mathrm{vol}}(K,W^2) \leq (1+t)^2-1 \leq 3t \leq 54\varepsilon.$$

We conclude (110) and (111), and in turn Theorem 1.3.

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