# Oriented matroid rigidity of multiplices

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## 1 Introduction

Multiplices were introduced in [2] as generalized simplices. A multiplex $M(d, n+1)$ is a $d$–polytope in $\mathbb{R}^d$, $d \geq 3$, with $n+1$ vertices $e_0, \ldots, e_n$ ordered $e_0 < e_1 < \cdots < e_n$, $n \geq d$. Set $e_i = e_0$ if $i < 0$, and $e_i = e_n$ if $i > n$. Then the facets of $M(d, n+1)$ are

$$F_i = [e_{i-d+1}, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{i+d-1}], \quad i = 0, \ldots, n,$$

where $[\cdot]$ denotes the convex hull operation. Observe that $F_0 = [e_0, \ldots, e_{d-1}]$, $F_1 = [e_0, e_2, \ldots, e_d]$, $F_{n-1} = [e_{n-d}, \ldots, e_{n-2}, e_n]$, and $F_n = [e_{n-d+1}, \ldots, e_n]$ are $(d-1)$–simplices.

We note that $M(d, d+1)$ is a $d$-simplex. As such, each of its facets and vertex figures is a $(d-1)$-simplex, and it is self-dual (combinatorially equivalent to its polar).

Let $n > d$. Then $M(d, n+1)$ is realizable (see Section 3 for the construction of $M(d, n+1)$ from $M(d, n)$), each of its facets and vertex figures is a

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$(d-1)$-multiplex, and it is self-dual. In fact, any $k$-face of $M(d,n+1)$ is a $k$-multiplex with at most $2k$ vertices under the induced ordering, $k \geq 1$, and any quotient polytope $G'/G$ is a multiplex. (We recall that $G \subset G'$ are faces of $M(d,n+1)$ and that $G'/G$ denotes any polytope whose face lattice is isomorphic to the lattice of the faces $F$ of $M(d,n+1)$ with $G \subseteq F \subseteq F'$.) Finally, we remark that reversing the ordering on the vertices of $M(d,n+1)$ yields again $M(d,n+1)$, and that $M(d,n+1)$ is the free join of lower dimensional polytopes if and only if $n \leq 2d-2$ (see [2]).

In view of the high degree of symmetry in the combinatorial structure of $M(d,n+1)$, it is natural to ask if it is reflected in its underlying affine structure (oriented matroid).

An *oriented matroid* on a finite non–empty set $V$ is a certain family $\mathcal{C}$ of signed sets called *circuits*; namely, each $C \in \mathcal{C}$ is an ordered pair $C = (C^+, C^-)$ where $C^+$ and $C^-$ are disjoint subsets of $V$. Here $C^+$ $(C^-)$ is the family of elements of the circuit with positive (negative) sign. We set $\underline{C} = C^+ \cup C^-$ and call it the *support* of $C$. The opposite circuit $-C$ is defined as $(-C)^+ = C^-$ and $(-C)^- = C^+$. Now the following properties characterize oriented matroids:

**(i)** $\underline{C} \neq \emptyset$ for each $C \in \mathcal{C}$.

**(ii)** If $C \in \mathcal{C}$ then $-C \in \mathcal{C}$.

**(iii)** If $\underline{C}_1 \subset \underline{C}_2$, $C_1, C_2 \in \mathcal{C}$, then either $C_1 = C_2$ or $C_1 = -C_2$.

**(iv) (weak elimination)** If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq -C_2$, and $e \in C_1^+ \cap C_2^-$ then there exists a $C_3 \in \mathcal{C}$ such that

$$C_3^+ \subset (C_1^+ \cup C_2^+) \backslash \{e\} \quad \text{and} \quad C_3^- \subset (C_1^- \cup C_2^-) \backslash \{e\}.$$

A basic example is the underlying oriented matroid of a polytope. Then the supports of the circuits are the minimal affinely dependent subsets of vertices, and the positive (negative) elements of the circuits are the ones with positive (negative) coefficient in the corresponding affine dependency (see [3]). In this paper, we are interested in determining whether any two realizations of $M(d,n+1)$ determine the same underlying oriented matroid, or in other words, is $M(d,n+1)$ rigid in the oriented matroid sense. We call this notaion o.m. rigidity so as to distinguish it from the better known notion of metric rigidity.

Rigidity of polytopes in this oriented matroid sense has been intensively investigated during the last two decades. The trivial examples are simplices, or more generally, pyramids over any rigid polytope. Much more interesting rigid polytopes are the cyclic polytopes in even dimension (see [5]), which maximize the number of $k$–faces for a fixed number of vertices. More complicated examples are Lawrence polytopes (see [1] or [6]), and polytopes which are glued together from rigid polytopes, as long as one glues in "necessarily flat facets" (see [4]).

Our main result is

**Theorem 1** *A multiplex $M(d, n+1)$ is o.m. rigid if and only if $n \leq 2d$.*

The circuits are described in Theorem 2 when $n \leq 2d-1$, and in Theorem 3 when $n = 2d$. Note that if $n \leq 2d-1$ then each circuit is *sequential*; namely, the initial and terminal element (according to the ordering) have the same sign, and the elements between have the opposite sign. Finally, Theorem 4 shows that $M(d, n+1)$ is not o.m. rigid if $n > 2d$.

In view of the fact that $M(d, 2d)$ and $M(d, 2d+1)$ are not the free join of lower dimensional polytopes, Theorem 1 yields non-trivial examples of o.m. rigid polytopes.

## 2 The o.m. rigid multiplices

Let $e_0 < e_1 < \ldots < e_n$ be the vertices of a multiplex $M(d, n+1)$ in $\mathbb{R}^d$. We recall some basic facts from [2] and [3]. When listing the elements of a circuit, we denote the negative elements by a bar. A crucial observation is that (see Lemma 3, [2])

$$\{e_i \overline{e_{i+1} e_{i+d}} e_{i+d+1} \mid i = 0, \ldots, n-d-1\} \subset \mathcal{C}. \tag{1}$$

where $\mathcal{C}$ is the set of circuits of the underlying oriented matroid of $M(d, n+1)$. In addition, both

$$[e_1, \ldots, e_n] \ \text{ and } \ [e_0, \ldots, e_{n-1}] \quad \text{are multiplices,} \tag{2}$$

and reversing the order of $e_0, \ldots, e_n$ gives again the same multiplex.

Next the rank of the underlying oriented matroid is $d+1$; that is, $d+2$ is the smallest number with the property that any subset of $e_0, \ldots, e_n$ of size $d+2$ contains $\underline{C}$ for some circuit $C$. An important property of our underlying oriented matroid is the following (this is formulated in terms of positive cocircuits in [3]): For any circuit $C \in \mathcal{C}$ and any facet $F_i$ of $M(d,n+1)$,

$$\underline{C} \backslash F_i \neq \emptyset \quad \text{implies that} \quad C^+ \backslash F_i \neq \emptyset \;\text{ and }\; C^- \backslash F_i \neq \emptyset. \tag{3}$$

Since a simplex is o.m. rigid (no circuits), we consider only the case $n \geq d+1$.

**Theorem 2** *Let $d \geq 3$ and $d+1 \leq n \leq 2d-1$. Then the circuits of the underlying oriented matroid of $M(d,n+1)$ are the signed subsets $C$ on $\{e_0,\ldots,e_n\}$ with the property that $|\underline{C}| = k$ for $4 \leq k \leq d+2$,*

**a)** *$C$ is sequential, and*

**b)** *$\{e_i, e_{i+k-3}, e_{i+d}, e_{i+d+k-3}\} \subset \underline{C}$ and*
*$|\underline{C} \cap \{e_j, e_{j+d}\}| = 1$ for $i+1 \leq j \leq i+k-4$ and $0 \leq i \leq n+3-d-k$.*

**Proof:** We prove the Theorem by induction on $d$. Denote by $\mathcal{C}$ the underlying oriented matroid of $M(d,n+1)$.

Let $d=3$. If $n=4$ then $F_2 = [e_0,e_1,e_3,e_4]$ is the only non-simplicial facet of $M(3,5)$, and the only circuits are $e_0\overline{e_1e_3}e_4$ and its opposite from [1]. So let $n=5$ and $C$ be a circuit. We may assume by (2) that $e_0, e_5 \in \underline{C}$ (otherwise, $\pm C \in \{e_0\overline{e_1e_3}e_4, e_1\overline{e_2e_4}e_5\}$). Now (3) applied to $F_2$ ($F_3 = [e_1,e_2,e_4,e_5]$) yields that $e_2 (e_3)$ is in $\underline{C}$. Clearly, $\{e_1,e_4\} \not\subset \underline{C}$. If $e_1 (e_4)$ is not in $\underline{C}$ then (3) applied to $F_4 = [e_2,e_3,e_5]$ ($F_1 = [e_0,e_2,e_3]$) yields that $e_4 (e_1)$ is in $\underline{C}$. Thus, $\underline{C}$ is $\{e_0,e_1,e_2,e_3,e_5\}$ or $\{e_0,e_2,e_3,e_4,e_5\}$. Assume that $e_5 \in C^+$. Then (3) applied successively to each $F_i$ yields that $C$ is sequential in either case.

Next consider the case $d \geq 4$, and assume that $C$ is a circuit of $\mathcal{C}$. Let $d+1 \leq n \leq 2d-2$. Then (3) applied to $F_{d-1} = [e_0,\ldots,e_{d-2},e_d,\ldots,e_n]$ yields that $\underline{C} \subset F_{d-1}$. Since $F_{d-1}$ is a $(d-1)$-multiplex with the induced ordering, the induction hypothesis readily yields that $C$ satifies a) and b).

Let $n = 2d-1$. By (2), we may assume that $\{e_0, e_{2d-1}\} \subset \underline{C}$. Then (3) applied to $F_{d-1}$ and $F_d$ yields that $\{e_{d-1}, e_d\} \subset \underline{C}$. Assume that $e_{d-1} \in C^+$.

Eliminate $e_{d-1}$ from $C$ and $e_{d-2}\overline{e_{d-1}e_{2d-2}}e_{2d-1}$, and denote by $\widetilde{C}$ the resulting circuit. Then $e_{d-1} \notin \underline{\widetilde{C}}$, $F_{d-1}$ and (3) yield that $e_{2d-1} \notin \underline{\widetilde{C}}$ and, as above, $\widetilde{C}$ satisfies a) and b). Since $e_{2d-1} \notin \underline{\widetilde{C}}$ and $\underline{\widetilde{C}} \cap \{e_{d-2}, e_{2d-2}\} \neq \emptyset$, it follows from (3) applied to $F_{d-2} = [e_0,\ldots,e_{d-3},e_{d-1},\ldots,e_{2d-3}]$ that

$\{e_{d-2}, e_{2d-2}\} \subset \widetilde{\underline{C}}$. A similar argument yields that $\underline{C} \cap \{e_{d-2}, e_{2d-2}\} \neq \emptyset$. Clearly, $|\underline{C} \cap \{e_{d-2}, e_{2d-2}\}| = 1$.

Let $k$ and $i$ be the integers associated with $\widetilde{C}$ in b). Then $|\widetilde{\underline{C}}| = k$, $e_{i+k-3} = e_{d-2}$ and $i = d+1-k$. Now if $i \geq 1$ then

$$G = [e_i, \ldots, e_{d-1}, e_{d+i}, \ldots, e_{2d-1}] = \bigcap_{j=d}^{d+i-1} F_j$$

is a $t$-multiplex with $t \leq d - i = k-1$ and $\widetilde{\underline{C}} \subset G$. Since

$$|\underline{C} \cap G| = (|\widetilde{\underline{C}}| - 1) + |\{e_{d-1}, e_{2d-1}\}| = k+1,$$

there is a support of a circuit of the underlying oriented matroid of $G$, and hence of $M(d, 2d)$, contained in $\underline{C}$; a contradiction by iii). So $\{e_0, e_{d-2}, e_d, e_{2d-2}\} \subset \widetilde{\underline{C}}$, which in turn yields for $C$ that

$$|\underline{C} \cap \{e_j, e_{j+d}\}| = 1 \text{ for } 1 \leq j \leq d-3.$$

Then b) is verified for $C$ with $i = 0$ and $k = d+2$.

In order to obtain the sign pattern, remember that $e_{d-1} \in C^+$. Then (3), applied first to $F_{d-1}$ and then to $F_{d-2}$, yields that $e_{2d-1} \in C^-$ and $\{e_{d-2}, e_{2d-2}\} \cap \underline{C}$ is in $C^+$. We deduce by (iv) that $e_{d-2} \in \widetilde{C}^+$, and the induction hypothesis now yields that $C$ is sequential with negative end points.

Finally, no signed subset of $\{e_0, \ldots, e_n\}$ satisfying a) and b) with at most $d+1$ elements, contains $e_0$ or $e_n$, so every such subset is a circuit by induction. Since any subset of $\{e_0, \ldots, e_n\}$ with $d+2$ elements contains a circuit (without sign), the same holds for the signed subsets of $\{e_0, \ldots, e_n\}$ with $d+2$ elements that satisfy a) and b). □

If $n = 2d$ then the proof is based on Theorem 2.

**Theorem 3** *Let $d \geq 3$. Then the circuits of the underlying oriented matroid of $M(d, 2d+1)$ are the signed subsets $C$ on $\{e_1, \ldots, e_n\}$ satisfying exactly one of a), b), c) or d) below.*

**a)** $|\underline{C}| = k$, $4 \leq k < d+2$, *$C$ is sequential,* $\{e_i, e_{i+k-3}, e_{i+d}, e_{i+d+k-3}\} \subset \underline{C}$ *and*
$|\underline{C} \cap \{e_j, e_{j+d}\}| = 1$ *for* $i+1 \leq j \leq i+k-4$ *and* $0 \leq i \leq d+3-k$.

**b)** $|\underline{C}| = d+2$, *$C$ is sequential,* $\{e_0, e_d, e_{2d}\} \subset \underline{C}$ *and* $|\underline{C} \cap \{e_j, e_{j+d}\}| = 1$ *for* $1 \leq j \leq d-1$.

**c)** $|\underline{C}| = d+2$, $\underline{C} \cap \{e_0, e_d, e_{2d}\} = \{e_0, e_{2d}\}$ *and for some* $1 \leq i \leq d-1$ *and* $\widetilde{C} \in \{C, -C\}$, $\{e_0, e_{i+d}\} \subset \widetilde{C}^+$, $\{e_i, e_{2d}\} \subset \widetilde{C}^-$, $\left|\widetilde{C}^- \cap \{e_l, e_{l+d}\}\right| = 1$ *for* $0 < l < i$ *and* $\left|\widetilde{C}^+ \cap \{e_l, e_{l+d}\}\right| = 1$ *for* $i < l < d$.

**d)** $|\underline{C}| = d+4+i-j$, $\underline{C} \cap \{e_0, e_d, e_{2d}\} = \{e_0, e_d\}$ *and for some* $1 \leq i < i+2 \leq j \leq d-1$ *and* $\widetilde{C} \in \{C, -C\}$, $\{e_0, e_{i+d}, e_{j+d}\} \subset \widetilde{C}^+$, $\{e_i, e_j, e_{2d}\} \subset \widetilde{C}^-$, $\left|\widetilde{C}^- \cap \{e_l, e_{l+d}\}\right| = 1$ *for* $0 < l < i$ *and* $\left|\widetilde{C}^+ \cap \{e_l, e_{l+d}\}\right| = 1$ *for* $j < l < d$.

**Proof:** Let $C$ be a circuit. If $\underline{C}$ does not contain $e_0$ or $e_{2d}$ then $C$ is of type a) by (2) and Theorem 2. Assume that $\{e_0, e_{2d}\} \subset \underline{C}$.

**Case 1:** $e_d \in \underline{C}$

Assume that $e_{2d} \in C^-$. Since $e_{d-1}\overline{e_d e_{2d-1}}e_{2d}$ is a circuit, we deduce by (iii) that $\underline{C}$ cannot contain both $e_{d-1}$ and $e_{2d-1}$. Now (3) applied to $F_{d-1}$ yields that $|C^+ \cap \{e_{d-1}, e_{2d-1}\}| = 1$.

Eliminate $e_{2d}$ from $C$ and $e_{d-1}\overline{e_d e_{2d-1}}e_{2d}$ using weak elimination, and denote by $C_0$ the resulting circuit. Now $\underline{C}_0 \cap \{e_{d-1}e_{2d-1}\} \neq \emptyset$, $e_{2d} \notin \underline{C}_0$ and (3) applied to $F_{d-1}$ readily yield that $e_{d-1} \in C_0^+$ and $e_{2d-1} \in C_0^-$. Next, Theorem 2 yields that there is a $0 \leq i < d-1$ such that $\{e_i, e_{2d-1}\} \subset C_0^-$, $\{e_{d-1}, e_{i+d}\} \subset C_0^+$, and $|C_0^+ \cap \{e_j, e_{j+d}\}| = 1$ for $i < j < d-1$. Since $e_d \in \underline{C}_0$ and $e_d \notin \{e_j, e_{j+d}\}$ for $1 < j < d-1$, we have that $e_d = e_{i+d}$, $e_d \in C_0^+$, $i = 0$ and $|\underline{C}_0| = d+2$.

Since the elements of $C_0$, different from $e_{d+1}$, $e_d$ and $\overline{e}_{2d-1}$, come from $C$, we have that $|\underline{C}| = d+2$, and $C$ is sequential if $|C^+ \cap \{e_{d-1}, e_{2d-1}\}| = 1$ and $e_d \in C^+$. The former is noted above, and the latter follows from $e_d \in C_0^+$ and $e_{d-1}\overline{e_d e_{2d-1}}e_{2d}$.

**Case 2:** $e_d \notin \underline{C}$

Assume that $e_{2d} \in C^-$, and hence, $e_0 \in C^+$ by (3) applied to $F_d$. Eliminate $e_{2d}$ from $C$ and $e_{d-1}\overline{e_d e_{2d-1}}e_{2d}$ using weak elimination, and denote by $C_1$ the resulting circuit.

**Case 2.1:** $e_d \in \underline{C}_1$

Then $e_d \in C_1^-$, and (3) applied to $F_d$ yield that $e_0 \in C_1^+$. Next, $e_{2d} \notin \underline{C}_1$ and Theorem 2 yield that there exists some $1 \leq i \leq d-1$ such that $\{e_i, e_d\} \subset C_1^-$, $e_{i+d} \in C_1^+$, and $|C_1^- \cap \{e_l, e_{l+d}\}| = 1$ for $0 < l < i$.

The conditions for weak elimination yield that $\{e_0, e_{i+d}\} \subset C^+$ and $C_1^- \backslash \{e_d\} \subset C^-$. Since the $(i+1)$–face $[e_0, \ldots, e_i, e_d \ldots, e_{i+d}]$ already contains $i+2$ elements of $\underline{C}$, it contains no other elements of $\underline{C}$ by (iii).

Now eliminate $e_0$ from $C$ and $-C_1$, and denote by $C_2$ the resulting circuit. Then $\underline{C}_1 \backslash \{e_d\} \subset \underline{C}$ yields that $e_d \in C_2^+$, and Theorem 2 yields some $1 \leq j \leq d-1$ such that $\{e_j, e_{2d}\} \subset C_2^-$, $\{e_d, e_{j+d}\} \subset C_2^+$, and $|C_2^+ \cap \{e_l, e_{l+d}\}| = 1$ for $j < l < d$. Clearly, $j \geq i$ and so, $e_j \in C^-$ and $C_2^+ \backslash \{e_d\} \subset C^+$. As above, $\underline{C}$ has no other elements in $\{e_j, \ldots, e_d, e_{j+d} \ldots, e_{2d}\}$. Thus, we obtain c) in the case $i = j$.

Finally, $\underline{C}$ has no other element than the ones we have listed, in the case $i < j$. This follows by eliminating $e_d$ from $C_1$ and $C_2$. Then the underlying set of the resulting circuit is a subset of $\underline{C}$ and so, it must be $\underline{C}$ by (iii). Finally, $j \neq i+1$ by (1) and we obtain d).

**Case 2.2:** $e_d \notin \underline{C}_1$

Since $\underline{C}_1$ contains an element outside of $\underline{C}$, we have that $\underline{C}$ does not contain both $e_{d-1}$ and $e_{2d-1}$, and one of them is in $\underline{C}_1$. Now, $e_{2d} \notin \underline{C}_1$ and (3) applied to $F_{d-1}$ yield readily that $e_{d-1} \in C_1^+$ and $e_{2d-1} \in C_1^-$. By Theorem 2, there exists $j \leq d-2$ such that $\{e_j, e_{2d}\} \subset C_1^-$, $\{e_d, e_{j+d}\} \subset C_1^+$ and $|C_1^+ \cap \{e_l, e_{l+d}\}| = 1$ if $j < l < d-1$. Now by reversing the order on $\{e_0, e_1, \ldots, e_n\}$ and taking the opposite of $C$, we argue as in case 2.1 and obtain c) and d).

Finally, the proof shows also that the signed sets described in a), b) and c) above are circuits of the underlying oriented matroid of $M(d, 2d+1)$. Let $X$ be a set of $d+4+i-j$ vertices described in d). Let $Y = X \cup \{e_{i+2}, \ldots, e_{j-1}\}$ if $j \neq i+2$, and $Y = X$ otherwise. Then $|Y| = d+2$ and $\underline{C} \subseteq Y$ for some $C \in \mathcal{C}$. It is now easy to check that $\underline{C} = X$. $\square$

# 3 The non–rigid multiplices

First, we recall the method in [2] to construct multiplices. The method is based on the circuits in (1). Assume that $e_0 < e_1 < \cdots < e_n$ are the vertices of a $d$-multiplex in $\mathbb{R}^d$. Then $[e_{n-d}, e_{n-d+1}, e_n]$ is a 2–face of the multiplex, and $[e_0, \ldots, e_n, e_{n+1}]$ is a $d$-multiplex if and only if

- $[e_{n-d}, e_{n-d+1}, e_n, e_{n+1}]$ is a quadrilateral and $[e_{n-d}, e_{n+1}]$ is a diagonal,

  and

- if $H$ is the affine hull of a facet of $[e_0, \ldots, e_n]$ not containing $e_n$ then $e_{n+1}$ lies on the same side of $H$ as $e_n$.

Let $n \geq 2d+1$. We construct two $d$-multiplices with $n+1$ vertices such that the two underlying oriented matroids are different.

Observe that given the underlying oriented matroid of a multiplex, the facets of the multiplex can be recovered (see [3]): the convex hull $F$ of a set vertices is a facet if and only if $F$ contains $d$ affinely independent vertices, and for any circuit $C$, $\underline{C} \backslash F \neq \emptyset$ yields that $C^+ \backslash F \neq \emptyset$ and $C^- \backslash F \neq \emptyset$. Therefore the underlying oriented matroid determines the order of the vertices up to reversing the order.

Let $[e_0, \ldots, e_{2d}]$ be any $d$-multiplex with $2d+1$ vertices.

In order to construct the multiplex $M_1 = [e_0, \ldots, e_n]$, choose $e_{2d+1}$ so that the lines $\text{aff}\{e_0, e_1\}$, $\text{aff}\{e_d, e_{d+1}\}$ and $\text{aff}\{e_{2d}, e_{2d+1}\}$ are either concurrent or parallel. Then $e_0\overline{e_1 e_{2d}}e_{2d+1}$ is a circuit. Now continue adding vertices so that $[e_0, \ldots, e_n]$ is a multiplex.

On the other hand, construct the multiplex $M_2 = [e_0, \ldots, e_{2d}, e'_{d+1}, \ldots, e'_n]$ so that the lines $\text{aff}\{e_i, e_{i+1}\}$, $\text{aff}\{e_{i+d}, e_{i+d+1}\}$ and $\text{aff}\{e_{i+2d}, e_{i+2d+1}\}$ are neither concurrent nor parallel for $i = 0, \ldots, n-2d-1$. Since $[e_i, e_{i+1}, e_{i+d}, e_{i+d+1}]$ and $[e_{i+d}, e_{i+d+1}, e_{i+2d}, e_{i+2d+1}]$ are 2–faces, we deduce that $e_i, e_{i+1}, e_{i+2d}, e_{i+2d+1}$ are not contained in a 2–plane. In particular, $e_i\overline{e_{i+1}e_{i+2d}}e_{i+2d+1}$ is not a circuit for $i = 0, \ldots, n-2d-1$, and hence the underlying matroids of $M_1$ and $M_2$ are different.

**Theorem 4** *Multiplices in $\mathbb{R}^d$ with at least $2d+2$ vertices are not o.m. rigid.*

# 4 Examples and Questions

First, we present a concrete set of vertices in $\mathbb{R}^3$ realizing the different oriented matroids of Theorem 4:

$$\begin{array}{lll} e_0 = (4,0,0), & e_1 = (8,0,0) & e_2 = (6,0,2), \\ e_3 = (0,4,0), & e_4 = (12,4,0), & e_5 = (7,2,3), \\ e_6 = (4,4,2), & e_7 = (8,4,2), & e'_7 = (8,4,3). \end{array}$$

Then $M_1 = [e_0, \ldots, e_6, e_7]$, $M_2 = [e_0, \ldots, e_6, e_7']$ are 3-multiplices, $e_0$, $e_1$, $e_6$ and $e_7$ are coplanar, $e_0$, $e_1$, $e_6$ and $e_7'$ are not coplanar, and the sets of circuits of $M_1$ and $M_2$ are different.

It is worthy of future consideration to determine if there is an algorithm to describe a set of circuits of $M(d, n+1)$ for $n \geq 2d+1$. Also, we know little about chirotopes for $M(d, n+1)$ and perhaps, they are simpler to describe than sets of circuits.

Next, we list the signed circuits (one from each pair of opposites) of $M(4, 9)$ with the vertices $e_0 < e_1 < \cdots < e_8$. For convenience, we denote an element $e_i(\overline{e}_i)$ of a circuit by $i(\overline{i})$. It is worth emphasizing that we are listing also, for example, the signed circuits of $M(4, 8)$ with vertices $e_0 < e_1 < \cdots e_7$ or $e_1 < e_2 < \cdots < e_8$. As a guidepost, we group the circuits as per Theorem 3.

a) $0\overline{14}5$, $1\overline{25}6$, $0\overline{124}6$, $0\overline{245}6$, $1\overline{235}7$, $1\overline{356}7$, $0\overline{1234}7$, $0\overline{1346}7$, $0\overline{2345}7$, $0\overline{3456}7$, $3\overline{47}8$, $2\overline{346}8$, $2\overline{467}8$, $1\overline{2345}8$, $1\overline{2457}8$, $1\overline{3456}8$, $1\overline{4567}8$

b) $0\overline{1234}8, 0\overline{2345}8, 0\overline{1346}8, 0\overline{1247}8, 0\overline{1467}8, 0\overline{2457}8, 0\overline{3456}8, 0\overline{4567}8$

c) $(i = 1)$ $\quad 0\bar{1}235\bar{8}, 0\bar{1}257\bar{8}, 0\bar{1}356\bar{8}, 0\bar{1}567\bar{8}$

$(i = 2)$ $\quad 0\bar{1}\bar{2}36\bar{8}, 0\bar{1}\bar{2}67\bar{8}, 0\bar{2}3\bar{5}6\bar{8}, 0\bar{2}\bar{5}67\bar{8}$

$(i = 3)$ $\quad 0\bar{1}\bar{2}\bar{3}7\bar{8}, 0\bar{1}\bar{3}\bar{5}7\bar{8}, 0\bar{2}\bar{3}\bar{5}7\bar{8}, 0\bar{3}\bar{5}67\bar{8}$

d) $(i = 1,\, j = 3)$ $\quad 0\bar{1}\bar{3}57\bar{8}$

Finally, the construction of $M(d, n+1)$ from $M(d, n)$ indicates that one can construct an infinite sequence $\{e_j\}_{j \in \mathbb{Z}}$ of points in $\mathbb{R}^d$ such that $[e_k, e_{k+1}, \ldots, e_m]$ is a $d$-multiplex for any $k, m \in \mathbb{Z}$ with $m > k + d$. We call the convex hull of $\{e_j\}_{j \in \mathbb{Z}}$, an *infinite d-multiplex*, and denote it by $M(d, \infty)$. Then the facets of $M(d, \infty)$ are

$$[e_{i-d+1}, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{i+d-1}], \quad i \in \mathbb{Z},$$

and the boundary complex of $M(d, \infty)$ is a well-defined $(d-1)$-dimensional polytopal cell complex.

As the boundary complex of $M(d, \infty)$ possesses the symmetry of shifting of indices by any integer, it raises the question if there is a realization of $M(d, \infty)$ similar to the realization of an "infinite" cyclic $d$-polytope by points on the curve $f(t) = (t, t^2, \ldots, t^d)$. The answer for $d = 3$ is most likely yes with the curve $f(t) = (\cos(mt)\sin t, \sin(mt)\sin t, \cos t)$ as a suitable candidate for a sufficiently large integer $m$.

Also, if $e_{j+1}$ is close to $e_j$ in the construction, then there is a convex body $K \subset \mathbb{R}^d$ such that $M(d, \infty)$ is obtained by removing two points from the boundary of $K$. From this, we deduce that the boundary of $M(d, \infty)$ gives rise to a cell decomposition of $S^{d-1}$ minus the two poles.

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