RANDOM PROJECTIONS OF REGULAR POLYTOPES

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Abstract. Based on an approach of Affentranger&Schneider we give an asymptotic formula for the expected number of k-faces of the orthogonal projection of a regular n-crosspolytope onto a randomly chosen isotropic subspace of fixed dimension, as n tends to infinity. In particular, we present a precise asymptotic formula for the (spherical) volume of spherical regular simplices, which generalizes Daniel's formula.

1. Introduction

Throughout the paper the n-dimensional Euclidean space equipped with inner product $\langle \cdot, \cdot \rangle$ is denoted by $\mathbb{R}^n$. For a polytope $P \subset \mathbb{R}^n$ the set of all k-faces is denoted by $\mathcal{F}_k(P)$ and its cardinality by $f_k(P)$, i.e., $f_k(P) = \#\mathcal{F}_k(P)$. The expected number of k-faces of an orthogonal projection of an n-dimensional polytope P onto a randomly chosen d-dimensional linear subspace with isotropic distribution is denoted by $E(f_k(\Pi_d P))$, $1 \leq d \leq n-1$, $0 \leq k \leq d-1$. It was proved by Affentranger&Schneider [AS92] that

$$E(f_k(\Pi_d P)) = 2 \sum_{s \geq 0} \sum_{F \in \mathcal{F}_k(P)} \sum_{G \in \mathcal{F}_{d-1-2s}(P)} \beta(F,G)\gamma(G,P), \tag{1.1}$$

or, equivalently,

$$E(f_k(\Pi_d P)) = f_k(P) - 2 \sum_{s \geq 0} \sum_{F \in \mathcal{F}_k(P)} \sum_{G \in \mathcal{F}_{d+1+2s}(P)} \beta(F,G)\gamma(G,P), \tag{1.2}$$

where $\beta(F,G)$ and $\gamma(G,F)$ denote the internal and external angle of G at its face F, respectively (cf. [Grü67]). By definition the internal or external angles are spherical volumes and therefore, in general, it is impossible to give an explicit formula for them. However, it was shown by Ruben [Rub60] (see also [Had79]) that for a regular n-simplex $T^n \subset E^n$

$$\gamma(T^k, T^n) = \sqrt{\frac{k+1}{\pi}} \int_{-\infty}^{\infty} e^{-(k+1)x^2} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-y^2} dy \right)^{n-k} dx.$$

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Using this representation Affentranger&Schneider established the asymptotic formula (cf. [AS92])

$$E(f_k(\Pi_d T^n)) \sim \frac{2^d}{\sqrt{d}}\binom{d}{k+1}\beta(T^k,T^{d-1})(\pi\ln n)^{(d-1)/2}, \quad n\to\infty. \tag{1.3}$$

This formula still involves the "unknown" internal angles $\beta(T^k,T^{d-1})$. For $k=0$ an asymptotic formula of $\beta(T^0,T^{d-1})$, $d\to\infty$, was given by Daniel in the context of densest sphere packings (cf. [Rog64]). In Section 2 we generalize the approach of Daniel and prove (see Corollary 2.1)

$$\beta(T^k,T^{d-1}) = \frac{(k+1)^{\frac{d-k-2}{2}}e^{\frac{d-3k-3}{2}}}{\sqrt{2}^{d-k}\sqrt{\pi}^{d-k-1}d^{\frac{d-k-2}{2}}}\cdot\left(1+O\left(\frac{k^2+1}{d}\right)\right).$$

Beside the regular simplex there are two more regular polytopes in arbitrary dimensions, namely the n-cube W^n and the regular n-crosspolytope C^n. It is easy to see that for $F\in\mathcal{F}_k(W^n)$, $G\in\mathcal{F}_l(W^n)$ with $F\subset G$ we have $\gamma(F,W^n)=(1/2)^{n-k}$ and $\beta(F,G)=\beta(F,W^l)=(1/2)^{l-k}$. Furthermore, $f_k(W^n)=2^{n-k}\binom{n}{k}$ and the number of l-faces containing a fixed k-face is equal to $\binom{n-k}{l-k}$. Thus we get by (1.1)

$$E(f_k(\Pi_d W^n)) = 2\binom{n}{k}\sum_{s\geq 0}\binom{n-k}{d-1-2s-k} \sim 2\frac{n^{d-1}}{(d-1-k)!k!}.$$

In Section 3 we complete the determination of $E(f_k(\Pi_d P))$ for regular polytopes by proving for the regular n-crosspolytope C^n

Theorem 1.1. *For any given integers $0\leq k<d\leq n-1$,*

$$E(f_k(\Pi_d C^n)) \sim \frac{2^d}{\sqrt{d}}\binom{d}{k+1}\beta(T^k,T^{d-1})(\pi\ln n)^{(d-1)/2}.$$

as n tends to infinity.

Observe that the asymptotic value of the expected number of k–faces is the same for the cross polytope C^n as for T^n in (1.3). At the moment, we are not aware of any direct argument leading to this coincidence. We note the following consequence of the proof of the estimates:

Corollary 1.1. *For fixed $k\in\mathbb{N}$, if $d/k^2\to\infty$ and $n/d\to\infty$ then*

$$E(f_k(\Pi_d T^n)) \sim E(f_k(\Pi_d C^n)) \sim \frac{\sqrt{2}^{d+k}\sqrt{\pi}^k(k+1)^{\frac{d-k-2}{2}}e^{\frac{d-3k-3}{2}}}{(k+1)!d^{\frac{d-3k-3}{2}}}\cdot(\ln n)^{\frac{d-1}{2}}.$$

We remark that by a result of Baryshnikov&Vitale, $E(f_k(\Pi_d T^n))$ coincides with the expected number of k-faces of the convex hull of a standard Gaussian sample in d-space (cf. [BV94]). They prove actually more: Let $v^1,\ldots,v^m$ be vectors in $\mathbb{R}^n$ with the same positive length such that each $\langle v^i,v^j\rangle$, $i\neq j$, is equal to the same non-positive value. Then the orthogonal projection of a random rotation of $v^1,\ldots,v^m$ onto $\mathbb{R}^d$, up to an independent affine transformation coincides in distribution with a standard Gaussian sample of m points in $\mathbb{R}^d$. If $\langle v^i, v^j \rangle = 0$ then the affine transformation can be chosen to be linear. We deduce choosing $m = n$ that

Remark. *$E(f_k(\Pi_d C^n))$ coincides with the expected number of k-faces of the convex hull of a standard Gaussian sample of n pairs $x^1, -x^1, \ldots, x^n, -x^n$ of points in $\mathbb{R}^d$.*

Finally, in Section 4 we give a list of some numerical computations of $E(f_k(\Pi_d C^n))$.

2. Spherical volumes of regular simplices

Let $B^n, S^{n-1} \subset \mathbb{R}^n$ denote the n-dimensional unit ball and n-dimensional unit sphere, respectively. For $\alpha \in (0,1)$ a regular n-cone of angle α is defined as the positive hull of n unit vectors $a^1, \ldots, a^n \in S^{n-1}$ satisfying

$$\langle a^i, a^j \rangle = \alpha, \quad i \neq j.$$

It is denoted by $\sigma(\alpha, n)$ and $T_*^{n-1}(\alpha) = \sigma(\alpha, n) \cap S^{n-1}$ is a regular spherical $(n-1)$-simplex of angle α.

By definition, the internal angle $\beta(F, G)$ is the "fraction" of the linear hull of $G - x^F$ taken up by the cone (positive hull) $\operatorname{pos}\{G - x^F\}$, where x^F is a relative interior point of the face F. Now, it is easy to check that for $F = T^k$ and $G = T^{d-1}$, $k < d$, the cone $\operatorname{pos}\{G - x^F\}$ can be written as a direct sum of $\operatorname{lin}\{F - x^F\}$ and a regular $(d-k)$-dimensional cone $\sigma(1/(k+2), d-k)$ of angle $1/(k+2)$. Thus

$$\beta(T^k, T^{d-1}) = \frac{V_*^{d-k-1}\left(T_*^{d-k-1}\left(\frac{1}{k+2}\right)\right)}{V_*^{d-k-1}(S^{d-k-1})}, \tag{2.1}$$

where $V_*^{d-k-1}(\cdot)$ denotes the spherical volume w.r.t. S^{d-k-1}, and the internal angle $\beta(T^k, T^{d-1})$ may be regarded as the normalized spherical volume of a regular $(d-k-1)$-simplex of angle $1/(k+2)$. In the following we will study the asymptotic behavior of the normalized volume of an arbitrary regular spherical simplex. To this end, for $n \in \mathbb{N}$ and $\alpha \in (0,1)$ let

$$\tau(\alpha, n) = \frac{V_*^{n-1}\left(T_*^{n-1}(\alpha)\right)}{V_*^{n-1}(S^{n-1})}.$$

The asymptotic behavior ($n \to \infty$) of $\tau(\alpha, n)$ for the special case $\alpha = 1/2$ has been investigated by Daniel (see [Rog64]). He proved

$$\tau(1/2, n) \sim \frac{e^{n/2-1}}{\sqrt{2}^{n+1}\sqrt{n}^{n-1}\sqrt{\pi}^{n}}.$$

Here we show the generalization

Lemma 2.1. *Let $0 < \alpha < 1$. If n tends to infinity then*

$$\tau(\alpha, n) = \sqrt{\frac{1-\alpha}{\alpha}}^{n-1} \frac{e^{\frac{n+2}{2} - \frac{1}{\alpha}}}{\sqrt{2}^{n+1}\sqrt{\pi}^{n}\sqrt{n}^{n-1}} \left(1 + O\left(\frac{1}{\alpha^2 n}\right)\right).$$

Proof. It is well known that $V_*^{n-1}(S^{n-1}) = n\pi^{n/2}/\Gamma(n/2+1)$, where $\Gamma(\cdot)$ denotes the Γ-function. Hence (cf. [Had79])

(2.2)
$$\int_{\sigma(\alpha,n)} e^{-\langle x,x\rangle}dx = \tau(\alpha,n)V_*^{n-1}(S^{n-1})\int_0^\infty e^{-r^2}r^{n-1}dr = \tau(\alpha,n)\pi^{n/2}.$$

Now let $a^1,\dots,a^n \in S^{n-1}$ be such that $\sigma(\alpha,n) = \mathrm{pos}\{a^1,\dots,a^n\}$ and let A be the $(n\times n)$-matrix with columns a^i. Calculating the volume of the simplex $\mathrm{conv}\{0,a^1,\dots,a^n\}$ yields that $\det A = \sqrt{1-\alpha+\alpha n}\sqrt{1-\alpha}^{n-1}$. Applying the linear transformation $x = Ay$ to the integral on the left hand side of (2.2) gives

$$\tau(\alpha,n) = \frac{\sqrt{(1-\alpha+\alpha n)(1-\alpha)^{n-1}}}{\sqrt{\pi}^n}\int_0^\infty\dots\int_0^\infty e^{-\langle Ay,Ay\rangle}\,dy_1\dots dy_n.$$

As $\langle Ay,Ay\rangle = \sum_{i=1}^n y_i^2 - 2\alpha\sum_{1\le i<j\le n} y_iy_j$, the substitution $z = \sqrt{\alpha}y$ leads to

(2.3)
$$\begin{aligned}\tau(\alpha,n) = {}& \frac{\sqrt{(1-\alpha+\alpha n)(1-\alpha)^{n-1}}}{\sqrt{\alpha\pi}^n}\times \\ & \int_0^\infty\dots\int_0^\infty e^{-\theta\sum_{i=1}^n z_i^2-\left(\sum_{i=1}^n z_i\right)^2}\,dz_1\dots dz_n\end{aligned}$$

with $\theta = (1-\alpha)/\alpha$. Let $\Phi(n)$ denote the integral on the right hand side. In the following we give an asymptotic formula for $\Phi(n)$ as n tends to infinity. To this end we fix an $s\in\mathbb{R}$. Since $\int_{-\infty}^\infty e^{-t^2}\,dt = \sqrt{\pi}$, integrating along the line $t-si$, $t\in(-\infty,\infty)$, shows

$$\sqrt{\pi}e^{-s^2} = \int_{-\infty}^\infty e^{-t^2+2its}\,dt.$$

We deduce

(2.4)
$$\begin{aligned}\Phi(n) &= \frac{1}{\sqrt{\pi}}\int_0^\infty\dots\int_0^\infty\int_{-\infty}^\infty e^{-\theta\sum_{i=1}^n z_i^2-t^2+2it\sum_{i=1}^n z_i}\,dtdz_1\dots dz_n \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-t^2}\left(\int_0^\infty e^{-\theta s^2+2its}\,ds\right)^n dt.\end{aligned}$$

Next we observe that the function $g(t) = e^{-t^2}(\int_0^\infty e^{-\theta s^2+2its}\,ds)^n$ regarded as complex function in the variable $t = \nu + i\chi$ is an entire function. Since

$$|g(\nu+i\chi)| \le e^{-\nu^2+\chi^2}\left(\int_0^\infty e^{-\theta s^2}ds\right)^n = \left(\frac{1}{2}\sqrt{\frac{\pi}{\theta}}\right)^n e^{-\nu^2+\chi^2}$$

for $\chi\ge 0$, the function $g(\nu+i\chi)$ tends to zero as ν tends to infinity for any fixed $\chi\ge 0$. Thus we may replace the integration $\int_{-\infty}^\infty g(t)dt$ along the real line with respect to t by integration along the line $\nu + \chi i$, $\nu \in (-\infty, \infty)$, $\chi = \sqrt{n/2}$, and get

$$\begin{aligned}\Phi(n) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\nu^2+\chi^2-2\chi\nu i} \left(\int_0^{\infty} e^{-\theta s^2 - 2\chi s + 2\nu s i}\, ds \right)^n \\ &= \frac{e^{n/2}}{\sqrt{\pi}(2n)^{n/2}} \int_{-\infty}^{\infty} e^{-\nu^2} \left(2\chi e^{-i\nu/\chi} \int_0^{\infty} e^{-\theta s^2 - 2\chi s + 2\nu s i}\, ds \right)^n d\nu.\end{aligned} \tag{2.5}$$

Let $\Psi(n, \nu)$ denote the expression taken to the n-th power (in the parentheses) on the right hand side. Integration by parts yields that

$$\begin{aligned}|\Psi(n,\nu)| &\leq 2\chi \int_0^{\infty} e^{-\theta s^2 - 2\chi s}\, ds \\ &= 2\chi \left[\frac{1}{2\chi} - \int_0^{\infty} \left(2\theta s\, e^{-\theta s^2} \right) \left(\frac{1}{2\chi} e^{-2\chi s} \right) ds \right] < 1.\end{aligned}$$

Since $e^{-\nu^2} < e^{-\nu}$ for $\nu > 1$, we have

$$\int_{\ln n}^{\infty} e^{-\nu^2} |\Psi(n,\nu)|^n\, d\nu < \frac{1}{n}. \tag{2.6}$$

Therefore let us assume that $|\nu| < \ln n$. By the corresponding formula in [Rog64] we deduce that, if the real part of $w \in \mathbb{C}$ is positive then

$$\int_0^{\infty} e^{-\theta s^2 - ws}\, ds = \frac{1}{w} - \frac{2\theta}{w^3} + O\left(\frac{\theta^2}{|w|^5} \right).$$

Applying this with $w = 2\chi - 2\nu i$ to $\Psi(n, t)$ yields that

$$\begin{aligned}\Psi(n,\nu) &= 2\chi \left[1 - \frac{i\nu}{\chi} - \frac{\nu^2}{n} + \frac{\nu^3 i}{6\chi^3} + O\left(\frac{\nu^4}{n^2} \right) \right] \times \\ &\quad \left[\frac{1}{2\chi - 2\nu i} - \frac{2\theta}{(2\chi - 2\nu i)^3} + O\left(\frac{\theta^2}{n^{5/2}} \right) \right] \\ &= 1 - \frac{\theta}{n} - \frac{\nu^2}{n} - \left(2\theta + \frac{2}{3}\nu^2 \right) \frac{\sqrt{2}\nu i}{n\sqrt{n}} + O\left(\frac{(1+\theta^2)(1+\nu^4)}{n^2} \right).\end{aligned}$$

Observe that $\Phi(n)$ is a real number, so no term of order $1/(n\sqrt{n})$ shows up in its expansion. We conclude by (2.5) and (2.6) that

$$\begin{aligned}\Phi(n) &= \frac{e^{n/2}}{\sqrt{\pi}(2n)^{n/2}} \times \\ &\quad \left(\int_{-\ln n}^{\ln n} e^{-2\nu^2 - \theta} \left(1 + O\left(\frac{(1+\theta^2)(1+\nu^4)}{n} \right) \right) d\nu + O\left(\frac{1}{n} \right) \right) \\ &= \frac{e^{(n/2)-\theta}}{\sqrt{2}(2n)^{n/2}} \left(1 + O\left(\frac{1+\theta^2}{n} \right) \right).\end{aligned}$$

Finally, the assertion follows by (2.3). □

By (2.1) and Lemma 2.1 we conclude

Corollary 2.1.

$$\beta(T^k, T^{d-1}) = \frac{(k+1)^{\frac{d-k-2}{2}} e^{\frac{d-3k-3}{2}}}{\sqrt{2}^{d-k}\sqrt{\pi}^{d-k-1} d^{\frac{d-k-2}{2}}} \cdot \left(1 + O\left(\frac{k^2+1}{d}\right)\right).$$

3. Random projections of regular crosspolytopes

In order to prove Theorem 1.1, we need the following statement about the asymptotic behavior of the external angles $\gamma(F^k, C^n)$ at a k-face of a regular n-crosspolytope C^n.

Lemma 3.1. *If n tends to infinity then*

$$\gamma(F^k, C^n) \sim \frac{1}{2} \frac{(k+1)!}{\sqrt{k+1}} \frac{(\pi \ln(n))^{k/2}}{n^{k+1}}.$$

Proof. The proof follows a proof of Vershik&Sporyshev [VS86] (see also [Ray70]) and is based on the following formula for $\gamma(F^k, C^n)$ (cf. [BH92])

$$\gamma(F^k, C^n) = \sqrt{\frac{k+1}{\pi}} \int_0^\infty e^{-(k+1)x^2} \left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy\right)^{n-k-1} dx. \tag{3.1}$$

Let $\Phi(x) = (2/\sqrt{\pi}) \int_0^x e^{-y^2} dy$. By the substitution $x = \Phi^{-1}(1-u)$ we get

$$\gamma(F^k, C^n) = \frac{\sqrt{k+1}}{2} \int_0^1 e^{-k(\Phi^{-1}(1-u))^2} (1-u)^{n-k-1} du.$$

Let

$$\xi_1(k,n) = \frac{\sqrt{k+1}}{2} \int_0^{1/2} e^{-k(\Phi^{-1}(1-u))^2} (1-u)^{n-k-1} du,$$
$$\xi_2(k,n) = \frac{\sqrt{k+1}}{2} \int_{1/2}^1 e^{-k(\Phi^{-1}(1-u))^2} (1-u)^{n-k-1} du.$$

Obviously, we have

$$0 \le \xi_2(k,n) \le \frac{\sqrt{k+1}}{2} \left(\frac{1}{2}\right)^{n-k} \tag{3.2}$$

and in the following we investigate the function $\xi_1(k,n)$. It is well known that $\Phi(x) = 1 - [e^{-x^2}/(x\sqrt{\pi})](1 + O(x^{-2}))$ as $x \to \infty$ (cf. [VS86]) and thus

$$u = \frac{e^{-x^2}}{x\sqrt{\pi}} (1 + O(x^{-2})), \quad u \to 0.$$

Taking the logarithm yields

$$x^2 = -\ln(u) - \ln(x) - \ln(\sqrt{\pi}) + \ln(1 + O(x^{-2})), \tag{3.3}$$

and so $x = \sqrt{-\ln(u)(1+O(x^{-1}))}$. Replacing this expression in (3.3) gives

$$x^2 = -\ln(u) - \frac{1}{2}\ln(-\ln(u)) - \frac{1}{2}\ln(1+O(x^{-1})) - \ln(\sqrt{\pi}) + \ln(1+O(x^{-2})).$$

Hence we may write

$$\xi_1(k,n) = \frac{\sqrt{k+1}}{2}\sqrt{\pi}^k \int_0^{1/2} u^k |\ln(u)|^{k/2}(1-u)^{n-k-1}(1+O(x^{-1}))du.$$

Observe, that $x \geq \Phi^{-1}(1/2)$. Applying the substitution $1-u = e^{-v}$ yields

$$\begin{aligned} \xi_1(k,n) &= \frac{\sqrt{k+1}}{2}\sqrt{\pi}^k \times \\ & \int_0^{\ln(2)} (1-e^{-v})^k |\ln(1-e^{-v})|^{k/2} e^{-(n-k)v}(1+O(x^{-1}))dv. \end{aligned}$$

Noting that $(1-e^{-v})^k = v^k(1+O(v))$ and $|\ln(1-e^{-v})|^{k/2} = |\ln(v)|^{k/2}(1+O(v))$ as $v \to 0$ we get

$$\xi_1(k,n) = \frac{\sqrt{k+1}}{2}\sqrt{\pi}^k \int_0^{\ln(2)} v^k |\ln(v)|^{k/2} e^{-(n-k)v}(1+O(x^{-1}))(1+O(v))dv.$$

The asymptotic behavior of such an integral was explicitly determined by Watson (cf. [VS86]) and applying that result gives

$$\begin{aligned} \xi_1(k,n) &\sim \frac{\sqrt{k+1}}{2}\sqrt{\pi}^k (n-k)^{-(k+1)}(\ln(n-k))^{k/2} k!(1+O(\ln(n)^{-1})) \\ &\sim \frac{1}{2}\frac{(k+1)!}{\sqrt{k+1}}\frac{(\pi\ln(n))^{k/2}}{n^{k+1}}. \end{aligned}$$

Together with (3.2) this shows $\gamma(F^k, C^n) \sim \xi_1(k,n)$ as n tends to infinity. □

Now we are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $F \in \mathcal{F}_k(C^n)$ and $G \in \mathcal{F}_l(C^n)$ with $F \subset G$. Since every l-face, $l < n$, of a regular n-crosspolytope is a regular simplex we get $\beta(F,G) = \beta(T^k, T^l)$. Furthermore, we have $f_k(C^n) = 2^{k+1}\binom{n}{k+1}$ (cf. [HRZ97]) and the number of l-faces containing a fixed k-face is equal to $2^{l-k}\binom{n-k-1}{l-k}$. By (1.1) we get

$$\begin{aligned} E(f_k(\Pi_d C^n)) = 2\sum_{s\geq 0} 2^{d-2s}\binom{n}{d-2s} \times \\ \binom{d-2s}{k+1}\beta(T^k, T^{d-1-2s})\gamma(T^{d-1-2s}, C^n) \end{aligned} \tag{3.4}$$

and by Lemma 3.1 we obtain for $n \to \infty$

$$2\binom{n}{d-2s}\gamma(T^{d-1-2s}, C^n) \sim \frac{(\pi\ln(n))^{(d-2s-1)/2}}{\sqrt{d-2s}}.$$

The number of nonzero summands in the sum does not depend on n and since the sum is dominated by the term $s = 0$, we obtain Theorem 1.1. $\square$

4. Remarks

As an easy application of (1.2) we determine the probability that the orthogonal projection of C^n onto a randomly chosen $(n-1)$-dimensional plane has $2n$ vertices. Let this probability be denoted by P_{2n} and let v be a vertex of C^n. By a result of McMullen [McM75] we have the angle-sum relation

$$\sum_{F \text{ is a face of } C^n} \beta(v,F)\gamma(F,C^n) = 1.$$

Since $\gamma(v,C^n) = 1/(2n)$ this is equivalent to

$$\beta(v,C^n) = \frac{2n-1}{2n} - \sum_{j=1}^{n-1} 2^j \binom{n-1}{j} \beta(v,T^j)\gamma(T^j,C^n) \tag{4.1}$$

and by (1.2) we obtain $E(f_0(\Pi_{n-1}C^n)) = 2n(1-2\beta(v,C^n))$. Thus

$$\begin{aligned} (1-P_{2n})(2n-2) + 2nP_{2n} &= E(f_0(\Pi_{n-1}C^n)) = 2n(1-2\beta(v,C^n)) \\ \iff \qquad P_{2n} &= 1 - 2n\beta(v,C^n). \end{aligned} \tag{4.2}$$

In particular for $n = 3$ we have $\beta(v,T^1) = 1/2$, $\beta(v,T^2) = \arccos(1/2)/(2\pi)$ and $\gamma(T^1,C^3) = \arccos(1/3)/(2\pi)$, $\gamma(T^2,C^3) = 1/2$. Hence by (4.1) we get $\beta(v,C^3) = 1/2 - \arccos(1/3)/\pi$ and therefore (cf. (4.2))

$$P_6 = 1 - 6\beta(v,C^3) \sim 0.35095.$$

The next tables contain some numerical values of $E(f_k(\Pi_d C^n))$ using (3.4), (3.1), (2.4) and (2.3). The calculations were carried out by the program *Maple V Release 4*[1].

n	$\mathbf{d=2}$; $k=0$	$\mathbf{d=3}$; $k=0$	$k=1$	$k=2$
10	6.66	12.15	30.46	20.31
20	7.68	16.21	42.62	28.41
30	8.23	18.68	50.05	33.37
40	8.61	20.47	55.42	39.95
50	8.89	21.88	59.64	39.76
60	9.12	23.04	63.12	42.08
70	9.31	24.02	66.08	44.05
80	9.47	24.88	68.65	45.76
90	9.61	25.64	70.93	47.28
100	9.73	26.326	72.97	48.65

Table 1.

[1] ©1981-1996 by Waterloo Maple Inc.

n	3	4	5	6	7	8	9	10
$\mathbf{d = n - 1}$ $k = n - 2$	4.70	10.67	23.61	51.40	110.54	233.57	498.46	1048.74

Table 2.

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