About the error term for best approximation with respect to the Hausdorff related metrics

Károly Böröczky*, Jr.

December 3, 2004

Abstract

Let M be a convex body with C^3_+ boundary in $\mathbb{R}^d$, $d \geq 3$, and consider a polytope P_n (or $P_{(n)}$) with at most n vertices (at most n facets) minimizing the Hausdorff distance from M. It has been long known that as n tends to infinity, there exist asymptotic formulae of order $n^{-\frac{2}{d-1}}$ for the Hausdorff distances $\delta_H(P_n, M)$ and $\delta_H(P_{(n)}, M)$. In this paper, a bound of order $n^{-\frac{5}{2(d-1)}}$ is given for the error of the asymptotic formulae.

This bound is clearly not the best possible, and P.M. Gruber (see [9]) conjectured that if the boundary of M is sufficiently smooth then there exist asymptotic expansions for $\delta_H(P_n, M)$ and $\delta_H(P_{(n)}, M)$. With the help of quasiconformal mappings, we show for the three dimensional unit ball that the error is at least $f(n) \cdot n^{-2}$ where $f(n)$ tends to infinity. Therefore in this case, no asymptotic expansion exists in terms of $n^{-\frac{2}{d-1}} = n^{-1}$.

1 Introduction

Assume that M is a convex body with C^2_+ boundary in $\mathbb{R}^d$, and δ is one of the notions of distance among the Hausdorff metric, the Banach–Mazur metric, or Schneider's notion of distance. Consider a polytope P_n (or $P_{(n)}$) with at most n vertices (at most n facets) minimizing $\delta(M, P_n)$ (or $\delta(M, P_{(n)})$). In the last two decades, much effort and many brilliant ideas have been put into obtaining asymptotic formulae for $\delta(M, P_n)$ and $\delta(M, P_{(n)})$ as n tends to infinity. After initial ideas by László Fejes Tóth (in dimensions 2 and 3, (*cf.* [6])) and results by McClure and Vitale in the plane, (*cf.* [13]), it was R. Schneider who provided the first time an asymptotic formula in higher dimensions (*cf.* [14]); namely, he considered the Hausdorff distance

*Supported by OTKA 031984, Hungary

for convex bodies with C^3_+ boundary. A new understanding of the problem came by the method developed by P.M. Gruber [7]. This method could handle all related questions (say for the Banach–Mazur distance), and the the assumption on the boundary could be relaxed to C^2_+. See P.M. Gruber's comprehensive surveys [8] and [9] for a detailed history of this field. The central problem of this paper is to estimate the error of these asymptotic formulae as n tends to infinity.

First we introduce the notions of distance of convex bodies K and P which we consider. Note that the support function of a convex body K is defined as $h_K(u) = \max_{x\in K}\langle x, u\rangle$.

Hausdorff metric: $\delta_H(K,P) = \max_{u\in S^{d-1}} |h_M(u) - h_P(u)|$; namely, the maximum of the distances of the points of K from P, and the distances of the points of P from K.

Banach–Mazur metric: Let the interiors of K and P contain the origin. Then $\delta_{BM}(K,P)$ is the minimum of $\ln\lambda$ such that there exists a linear transformation T satisfying $TP \subset K \subset \lambda\cdot TP$.

Schneider's distance: Let P be contained in K. Then $\delta_{SCH}(K,P)$ is the maximal volume of a cap cut off from K by a supporting hyperplane of P.

Observe that the Hausdorff metric is a metric on the space of convex bodies, while the Banah–Mazur distance is metric for convex bodies which contain the origin in their interiors if convex bodies and their linear images are identified. On the other hand, Schneider's notion of distance is not a metric, and actually not so important as the other two notions.

Let us consider a convex body M. Then ∂M can be covered by a family of open neighbourhoods on ∂M such that each of them is the graph of some convex function in $d-1$ variables. If each function is C^k, $k\geq 2$, then ∂M is a C^k manifold. In this case, the second fundamental form Q_x can be defined at each $x\in\partial M$, and the Gauß curvature $\kappa(x)$ is the determinant of Q_x. The convexity of M yields that Q_x is positive semidefinite, and we say that M is C^k_+ if the second fundamental form Q_x is positive definite at each $x\in\partial M$; or equivalently, if $\kappa(x)$ is positive at each x. The basic reference about the properties of smooth convex bodies is the book [15] of R. Schneider, and see Section 2.2 for the properties we need.

Next we list the known asymptotic formulae where P_n ($P_{(n)}$) denotes a best approximating polytope with at most n vertices (facets) with respect to the given metric. If ∂M is C^2_+ then these results were proved by P.M. Gruber (see [8] and [9] for references). The integration is with respect to the induced measure on ∂M as a C^2 hypersurface in $\mathbb{R}^d$, which measure coincides with the Hausdorff $(d-1)$–measure. The exterior unit normal at a point $x\in\partial M$ is denoted by $\nu(x)$. Finally, we denote the minimal density of a covering of $\mathbb{R}^{d-1}$ with congruent balls by ϑ_{d-1}, and the volume of the $(d-1)$–dimensional unit ball by κ_{d-1}. Then

$$\begin{aligned}
\delta_H(M,P_n),\, \delta_H(M,P_{(n)}) &\sim \frac{1}{4}\left(\frac{\vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{2}{d-1}}\left(\int_{\partial M}\kappa(x)^{\frac{1}{2}}dx\right)^{\frac{2}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}};\\
\delta_{BM}(M,P_n),\, \delta_{BM}(M,P_{(n)}) &\sim \frac{1}{2}\left(\frac{\vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{2}{d-1}}\left(\int_{\partial M}\frac{\kappa(x)^{\frac{1}{2}}}{h_M(\nu(x))^{\frac{d-1}{2}}}dx\right)^{\frac{2}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}};\\
\delta_{SCH}(M,P_n),\, \delta_{SCH}(M,P_{(n)}) &\sim \frac{\kappa_{d-1}}{d+1}\cdot\left(\frac{\vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{d+1}{d-1}}\left(\int_{\partial M}\kappa(x)^{\frac{1}{d+1}}dx\right)^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{d+1}{d-1}}}.
\end{aligned}$$

In case of the Hausdorff metric, the same asymptotic formulae are proved assuming that P_n is inscribed, or that P_n is circumscribed (only with double constant). The integral which appears in the formula for the Banach–Mazur distance, is the so called centro affine surface area, and it is invariant under linear transformations (see W. Blaschke [3] or K. Leichtweiß [11]). In addition, the integral which appears in the formula for Schneider's distance, is the so called affine surface area, and it is invariant under volume preserving affine transformations (see W. Blaschke [3] or K. Leichtweiß [11]). The goal of this paper is to estimate the the error term for the formulae. We write $f=O(g)$ if $|f|\leq c\cdot g$ where c depends only on M.

THEOREM 1.1 *Let M be a convex body with C^3_+ boundary, and assume that P_n is some polytope with at most n vertices.*

(i) *If P_n minimizes $\delta_H(M,P_n)$ then*

$$\delta_H(M,P_n)=\frac{1}{4}\left(\frac{\vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{2}{d-1}}\left(\int_{\partial M}\kappa(x)^{\frac{1}{2}}dx\right)^{\frac{2}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}}+O\left(\frac{1}{n^{\frac{5}{2(d-1)}}}\right).$$

If P_n is inscribed or circumscribed then $\delta_H(M,P_n)$ is twice the value as above.

(ii) *If $o\in \mathrm{int}M, \mathrm{int}P_n$ and P_n minimizes $\delta_{BM}(M,P_n)$ then*

$$\delta_{BM}(M,P_n)=\frac{1}{2}\left(\frac{\vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{2}{d-1}}\left(\int_{\partial M}\frac{\kappa(x)^{\frac{1}{2}}}{h(\nu(x))^{\frac{d-1}{2}}}dx\right)^{\frac{2}{d-1}}\cdot\frac{1}{n^{\frac{2}{d-1}}}+O\left(\frac{1}{n^{\frac{5}{2(d-1)}}}\right).$$

(iii) *If $P_n\subset M$ and P_n minimizes $\delta_{SCH}(M,P_n)$ then*

$$\delta_{SCH}(M,P_n)=\frac{\kappa_{d-1}}{d+1}\left(\frac{\vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{d+1}{d-1}}\left(\int_{\partial M}\kappa(x)^{\frac{1}{d+1}}dx\right)^{\frac{d+1}{d-1}}\cdot\frac{1}{n^{\frac{d+1}{d-1}}}+O\left(\frac{1}{n^{\frac{2d+3}{2(d-1)}}}\right).$$

The same formulae hold if P_n is replaced by some polytope $P_{(n)}$ with at most n facets.

It is clear from the proof of Theorem 1.1 that Theorem 1.1 holds even if the only assumption on the boundary of M is that it is C^2_+, and the coefficients of the second fundamental form Q_x are Lipschitz functions of the local parameters.

We also show (see Corollary 3.1) that the analogous formulae hold for convex hypersurfaces which are open subsets of the boundary of some convex body, only one needs the appropriate definition of approximating polytopal hypersurfaces.

Next let us discuss possible improvements on the error term. For a convex domain M in $\mathbb{R}^2$, let P_n be some convex polygon with at most n vertices containing M. If the boundary of M is C^4_+ and P_n minimizes the Hausdorff distance from M then

$$\delta_H(M, P_n) = a_2(M) \cdot \frac{1}{n^2} + a_4(M) \cdot \frac{1}{n^4} + o\left(\frac{1}{n^4}\right)$$

holds (see M. Ludwig [12]) where $a_2(M)$ is given above, and $a_4(M)$ depends on M. We remark that if the boundary of M is C^∞_+ and the area of P_n is maximal then there exists an infinity expansion for the area of $M \backslash P_n$ in terms of $\frac{1}{n^2}$ (see S. Tabachnikov [16]). These facts point to the conjecture of P.M. Gruber (see [9]) that for sufficiently smooth convex bodies with positive curvature in $\mathbb{R}^d$, there exists an asymptotic expansion of $\delta(M, P_n)$.

In order to have more information about the expected order of the error term in higher dimensions, assume that M is the d–dimensional unit ball B^d, $d \geq 3$. Let P_n ($P_{(n)}$) be the polytope with at most n vertices (facets) which minimizes $\delta_H(B^2, P_n)$ ($\delta_H(B^2, P_{(n)})$). We prove that

$$\delta_H(B^d, P_n) = \frac{1}{4}\left(\frac{d\kappa_d \vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{2}{d-1}} \cdot \frac{1}{n^{\frac{2}{d-1}}} + O\left(\frac{1}{n^{\frac{8}{3(d-1)}}}\right). \tag{1}$$

We note that the analogous estimates follow directly for the Banach–Mazur distance or Schneider's notion of distance. If $d = 3$ then an equivalent estimate was proved in the Masters thesis [5] of G. Fejes Tóth utilizing the method of B.L. van der Waerden [17].

As a lower bound on the error term, we prove that

$$\delta_H(B^3, P_n) > \frac{2\pi}{3\sqrt{3}} \cdot \frac{1}{n} + \frac{f(n)}{n^2} \tag{2}$$

where $f(n)$ tends to infinity. This shows that $\delta_H(B^3, P_n)$ can not have a series expansion in terms of $n^{\frac{-2}{d-1}} = \frac{1}{n}$. It seems to be plausible that the right order of the error term in Theorem 1.1 is $n^{\frac{-3}{d-1}}$ if $d \geq 3$. Exactly the same can be said about the error term if the polytopes are assumed to be inscribed or circumscribed.

Remark about packing and covering: Approximation of B^d by polytopes in the Hausdorff sense is intimately connected to the problem of covering S^{d-1} with equal balls. Let $R_n(S^{d-1})$ ($r_n(S^{d-1})$) be the minimal (maximal) radius of n equal balls in S^{d-1} which form a covering (packing). Denote the maximal density of packing equal balls into $\mathbb{R}^{d-1}$ by δ_{d-1}. The method of proving the results above yields the following statements:

$$R_n(S^{d-1}) = \left(\frac{d\kappa_d\vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{1}{d-1}} \cdot \frac{1}{n^{\frac{1}{d-1}}} + O\left(\frac{1}{n^{\frac{5}{3(d-1)}}}\right) \tag{3}$$

$$r_n(S^{d-1}) = \left(\frac{d\kappa_d\delta_{d-1}}{\kappa_{d-1}}\right)^{\frac{1}{d-1}} \cdot \frac{1}{n^{\frac{1}{d-1}}} + O\left(\frac{1}{n^{\frac{5}{3(d-1)}}}\right). \tag{4}$$

On the other hand, we prove that

$$R_n(S^2) > \sqrt{\frac{8\pi}{3\sqrt{3}}} \cdot \frac{1}{\sqrt{n}} + \frac{f_0(n)}{n^{\frac{3}{2}}} \quad \text{and} \quad r_n(S^2) < \sqrt{\frac{2\pi}{\sqrt{3}}} \cdot \frac{1}{\sqrt{n}} - \frac{f_0(n)}{n^{\frac{3}{2}}} \tag{5}$$

where $f_0(n)$ tends to infinity. While the upper estimates for the error terms have been known if $d = 3$ (see B.L. van der Waerden [17] and G. Fejes Tóth [5]), the trivial lower estimate of order $n^{-\frac{3}{2}}$ had not been improved before in spite of various efforts.

The paper is organized as follows: First we prove Theorem 1.1, then we extend it to Jordan measurable subsets of the boundary of ∂M. Finally, we verify the lower bound on the error term.

Acknowledgment: I would like to thank Monika Ludwig and Peter Gruber for many useful comments on polytopal approximation, and I am grateful to István Sigray for explaining the basic properties of quasiconformal mappings.

2 The proof of Theorem 1.1

We start with coverings by congruent ellipsoids in $\mathbb{R}^{d-1}$, what is followed by constructing almost flat patches on ∂M. We verify Theorem 1.1 first for the Hausdorff metric and for P_n in order to present the main ideas, then we consider the other metrics, and we close the section by explaining the necessary alterations for $P_{(n)}$.

2.1 Coverings in $\mathbb{R}^{d-1}$

Let us consider coverings of a Jordan measurable set J in $\mathbb{R}^{d-1}$. For $r > 0$, the Minkowski sum $J+rB^{d-1}$ is the parallel domain with radius r, and if J contains some ball of radius r then let $J \ominus rB^{d-1}$ be the family of the centers of all r dimensional balls which are contained in J. The Hausdorff $(d-1)$–measure is denoted by $|\cdot|$.

LEMMA 2.1 *Assume that a closed Jordan measurable set J in $\mathbb{R}^{d-1}$ contains some ball of radius $r > 0$, and denote the minimal number of balls with radius r which cover J by $N_J(r)$. Then*

$$\frac{\vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{|J \ominus rB^{d-1}|}{r^{d-1}} \leq N_J(r) \leq \frac{\vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{|J + rB^{d-1}|}{r^{d-1}}.$$

Proof: For $a > 0$, denote by $\mathcal{T}_a^{d-1}$ the torus $\mathbb{R}^{d-1}/a\mathbb{Z}^{d-1}$, equipped with the Haar measure induced by the Lebesgue measure in $\mathbb{R}^{d-1}$. If J' is a Jordan measurable subset and Γ is a finite subset of $\mathcal{T}_a^{d-1}$ then

$$\int_{\mathcal{T}_a^{d-1}} \#\left((x + J') \cap \Gamma\right) \, dx = |J'| \cdot \#\Gamma. \tag{6}$$

For $a > 2r$, let $N_a(r)$ be the minimal number of r balls in $\mathcal{T}_a^{d-1}$ which cover $\mathcal{T}_a^{d-1}$, and denote by $\Gamma_a(r)$ a set of centers of $N_a(r)$ balls with radius r in $\mathcal{T}_a^{d-1}$ which cover $\mathcal{T}_a^{d-1}$. We use the following property of ϑ_{d-1}: For any $\varepsilon > 0$, there exists a positive $a_0(\varepsilon)$ such that if $a > a_0(\varepsilon) \cdot r$ then

$$(1-\varepsilon) \cdot \frac{\vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{a^{d-1}}{r^{d-1}} \leq N_a(r) \leq (1+\varepsilon) \cdot \frac{\vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{a^{d-1}}{r^{d-1}}. \tag{7}$$

In the argument below, after choosing ε, we always fix some $a > a_0(\varepsilon) \cdot r$ with $a > \mathrm{diam} J + 4r$, and assume that $J + rB^{d-1}$ is embedded into $\mathcal{T}_a^{d-1}$.

Let $\varepsilon > 0$. Now (6) yields that there exists a copy of J which is covered by at most $a^{-(d-1)} \cdot |J + rB^{d-1}| \cdot N_a(r)$ balls of radius r, and the upper bound for $N_J(r)$ is a consequence of (7) and of the arbitrariness of ε.

Next, we suppose indirectly that

$$N_J(r) < (1-\varepsilon) \cdot \frac{\vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{|J \ominus rB^{d-1}|}{r^{d-1}}$$

for some positive ε. Now (6) and (7) yield that there exists a copy of $x + J \ominus rB^{d-1}$ which contains at least

$$a^{-(d-1)} \cdot |J \ominus rB^{d-1}| \cdot N_a(r) \geq (1-\varepsilon) \cdot \frac{\vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{|J \ominus rB^{d-1}|}{r^{d-1}}$$

centers. Replace the balls whose center is in $x + J \ominus rB^{d-1}$ with $N_J(r)$ balls with radius r which cover $x + J$. This way we obtain a new family of balls covering $\mathcal{T}_a^{d-1}$, and the indirect assumption yields that the cardinality of the new family is less than $N_a(r)$. This contradiction finally proves the lemma.

Q.E.D.

Proposition 2.1 was proved in U. Betke, M. Henk and J.M. Wills [2] using a more complicated argument.

Let q be a positive definite quadratic form in $\mathbb{R}^{d-1}$, and we denote the set of points satisfying $q(x) \le 1$ by B_q. If C is a convex body then we denote the inradius with respect to q by $\varrho_q(C)$; namely, $\varrho_q(C)$ is the maximum of ϱ such that a translate of ϱB_q is contained in C. In addition, $N_{C,q}(r)$ denotes the minimal number of translates of $r \cdot B_q$ which cover C.

The inradius with respect to the Euclidean metric is denoted by $\varrho(C)$. If $r < \varrho(C)$ then readily

$$\left(1 - \frac{r}{\varrho(C)}\right)^{d-1} \cdot |C| \le |C \ominus rB^{d-1}| < |C + rB^{d-1}| \le \left(1 + \frac{r}{\varrho(C)}\right)^{d-1} \cdot |C|.$$

Therefore transforming q into the quadratic form $q_0(u) = \langle u, u \rangle$ and Lemma 2.1 yield

COROLLARY 2.2 *Let C be a convex body and let q be a positive definite quadratic form in $\mathbb{R}^{d-1}$. If $r < \frac{1}{2} \cdot \varrho_q(C)$ then*

$$N_{C,q}(r) = \left(1 + O\left(\frac{r}{\varrho_q(C)}\right)\right) \cdot \frac{\vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{\sqrt{\det q} \cdot |C|}{r^{d-1}}.$$

2.2 Subdividing the boundary

Let M be a convex body with C^2 boundary. For some $x \in \partial M$, denote the tangent hyperplane at x by H_x, and hence an open neighbourhood U of x in ∂M is the graph of a convex C^2 function f defined in the projection V_x of U into H_x. Denote the derivative of f at z by l_z and the quadratic form representing the second derivative of f by q_z. We deduce using the Taylor expansion of f that if $y, z \in V_x$ then

$$f(z) = f(y) + l_y(z - y) + \frac{1}{2} q_w(z - y) \tag{8}$$

where $w = y + t(z - y)$ for some $0 < t < 1$. In particular, the second fundamental form at x is $Q_x = q_x$.

Now let ∂M be C^2_+, and assume that the coefficients of the second fundamental form are Lipschitz functions of the local parameters. Then q_z is positive definite for $z \in V_x$, and q_z is a Lipschitz function of z. Equip the tangent space at $x \in \partial M$

with the second fundamental form Q_x, and consider ∂M as a Riemannian manifold with this metric. For $x, y \in \partial M$, we denote the distance of x and y in this metric by $d_{\partial M}(x, y)$. In addition, we denote the Euclidean distance by $d(\cdot,\cdot)$. We note that there exists a $d_0 > 0$ such that $x, y \in \partial M$ and $d(x, y) < d_0$ yield that

$$d_{\partial M}(x,y) \approx d(x,y) \quad \text{and} \quad \angle\,(\nu(x),\nu(y)) \approx d(x,y). \tag{9}$$

For small $\omega > 0$, fix a maximal system $\{x_1, \ldots, x_{k_\omega}\}$ of points on ∂M such that $d(x_i, x_j) \geq \omega$. Then for any $x \in \partial M$ there exists an x_i satisfying $d(x, x_i) < 2\omega$. We define P_ω as the polytope bounded by the tangent hyperplanes H_{x_i}, $i = 1, \ldots, k_\omega$, and we denote the facet of P_ω touching at x_i by F_i, and the image of F_i and $2(F_i - x_i) + x_i$ by the closest point map into ∂M by Ω_i and by $\widetilde{\Omega}_i$, respectively.

LEMMA 2.3 *Let M be a convex body with C^2_+ boundary, and assume that the coefficients of Q_x are Lipschitz functions of the local parameters on ∂M. Let $\omega > 0$ be small, and hence $k_\omega \approx \omega^{-(d-1)}$. In addition, fix an $i \leq k_\omega$, and denote the orthogonal projection from $\widetilde{\Omega}_i$ into affF_i by $'$. Then*

(i) $\varrho(F_i) > \frac{1}{3}\omega$ *and* $|F_i| \approx \omega^{d-1}$*;*

(ii) $(1 - O(\omega^2)) \cdot (F_i - x_i) + x_i \subset \Omega_i' \subset F_i$*;*

(iii) *If $x, y \in \widetilde{\Omega}_i$ then*

$$d(x,y) = (1+O(\omega^2)) \cdot d(x',y') \quad \text{and} \quad d_{\partial M}(x,y) = (1+O(\omega)) \cdot \sqrt{Q_{x_i}(x'-y')};$$

(iv) $|\Omega_i| = (1 + O(\omega^2)) \cdot |F_i|$ *and*

$$\int_{\Omega_i} \sqrt{\kappa(x)}\, dx = (1+O(\omega)) \cdot \sqrt{\det Q_{x_i}} \cdot |F_i|.$$

Proof: Since $\angle(\nu(x), \nu(x_i)) = O(\omega)$ holds for $x \in \widetilde{\Omega}_i$ by (9), we deduce that the inradius of F_i is at least $\frac{1}{3}\omega$, and the circumradius is at most 3ω for small ω. This yields $k_\omega \approx \omega^{-(d-1)}$ and **(i)**. In addition, **(ii)** follows as $d(x, x') = O(\omega^2)$ holds for $x \in \widetilde{\Omega}_i$. Finally, (9) yields that $\langle \nu(x), \nu(x_i) \rangle = 1 - O(\omega^2)$ holds for $x \in \widetilde{\Omega}_i$, and hence **(iii)** and **(iv)** are consequences of (8), and the fact that the second fundamental form is Lipschitz.

Q.E.D.

2.3 Polytopal approximation

Let M be a smooth convex body and P be a polytope in $\mathbb{R}^d$, and assume that $x \in \partial M$ and $y \in \partial P$ satisfy that $d(x,y) = \delta_H(M,P)$. We are going to use without mentioning the following

Fact: *$y - x$ is a normal to ∂M at x. If $y \notin M$ then y is a vertex of P. If $y \in M$ then y lies in the relative interior of a facet of P and $y - x$ is a normal to this facet.*

First we consider approximation of the unit ball B^d because this case is much simpler.

LEMMA 2.4 *If P_n is a polytope with at most n vertices which minimizes the Hausdorff distance from B^d then*

$$\delta_H(P_n, B^d) = \left(1 + O\left(n^{\frac{-2}{d-1}}\right)\right) \cdot \frac{1}{4} \cdot R_n(S^{d-1})^2.$$

If P_n is assumed to be inscribed or circumscibed then $\frac{1}{4}$ should be replaced by $\frac{1}{2}$.

Proof: We observe that if the spherical distance of $x, y \in S^{d-1}$ is R then

$$\begin{aligned} \langle \nu(x), \nu(y) \rangle &= 1 - O(R^2) \\ \langle x - y, \nu(x) \rangle &= (1 + O(R^2)) \cdot \tfrac{1}{2} R^2. \end{aligned} \tag{10}$$

In addition, we have $R_n(S^{d-1}) \approx n^{\frac{-1}{d-1}}$, and let Σ_n be a family of centers in a covering of S^{d-1} by n spherical balls of radius $R_n(S^{d-1})$.

First assume that $P_n \subset B^d$, and hence the vertex set is contained in S^{d-1}. The affine hull of each facet of P_n cuts off a spherical ball from S^{d-1}. Since the maximal radius of these balls equals the minimal radius of spherical balls centered at the vertices which cover S^{d-1}, we deduce

$$\delta_H(P_n, B^d) \geq \frac{1}{2} \cdot R_n(S^{d-1})^2 \cdot \left(1 + O\left(n^{\frac{-2}{d-1}}\right)\right) \tag{11}$$

by (10). The convex hull of Σ_n yields an upper bound of the same order for $\delta_H(P_n, B^d)$.

Next, assume that P_n is circumscribed around B^d. Let $\Sigma \subset S^{d-1}$ be the set of radial projections of the vertices of P_n, and hence there exists an $x \in \Sigma$ such that the spherical distance of x from any points of Σ is at least $R_n(S^{d-1})$. Now there exist vertices $v_1, \ldots, v_d$ of P_n such that their convex hull lies on ∂P_n, and the radial projection of the convex hull contains x. Denote by x_i the radial projection of v_i into S^{d-1}. Since $\mathrm{aff}\{v_1, \ldots, v_d\}$ avoids $\mathrm{int}B^d$, we may assume that $\langle x - v_1, \nu(x) \rangle \leq 0$, and hence (10) yields (11). In order to construct a well approximating polytope, move each point of Σ_n outwards radially with a distance of

$$\frac{1}{2} \cdot R_n(S^{d-1})^2 \cdot \left(1 + c \cdot n^{\frac{-2}{d-1}}\right)$$

for a suitable positive constant c, and let Q_n be the convex hull of these points. If c is chosen large enough then (10) yields that Q_n contains B^d, and also yields the required upper bound for $\delta_H(Q_n, B^d)$.

Finally, in case of general approximation, construct Σ, x and $v_1, \ldots, v_d$ as above. If there exists a v_i with

$$\langle x - v_i, \nu(x)\rangle \leq \frac{1}{4} \cdot R_n(S^{d-1})^2$$

then the distance of this v_i from B^d is suitably large, and the distance of $\operatorname{aff}\{v_1, \ldots, v_d\}$ and x is suitably large otherwise. For an upper bound on $\delta_H(P_n, B^d)$, move each point of Σ_n outwards radially with a distance of $\frac{1}{4} \cdot R_n(S^{d-1})^2$, and take the convex hull.

Q.E.D.

For $r < 1$, denote by $N_{S^{d-1}}(r)$ the minimal number of spherical balls of radius r which cover S^{d-1}. We deduce by Lemma 2.4 that (1) is equivalent with the estimate

$$N_{S^{d-1}}(r) = \left(1 + O\left(r^{\frac{2}{3}}\right)\right) \cdot \frac{d\kappa_d \vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{1}{r^{d-1}}. \tag{12}$$

We apply Lemma 2.3 with $\omega = r^{\frac{1}{3}}$ and $M = B^d$, and we consider the P_ω given by Lemma 2.3. Denote the unit $(d-1)$–ball centered at the origin and parallel to F_i by B_i.

Let us assume that S^{d-1} is covered by $N_{S^{d-1}}(r)$ spherical balls of radius r. Project the n_i balls which are completely contained in Ω_i orthogonally into F_i. Then the projection of each ball can be covered by a $(d-1)$-ball of radius $\left(1 + O\left(r^{\frac{2}{3}}\right)\right) \cdot r$ by Lemma 2.3 **(iii)**, and the projections cover $F_i \ominus c \cdot rB_i$ for a c depending only on d by Lemma 2.3 **(ii)**. We deduce by Corollary 2.2 and Lemma 2.3 **(i)** that

$$n_i = \left(1 + O\left(r^{\frac{2}{3}}\right)\right) \cdot \frac{\vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{|F_i|}{r^{d-1}}.$$

On the other hand, readily $\sum n_i \leq N_{S^{d-1}}(r)$, and hence Lemma 2.3 **(iv)** yields that

$$N_{S^{d-1}}(r) \geq \left(1 + O\left(r^{\frac{2}{3}}\right)\right) \cdot \frac{\vartheta_{d-1}}{\kappa_{d-1}} \cdot \frac{d\kappa_d}{r^{d-1}}.$$

In order to obtain a upper bound on $N_{S^{d-1}}(r)$, we may choose a c depending on d with the help of Lemma 2.3 **(iii)** such that if the spherical distance of $x, y \in \widetilde{\Omega}_i$

is r then $d(x',y') > (1-c\cdot r^{\frac{2}{3}})\cdot r$. Let us cover each F_i by the minimal number n_i of balls of radius $(1-c\cdot r^{\frac{2}{3}})\cdot r$. Then projecting the centers orthogonally to affF_i into $\widetilde{\Omega}_i$, we obtain a covering of Ω_i by n_i spherical balls of radius r. Here the same formula holds for n_i as above, therefore we conclude (12), and in turn (1).

Next, let M be a convex body with C^2_+ boundary, and assume that the coefficients of Q_x are Lipschitz functions of the local parameters on ∂M. We denote by $R_n(\partial M)$ the minimal R such that on ∂M, some n balls of radius R with respect to the metric determined by the second fundamental form cover ∂M.

LEMMA 2.5 *If P_n is a polytope with at most n vertices which minimizes the Hausdorff distance from M then*

$$\delta_H(P_n,M) = \left(1+O\left(n^{\frac{-1}{d-1}}\right)\right)\cdot\frac{1}{4}\cdot R_n(\partial M)^2.$$

If P_n is assumed to be inscribed or circumscribed then $\frac{1}{4}$ should be replaced by $\frac{1}{2}$.

Proof: We deduce by (8) that if $d_{\partial M}(x,y)=R$ is small then

$$\begin{array}{rcl} \langle \nu(x),\nu(y)\rangle & = & 1-O(R^2) \\ \langle x-y,\nu(x)\rangle & = & (1+O(R))\cdot\frac{1}{2}\,R^2. \end{array} \tag{13}$$

Since $R_n(\partial M)\approx n^{\frac{-1}{d-1}}$, the proof can be completed analogously to the proof of Lemma 2.4.

Q.E.D.

For small r, denote by $N_{\partial M}(r)$ the minimal number of balls of radius r with respect to the metric determined by the second fundamental form which cover ∂M. We deduce by Lemma 2.5 that Theorem 1.1 for P_n and for the Hausdorff metric is equivalent with the estimate

$$N_{\partial M}(r) = \left(1+O\left(r^{\frac{1}{2}}\right)\right)\cdot\frac{\vartheta_{d-1}}{\kappa_{d-1}}\cdot\int_{\partial M}\kappa(x)^{\frac{1}{2}}dx\cdot\frac{1}{r^{d-1}}. \tag{14}$$

For both the upper bound and the lower bound on $N_{\partial M}(r)$, apply Lemma 2.3 with $\omega = r^{\frac{1}{2}}$. Then (14) can be verified analogously to (12).

2.4 The Banach–Mazur metric and Schneider's distance

These notions of distance can be dealt with in a similar way as was done for the Hausdorff metric. If $M=B^d$ then the formulae follow directly from the estimates for the Hausdorff metric. So let ∂M be C^2_+, and assume that the coefficients of the second fundamental form are Lipschitz functions of the local parameters.

Let us consider the Banach–Mazur distance. First we write it in a form which is more suitable in our context: Let P be some polytope containing the origin in its interior. Assume that $\{F_j\}$ is the family of facets of P, and $x_j \in \partial M$ is the point where the exterior normals coincide with the exterior normals to F_j. Then $M \subset P$ is equivalent to saying that for each j, we have $h_M(\nu(x_j)) \leq h_P(\nu(x_j))$. Now assume that $P \subset M$ and s_j is the distance of x_j from affF_j (the depth of the corresponding cap). Therefore the minimal λ satisfying $M \subset \lambda \cdot P$ is

$$\lambda = 1 + \max_j \left(1 - \frac{s_j}{h_M(\nu(x_j))}\right)^{-1} \cdot \frac{s_j}{h_M(\nu(x_j))}. \tag{15}$$

For $x \in \partial M$, consider the quadratic form $\widetilde{Q}_x = \frac{1}{h_M(\nu(x))} \cdot Q_x$ on the tangent hyperplane, whose coefficients are Lipschitz functions of the local parameters. We deduce by (15) that the same argument as for the Hausdorff metric gives the result for the Banach–Mazur distance, only the Riemannian metric on ∂M determined by Q_x should be replaced by the Riemannian metric determined by $\widetilde{Q}_x$.

Let us turn to Schneider's notion of distance. Assume that $P_n \subset M$ is a polytope with at most n vertices such that $\delta_{SCH}(M, P_n)$ is minimal. It follows that for large n, the cap avoiding intP_n whose volume is $\delta_{SCH}(M, P_n)$ is cut off by the affine hull of a facet of P_n. We denote the family of facets of P_n by $\{F_j\}$, and the furthest point of the cap cut off by affF_j from affF_j by x_j. Let $d_{\partial M}(x_j, y_j) = R_j$ for some $y_j \in \text{aff}F_j \cap \partial M$. Then (8) yields that

$$\delta_{SCH}(M, P_n) = \left(1 + O\left(n^{\frac{-1}{d-1}}\right)\right) \cdot \frac{\kappa_{d-1}}{2^{\frac{d-1}{2}}(d+1)} \cdot \max_j \frac{R_j^{d+1}}{\sqrt{\kappa(x_j)}}.$$

Therefore the case of Schneider's notion of distance can be handled as the case of the Hausdorff metric, only the Riemannian metric on ∂M determined by Q_x should be replaced by the Riemannian metric determined by the quadratic form $\kappa(x)^{\frac{-1}{d+1}} Q_x$.

2.5 The case of the facets

Let ∂M be C^2_+, and assume that the coefficients of the second fundamental form are Lipschitz functions of the local parameters.

For large n, consider an $x \in \partial M$, and a hyperplane H parallel to the tangent hyperplane at x and intersecting M such that distance r of H from x is $O\left(n^{\frac{-2}{d-1}}\right)$. Let $z \in H$, and assume that the closest point y of ∂M to z satisfies that $d_{\partial M}(x, y) = R$ with $R = O\left(n^{\frac{-1}{d-1}}\right)$. Then (8) yields that

$$\pm d(y, z) = \left(1 + O\left(n^{\frac{-1}{d-1}}\right)\right) \cdot \frac{R^2}{2} - \left(1 + O\left(n^{\frac{-1}{d-1}}\right)\right) \cdot r, \tag{16}$$

and if $M = B^d$ then $O\left(n^{\frac{-1}{d-1}}\right)$ can be replaced by $O\left(n^{\frac{-2}{d-1}}\right)$ in (16). We note that the two sides of (16) are non–positive exactly if $z \in M$.

Let $P_{(n)}$ be a polytope with at most n facets minimizing $\delta_H(M, P_{(n)})$. We claim that

$$\delta_H(P_{(n)}, M) = \left(1 + O\left(n^{\frac{-1}{d-1}}\right)\right) \cdot \frac{1}{4} \cdot R_n(\partial M)^2, \tag{17}$$

and if $P_{(n)}$ is assumed to be inscribed or circumscribed then $\frac{1}{4}$ should be replaced by $\frac{1}{2}$.

We may assume that $P_{(n)}$ has n facets, and each facet intersects M. We denote the facets by $F_1, \ldots, F_n$, and let $x_j \in \partial M$ be the point where the outer normal is the same as for F_j, and we denote by r_j the distance of x_j from $\mathrm{aff} F_j$ (here $r_j = 0$ if $P_{(n)}$ is circumscribed). There exists an $y \in \partial M$ such that $d_{\partial M}(y, x_j) \geq R_n(\partial M)$ holds for $j = 1, \ldots, n$, and denote by z_j the point of $\mathrm{aff} F_j$ such that $y - z_j$ is normal to ∂M. It is readily enough to consider x_j satisfying that $d_{\partial M}(x_j, y) = O\left(n^{\frac{-1}{d-1}}\right)$, and hence (16) yields that $\delta_H(M, P_{(n)})$ is at least the expression in (17). On the other hand, well approximating polytopes can be constructed choosing $x_1, \ldots, x_n \in \partial M$ so that for any $y \in \partial M$ there exists an x_j with $d_{\partial M}(y, x_j) \leq R_n(\partial M)$, and letting $r_j = 0$ if the polytope should be circumscribed,

$$r_j = \left(1 + \frac{c}{n^{\frac{1}{d-1}}}\right) \frac{1}{2}\, R_n(\partial M)^2$$

with a suitable c if the polytope should be inscribed, and $r_j = \frac{1}{4}\, R_n(\partial M)^2$ otherwise. Therefore Theorem 1.1 follows from (14) in case of the Hausdorff metric (and naturally, with the improved error term for the ball).

Finally, the same alterations of the Riemannian metric on ∂M are needed as for P_n in the cases of the Banach–Mazur metric and Schneider's notion of distance.

3 Convex C^2_+ hypersurfaces

Let M be a convex body with C^2 boundary. We consider an open, Jordan measurable subset X of ∂M such that $\kappa(x) > 0$ if $x \in \partial M$ is contained in a neighborhood of the closure of X.

Let Y be a Jordan measurable subset of a polytope P, which we call as a convex polytopal hypersurface. We say that Y approximates X if Y consists of the points y of ∂P such that $\nu(x)$ is an exterior normal at y for some $x \in X$. We say that Y is inscribed if $Y \subset M$, and Y is circumscribed if $Y \cap \mathrm{int} M = \emptyset$. The faces of Y are the intersections of the faces of P with the interior of Y.

Now we extend the notions of distance to X and Y. Observe that for $x \in X$, we have

$$h_P(\nu(x)) - h_M(\nu(x)) = \max_{y \in Y} \langle \nu(x), y - x \rangle.$$

Hausdorff metric: $\delta_H(X, Y) = \max_{x \in X} \left| \max_{y \in Y} \langle \nu(x), y - x \rangle \right|$.

Banach–Mazur metric: Assume that Y is inscribed, and that both $\mathrm{int}M$ and $\mathrm{int}P$ contain the origin. Then $\delta_{BM}(X, Y)$ is the minimum of $\ln \lambda$ such that λY is circumscribed.

Schneider's distance: If Y is inscribed into X then $\delta_{SCH}(X, Y)$ is the maximum volume of a cap cut off from M by the affine hull of a facet of Y.

Let us justify the definitions above. Observe that if $X = \partial M$ (and hence $Y = \partial P$) then $\delta_H(X, Y) = \delta_H(M, P)$. In case of Schneider's notion of distance we have to assume that ∂P is close to ∂M.

The only real exception is the Banach–Mazur distance, in this case the definition has been substantially altered. On the other hand, the lack of linear transformation is irrelevant in our context because linear transformations keep the family of faces.

When applying the same argument for X as for ∂M, everything works simply substituting X for ∂M except for the fact that some patches Ω_i intersect the boundary of X. Since X is Jordan measurable, the proportion of the total area of these patches tends to zero, and hence the analogous asymptotic formula holds for best approximation of X.

In order to obtain some estimate on the error of the asymptotic formula, we need better control on the total area of the patches near the boundary of X. So call a set $\sigma \subset \mathbb{R}^d$ rectifiable if it is the finite union of images of subsets of $\mathbb{R}^{d-2}$ by Lipschitz maps. If σ is a rectifiable subset of ∂M then for small t (see K. Falconer [4],p. 42–44, but it is rather easy to prove),

$$|(\sigma + tB^d) \cap \partial M| < \gamma \cdot t \tag{18}$$

where γ depends on σ. Therefore we conclude

COROLLARY 3.1 *Let X be an open, Jordan measurable subset of the boundary of a convex body M where ∂M is C^2 and $\kappa(x) > 0$ in a neighborhood of the closure of X. Let Y_n be the polytopal hypersurface with at most n vertices minimizing $\delta_H(X, Y_n)$. Then*

$$\delta_H(X, Y_n) \sim \frac{1}{4} \left(\frac{\vartheta_{d-1}}{\kappa_{d-1}} \right)^{\frac{2}{d-1}} \cdot \left(\int_X \kappa(x)^{\frac{1}{2}} dx \right)^{\frac{2}{d-1}}.$$

If in addition, the coefficients of the second fundamental form are Lipschitz functions of the local parameters and the boundary of X is rectifiable then

$$\delta_H(X, Y_n) = \frac{1}{4}\left(\frac{\vartheta_{d-1}}{\kappa_{d-1}}\right)^{\frac{2}{d-1}} \cdot \left(\int_X \kappa(x)^{\frac{1}{2}} dx\right)^{\frac{2}{d-1}} + O\left(\frac{1}{n^{\frac{5}{2(d-1)}}}\right).$$

Remark: If the Y_n's are assumed to be inscribed or circumscribed then $\delta_H(X, Y_n)$ is twice the value as above. In addition, the same formulae hold if the number of facets is at most n.

If $o \in \mathrm{int} M$ and the metric is the Banach–Mazur metric, or $Y_n \subset M$ and the distance is Schneider's notion of distance then the analogous formulae hold.

4 Lower bound on the error term

In this section, we prove (2), so let P_n be the 3–polytope with at most n vertices such that $\delta_H(B^3, P_n)$ is minimal. Suppose that indirectly, there exist an infinite sequence $\{n_k\}$ and a constant c_1 such that

$$\delta_H(B^3, P_{n_k}) < \frac{2\pi}{3\sqrt{3}} \cdot \frac{1}{n_k} + \frac{c_1}{n_k^2}.$$

Denote the regular triangle in $\mathbb{R}^2$ with area $\frac{2\pi}{n}$ by $\widetilde{T}_n$. Then the side of $\widetilde{T}_n$ is $e_n = \frac{\sqrt{8\pi}}{3^{1/4}} \cdot \frac{1}{\sqrt{n}}$, while the circumradius is $R_n = \frac{\sqrt{8\pi}}{3^{3/4}} \cdot \frac{1}{\sqrt{n}}$. Therefore Lemma 2.4 yields that

$$R_{n_k}(S^2) < \left(1 + \frac{c_2}{n_k}\right) \cdot R_{n_k} \tag{19}$$

for some constant c_2. In each formula below, the implied constant in $O(\cdot)$ depends only on c_2.

The way to find a contradiction is to construct an isometry of open subsets of S^2 and $\mathbb{R}^2$ with the help of (19). The construction is based on the theory of quasiconformal mappings, which idea goes back essentially to Gauß.

Assume that g is a function defined on a subset of S^2 into $\mathbb{R}^2$. We call g an $(1+\varepsilon)$–isometry if for any two x, y in the domain of definition, we have

$$(1+\varepsilon)^{-1} d_{S^2}(x, y) \leq d(f(x), f(y)) \leq (1+\varepsilon) \cdot d_{S^2}(x, y).$$

Now for a spherical triangle T with vertices v_1, v_2, v_3, we construct a map $g_{T,n}$: $T \to \widetilde{T}_n$ as follows: Fix a linear bijective map from $\mathrm{conv}\{v_1, v_2, v_3\}$ onto $\widetilde{T}_n$, and define $g_{T,n}$ as the composition of this linear map and the radial projection from T onto $\mathrm{conv}\{v_1, v_2, v_3\}$. We say that T is $(1+\varepsilon_0)$–isomorphic to $\widetilde{T}_{n_k}$ if $g_{T,n}$ is an $(1+\varepsilon)$–isometry.

We note that among the spherical triangles contained in a fixed spherical circle, the inscribed regular triangle and only that maximizes the area. We deduce

PROPOSITION 4.1 *Assume that the spherical circumradius of the spherical triangle T is at most $\left(1 + O\left(\frac{1}{n}\right)\right) \cdot R_n$. Then*

(i) $|T| \le \left(1 + O\left(\dfrac{1}{n}\right)\right) \cdot |\widetilde{T}_n|$*;*

(ii) *for any $\varepsilon > 0$ there exists a $c(\varepsilon) > 0$ such that if n is large and T is not $(1+\varepsilon)$–isomorphic to $\widetilde{T}_n$ then $|T| \le (1 - c(\varepsilon)) \cdot |\widetilde{T}_n|$;*

(iii) *there exists an $0 < \varepsilon_0 < 0.1$ such that if n is large and T is $(1+\varepsilon_0)$–isomorphic to $\widetilde{T}_n$ then each angle of T is strictly between $\frac{2\pi}{7}$ and $\frac{2\pi}{5}$.*

Let Σ_n be the family of centers of n spherical circles with radius $R_n(S^2)$ which circles cover S^2. Triangulate the surface of convΣ_n using the vertices, and project radially the triangles into S^2. This way we obtain a triangulation Ω_n of S^2, a so called Delone triangulation corresponding to Σ_n. The circumradius of each triangle in Ω_n is at most $R_n(S^2)$.

PROPOSITION 4.2 *For any $0 < \varepsilon \le \varepsilon_0$ there exists a $C(\varepsilon)$ with the following property: If n_k is large then all but $C(\varepsilon)$ triangles in Ω_{n_k} are $(1+\varepsilon)$–isomorphic to $\widetilde{T}_{n_k}$.*

Proof: Denote the number of triangles in Ω_{n_k} which are not $(1+\varepsilon)$–isomorphic to $\widetilde{T}_{n_k}$ by N_{n_k}. For any triangle T in Ω_{n_k}, we call the quantity $|T| - \left|\widetilde{T}_{n_k}\right|$ the excess of T. The sum of the excesses is readily zero, therefore (19) and Proposition 4.1 yield that

$$-N_{n_k} \cdot c(\varepsilon) \cdot \left|\widetilde{T}_{n_k}\right| + n_k \cdot O\left(\frac{1}{n_k}\right) \cdot \left|\widetilde{T}_{n_k}\right| \ge 0.$$

In turn, we conclude the existence of $C(\varepsilon)$.

Q.E.D.

We call a vertex in Ω_{n_k} an ε*–skew vertex* if it is the vertex of a triangle in Ω_{n_k} which is not $(1+\varepsilon)$–isomorphic to $\widetilde{T}_{n_k}$. There exist at most $3C(\varepsilon)$ ε–skew vertices, and $\varepsilon \le \varepsilon_0$ yields that each non ε–skew vertex in Ω_{n_k} has degree six by Proposition 4.1 (iii). We may assume by Proposition 4.2 and possibly taking a subsequence of $\{n_k\}$ that there exist $p_1, \dots, p_m \in S^2$, $m \le 3C(\varepsilon_0)$, such that the set of ε_0–skew vertices of Ω_{n_k} tends to $\{p_1, \dots, p_m\}$. Choose $r_0 > 0$ and $q \in S^2$ such that the spherical distance of q from $\{p_1, \dots, p_m\}$ is larger than r_0. After rotating S^2 slightly for each Ω_{n_k}, we may assume that q is a vertex of each Ω_{n_k}, and the set of ε_0–skew vertices of Ω_{n_k} still tends to $\{p_1, \dots, p_m\}$.

If Ω is a triangulation of S^2 or $\mathbb{R}^2$, and u, v are vertices of Ω, then the graph theoretic distance of u and v is defined as the smallest cardinality of a set $\{v_1, \dots, v_k\}$

of vertices such that $v_k = v$ and each pair $\{u, v_1\}$ and $\{v_i, v_{i+1}\}$, $i = 1, \ldots, k-1$, is formed by the endpoints of some edge. Let $\widetilde{\Omega}_n$ be a triangulation of $\mathbb{R}^2$ such that the origin o is a vertex, and each triangle is congruent to $\widetilde{T}_n$, and let H_n ($\widetilde{H}_n$) denote the union of the the triangles of Ω_n ($\widetilde{\Omega}_n$) such that the graph theoretic distance of each vertex of the triangle from q (from o) is at most $\frac{r_0}{2e_n}$.

We may assume that each triangle T of Ω_{n_k} in H_{n_k} is $(1+\varepsilon_0)$–isomorphic to $\widetilde{T}_{n_k}$, and any vertex of T has degree six in Ω_{n_k}. We deduce that H_{n_k} is a subset of the spherical circle of center q and radius r_0, and there exists a sense preserving bijection $h_{n_k} : H_{n_k} \to \widetilde{H}_{n_k}$ such that the restriction of h_{n_k} to any triangle T of Ω_{n_k} in H_{n_k} is the composition of a congruence and g_{T,n_k}. In particular, h_{n_k} is an $(1+\varepsilon_0)$–isometry on these triangles.

The union of the edges of Ω_{n_k} such that both endpoints have graph theoretic distance $\left[\frac{r_0}{2e_{n_k}}\right]$ from q, is a simple closed curve γ_{n_k}. Denote a closest point of γ_{n_k} to q by q_{n_k}. Then the triangle inequality yields that h_{n_k} is an $(1+\varepsilon_0)$–isometry of the spherical circle with center q and radius $d_{S^2}(q_{n_k}, q)$, and hence

$$d_{S^2}(q_{n_k}, q) \geq \frac{1}{1+\varepsilon_0} \cdot \varrho\left(\widetilde{H}_{n_k}\right) = \frac{1}{1+\varepsilon_0} \cdot \frac{\sqrt{3}e_{n_k}}{2} \cdot \left[\frac{r_0}{2e_{n_k}}\right] > \frac{r_0}{4}.$$

Therefore each h_{n_k} is an $(1+\varepsilon_0)$–isometry of the open spherical circle B with center p and radius $\frac{1}{4}\, r_0$.

It is time to introduce quasiconformal mappings. We need only the very basic properties, which can be found say in L.V. Ahlfors [1] or in O. Lehto [10]. We note that using stereographic projection from any $y \in S^2$ to the tangent plane at the opposite point, S^2 inherits a natural complex structure. For us, a domain is a connected open subset of S^2 or $\mathbb{R}^2$, and a mapping is a sense preserving homeomorphism between open domains S^2 or $\mathbb{R}^2$. Assume that φ is a Lipschitz mapping, and hence it is differentiable almost everywhere. Let $K \geq 1$. Then φ is K*–quasiconformal* (see O. Lehto [10], Theorem 3.5) if for almost every z where $\varphi'(z)$ exists and for any pair u, v of unit vectors in the tangent space at z,

$$\|\varphi'(z)u\| \leq K \cdot \|\varphi'(z)v\|.$$

We note that an $(1+\varepsilon)$–isometry is $(1+\varepsilon)^2$–quasiconformal, and a mapping which is K–quasiconformal for any $K > 1$ is conformal.

We recall two basic properties of quasiconformal mappings, where the first is stated on p. 13. of O. Lehto [10].

LEMMA 4.3 *Assume that φ is a sense preserving homeomorphism of the domain A, and φ is K–quasiconformal on $A \backslash \Sigma$ where Σ is a finite set. Then φ is K–quasiconformal on A.*

We note that a sequence $\{\varphi_n\}$ of functions on the domain A tends locally uniformly to some function φ if $\{\varphi_n\}$ tends uniformly to φ on any compact subset of A. Now combination of Theorem 2.1 and Theorem 2.2 in O. Lehto [10] yields

LEMMA 4.4 *Assume that $\{\varphi_n\}$ is a sequence of uniformly bounded K–quasiconformal mappings of the domain $A \subset S^2$ into $\mathbb{R}^2$. If $d(\varphi_n(x), \varphi_n(y)) \geq \frac{1}{2}\, d_{S^2}(x,y)$ for any $x, y \in A$ then there exists a subsequence of $\{\varphi_n\}$ which tends locally uniformly to some K–quasiconformal mapping of A.*

In particular, we may assume that h_{n_k} tends locally uniformly to some $(1+\varepsilon_0)^2$–quasiconformal mapping h of B onto an open subset $\widetilde{B}$ of $\mathbb{R}^2$. Let $0 < \varepsilon < \varepsilon_0$. Then there exists a finite subset Σ of S^2 such that a suitable subsequence of the family of the at most $3C(\varepsilon)$ ε–skew vertices in Ω_{n_k} tends to Σ. We deduce by Lemma 4.4 that h is $(1+\varepsilon)^2$–quasiconformal on $B \backslash \Sigma$, and hence $(1+\varepsilon)^2$–quasiconformal on B by Lemma 4.3. This holds for any $\varepsilon > 0$, therefore h is actually conformal.

Now let A be a closed spherical circle contained in B, and choose $0 < \varepsilon < \varepsilon_0$. There exist compact subsets $\widetilde{A}_1, \widetilde{A}_2$ of $\mathbb{R}^2$ bounded by simple, closed Jordan curves with the properties that $\widetilde{A}_1 \subset \mathrm{int} h(A)$, $h(A) \subset \mathrm{int}\widetilde{A}_2$, and

$$(1+\varepsilon)^{-1} \cdot |h(A)| < |\widetilde{A}_1| < |\widetilde{A}_2| < (1+\varepsilon) \cdot |h(A)|.$$

The local uniform convergence yields that $h_{n_k}(\partial A)$ lies in $\widetilde{A}_2 \backslash \widetilde{A}_1$ for large n_k, and hence

$$(1+\varepsilon)^{-1} \cdot |h_{n_k}(A)| < |h(A)| < (1+\varepsilon) \cdot |h_{n_k}(A)|.$$

The area of each triangle in Ω_{n_k} is $O\left(\frac{1}{n_k}\right)$. In addition, all but $C(\varepsilon)$ triangles are $(1+\varepsilon)$–isomorphic to $\widetilde{T}_{n_k}$, and at most $\frac{c'}{\sqrt{n_k}}$ triangles meet the boundary of A where c' depends on A. We deduce that for large n_k,

$$(1+\varepsilon)^{-2} \cdot |A| \leq |h_{n_k}(A)| \leq (1+\varepsilon)^2 \cdot |A|,$$

which in turn yields that $|h(A)| = |A|$ by the arbitrariness of ε. In particular, h is conformal and preserves area, and hence an isometry between B and $\widetilde{B}$. This is absurd, thus we have obtained a contradiction to (19). In other words, (2) has been finally verified.

Remark: The corresponding properties for $r_n(S^d)$ can be verified analogously. The only essential alterations are needed in the argument for (5). If (5) does not hold for $r_n(S^2)$ then there exists a sequence of saturated packings with n_k equal spherical circles which show that

$$r_{n_k}(S^2) > \sqrt{\frac{2\pi}{\sqrt{3}}} \cdot \frac{1}{\sqrt{n_k}} - \frac{c_3}{{n_k}^{\frac{3}{2}}} \tag{20}$$

for some constant c_3. Here saturated means that no circle of the same radius can be added to the packing.

We note that each side of a Delone triangle T is at least $2r_{n_k}(S^2)$, while the circumradius is less than $2r_{n_k}(S^2)$, and the area of a Delone triangle is at least the area of the regular spherical triangle T_{n_k} with edge length $2r_{n_k}(S^2)$. But unfortunately, there is a triangle whose circumradius and two sides are $2r_{n_k}(S^2)$, and whose area is $|T_{n_k}|$. For large n_k, the angles of this triangle are arbitrarily close to $\frac{\pi}{6}$ and $\frac{2\pi}{3}$. Therefore in the analogue of Proposition 4.1, one assumes that at least two angles of T is at least $\frac{2\pi}{11}$. We note that if two angles of the Delone triangle T are at most $\frac{2\pi}{11}$ then the Delone triangle sharing the longest side has large area, and hence the argument can be completed as for $R_n(S^2)$.

References

[1] L.V. Ahlfors: *Lectures on quasiconformal mappings.* Van Nostrand, 1966.

[2] U. Betke, M. Henk and J.M. Wills: *A new approach to covering.* Mathematika, 42: 251-263, 1995.

[3] W. Blaschke: *Affine Differentialgeometrie.* Springer, Berlin, 1923.

[4] K. Falconer: *Fractal geometry.* John Wiley & Sons, New York, 1990.

[5] G. Fejes Tóth: *The loosest packing of the sphere by circles.* MSc. thesis, Roland Eötvös University, Budapest, 1970, (in Hungarian).

[6] L. Fejes Tóth: *Lagerungen in der Ebene, auf der Kugel und im Raum.* Springer-Verlag, Berlin, 2nd edition, 1972.

[7] P.M. Gruber: *Asymptotic estimates for best and stepwise approximation of convex bodies I.* Forum Math., 5: 281–297, 1993.

[8] P.M. Gruber: *Aspects of approximation of convex bodies.* In: Handbook of convex geometry A, 319–345. North-Holland, Amsterdam, 1993.

[9] P.M. Gruber: *Comparisons of best and random approximation of convex bodies by polytopes.* Rend. Circ. Mat. Palermo, 50: 189–216, 1997.

[10] O. Lehto: *Univalent functions and Teichmüller spaces.* Springer–Verlag, Berlin, 1987.

[11] K. Leichtweiß: *Affine Geometry of Convex Bodies.* Johann Ambrosius Barth Verlag, Heidelberg, Leipzig, 1998.

[12] M. Ludwig: *Asymptotic approximation of convex curves: the Hausdorff metric case.* Arch. Math., 70: 331–336, 1998.

[13] D.E. McClure and R.A. Vitale: *Polygonal approximation of plane convex bodies.* J. Math. Anal. Appl., 51: 326–358, 1975.

[14] R. Schneider: *Zur optimalen Approximation konvexer Hyperflächen durch Polyeder.* Math. Ann., 256: 289–301, 1979.

[15] R. Schneider: *Convex bodies: the Brunn–Minkowski theory.* Cambridge Univ. Press, Cambridge, 1993.

[16] S. Tabachnikov: *On the dual billiard problem.* Adv. Math., 115: 221–249, 1995.

[17] B.L. van der Waerden: *Punkte auf der Kugel. Drei Zusatze.* Math. Ann., 125: 213–222, 1952.

Károly Böröczky, Jr.
Rényi Institute of Mathematics
H-1364 Budapest Pf. 127
Hungary
carlos@renyi.hu