On Generalized Kissing Numbers and Blocking Numbers

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Abstract

Let K be an n-dimensional convex body, and let $\alpha > 0$. The paper discusses two related topics: the generalized kissing number is the maximal number of non–overlapping translates of αK such that each touches K, while the generalized Blocking number is the minimal number of non–overlapping translates of αK such that each touches K, and no additional translate of αK with these properties can be inserted.
If K is a polytope then the asymptotics has been known as α tends to zero. We extend this result to any convex body. In addition, various other results are verified.

1 Introduction

Let K be an n-dimensional convex body, and let $N(K)$ be the maximal number of non-overlapping translates of K which can touch K at its boundary. This notion was introduced by H. Hadwiger in [12], and it is known as the kissing number or the Hadwiger number of K (see L. Fejes Tóth [10]). Note that some papers refer to kissing number if not only translates of K but also congruent copies of K are allowed. The history of the problem is covered by the surveys G. Fejes Tóth, W. Kuperberg [8] and Zong [24]. As a counterpart of $N(K)$, define the blocking number $B(K)$ to be the smallest number of nonoverlapping translates of K, all of them touch K at its boundary, which block any other from touching K (see Zong [22]). For the known results about $B(K)$, we refer to Dalla, Larman, Mani-Levitska, and Zong [6] and Zong [23].

In this paper, we are concerned with a genealization of these notions: for a positive number α, define $N_\alpha(K)$ the maximal number of nonoverlapping translates of αK which can touch K at its boundary. In addition, let $B_\alpha(K)$ be the smallest number of non–overlapping translates of αK, all of which touch K at its boundary, which can block any other translates of αK from touching K.

Now $N_\alpha(K)$ for small α was first investigated by L. Fejes Tóth [10] (he proved Theorem 3 for polytopes). Rather recently, I. Talata [19] considered the case of large α. With the help of the QS theorem of V.D. Milman, he proved that if $\alpha < \sqrt{2}+1$ then there exists a $c > 0$ (depending on α) such that the inequality $N_\alpha(K) \geq 2^{cn}$ holds for any centrally symmetric K. Note that the bound $\sqrt{2}+1$ is sharp, as it is shown by the example of the ball.

2 The kissing number

In order to handle the equality case in Theorem 1, we consider tilings of a convex body by translates of a convex body K. We write $K = P \times M$ if K is the direct sum of compact, convex sets P and M.

Lemma 1 *Let K be a convex body in $\mathbb{R}^n$. If the union of some non–overlapping translates $x_1 + K, \ldots, x_m + K$, $m \geq 2$, is convex then $K = P \times M$, where P is a $d \geq 1$ dimensional parallelotope and $x_1 + P, \ldots, x_m + P$ is part of the facet to facet tiling of $\mathbb{R}^d$ by translates of P.*

Proof: We prove the Lemma by induction on n, where the case $n = 1$ trivially holds.

Denote by M the union of $x_1 + K, \ldots, x_m + K$. We deduce from $m \geq 2$ that at least two translates intersects the boundary of M, and hence there there exist a $y \in \partial M$ which is contained say in $x_1 + K$ and $x_2 + K$. Denote by H some hyperplane through y, which separates $x_1 + K$ and $x_2 + K$, and by l line through o parallel with $x_1 - x_2$.

Since $x_1 + K$ is sandwiched between H and $H + x_1 - x_2$, we deduce that if $x \in K$ then the length of $(x + l) \cap K$ is at most $\|x_1 - x_2\|$.

Denote by H_0 some supporting hyperplane of M at y, and by u the exterior unit normal to H. Number the translates so that $x_1 + K, \dots, x_k + K$ are the translates intersecting H.

Observe that the induction hyphothesis applied in H_0 yields that for any $x \in H_0 \cap (x_1 + K)$, the length of $(x + l) \cap (x_1 + K)$ is $\|x_1 - x_2\|$. Set $\alpha_0 = \langle u, y \rangle$, and let α_1 be the infimum of all α such that if $\langle u, x \rangle = \alpha$ and $x \in x_1 + K$, then the length of $(x + l) \cap (x_1 + K)$ is at least $\|x_1 - x_2\|$. Suppose that $\alpha_0 - \alpha_1$ is less than the width of K in the direction of u. Choose some $\varepsilon > 0$ such that if $\langle u, x \rangle = \alpha_1 - \varepsilon$ and $x \in x_1 + K$ then the length of $(x + l) \cap (x_1 + K)$ is at least $\frac{2}{3} \|x_1 - x_2\|$, while there exist $z_i \in \partial x_i + K$, $i = 1, 2$, such that

$$\langle u, z_1 \rangle = \langle u, z_2 \rangle = \alpha_1 - \varepsilon \quad \text{and} \quad \|z_1 - z_2\| \leq \tfrac{1}{3} \cdot \|x_1 - x_2\|.$$

Then the intersection of aff $\{z_1, z_2\}$ with each $x_j + K$, $j > k$, is either empty or has length at least $\frac{2}{3} \|x_1 - x_2\|$. This yields that the interior points of conv $\{z_1, z_2\}$ are not in M, which is absurd. Therefore

$$K = \text{conv}\,\{x_1, x_2\} \times M$$

for some $(n-1)$ dimensional convex, compact set M.

Now applying the induction hyphothesis for sections of M parallel to H yields the Lemma, as these sections change in a continuous way as the parallel hyperplanes sweep through M.

Q.E.D.

Set $DK = \frac{1}{2}(K - K)$, and denote by $\|\cdot\|_{DK}$ the norm with respect to the o-symmetric convex body DK. Observe that the generalized kissing number $N_\alpha(K)$ is the maximal number m of points $x_1, \dots, x_m$ on the boundary of $K - \alpha K$ such that $x_1 + \alpha\, DK, \dots, x_m + \alpha\, DK$ is a packing.

Theorem 1

$$N_\alpha(C) \leq \frac{(1 + 2\alpha)^n - 1}{\alpha^n}.$$

Equality holds if and only if C is a parallelotope and $\alpha = \frac{1}{k}$ for a positive integer k.

Proof: Let $x_1, \dots, x_m \in \partial(C - \alpha C)$ such that $\|x_i - x_j\|_{DK} \geq 2\alpha$ for $i \neq j$. Set

$$M = \left(1 - \frac{\alpha}{2}\right) \cdot K - \frac{\alpha}{2} \cdot K.$$

Then the sets $x_i + \alpha \cdot DK$ do not overlap, even do not overlap M, and each is contained in $M + 2\alpha \cdot DK$. On the other hand, the general Brunn–Minkowski inequality (see [5]) yields that

$$V(M, DK; i) \leq V(M, DK; i), \quad i = 1, \dots, n. \tag{1}$$

Combining these two facts results in the inequalities

$$n \leq \frac{V(M+2\alpha \cdot DK) - V(M)}{\alpha^d \cdot V(DK)} \leq \frac{V(DM+2\alpha \cdot DK) - V(DM)}{\alpha^d \cdot V(DK)}.$$

Therefore the upper bound of the Theorem is a consequence of $DM = DK$.

Assume that equality holds in the upper bound. Then equality holds also for DK, and hence $\{x_i + \alpha \cdot DK\}$ and DK together tile $DK + 2\alpha \cdot DK$. Since DK lies in the interior and it can be separated by hyperplanes from the translates of $\alpha\, DK$, we deduce that DK is a polytope.

Let F be a facet of DK. Then the facet $(1+2\alpha)F$ of $(1+2\alpha)DK$ is tiled by some translates of $\alpha\, F$. We deduce from Lemma 1 that $F = P \times M$ for some parallelotope P, $\dim P \geq 1$, and the tiling of $(1+2\alpha)F$ is determined by the facet to facet tiling of $\alpha\, P$. Now the width of $(1+2\alpha)F$ is larger than the width of $\alpha\, F$ in any direction, therefore $F = P$ is an $(n-1)$–parallelotope.

It follows that $\frac{1+2\alpha}{\alpha}$ is an integer, and hence $\alpha = \frac{1}{k}$ for some positive integer k. On the other hand, there is a translate of $\alpha\, F$ in the tiling of $(1+2\alpha)F$, which lies in the interior of $(1+2\alpha)F$. Since this tile can be separated from the other tiles, and the same holds for the corresponding translates of αDK, we deduce that each pair of opposite $(n-2)$–faces of F is contained a pair of parallel supporting hyperplanes of DK. Thus DK is a parallelotope, which in turn yields that K is a parallelotope, as well.

Q.E.D.

Remark: If $\frac{1}{k+1} < \alpha \leq \frac{1}{k}$ for some positive integer k then the generalized kissing number of the n–cube I_n is

$$N_\alpha(I_n) = k^n \left[\left(1 + \frac{2}{k}\right)^n - 1 \right].$$

This can be shown by choosing n pairwise non–parallel facets of $(1+2\alpha)I_n$, and moving step by step the translates of αI_n as close to these facets as possible.

Next we give a lower bound for N_α which is exponential in n, assuming that α is not too large.

We say that the o–symmetric convex sets M and N are ε–close to each other if $(1+\varepsilon)^{-1}M \subset N \subset (1+\varepsilon)M$. The argument for the exponential lower bound is based on the celebrated QS–theorem of V.D. Milman (QS=quotient and section, see V.D. Milman [13]), which we quote here as

Lemma 2 *For every $\varepsilon > 0$ there exists a $c(\varepsilon) > 0$ with the following property: Let C be an o–symmetric convex body in $\mathbb{R}^n$, n large. Then there exist linear spaces $L \subset H$ such that $d \geq c(\varepsilon) \cdot n$ holds for $d = \dim L$, and the orthogonal projection of $H \cap C$ into L is ε–close to some ellipsoid in L.*

Let $1 \le \alpha < \sqrt{2}$, and choose some $\varepsilon > 0$ such that $(1+\varepsilon)\alpha < \sqrt{2}$. Shannon [17] and Wyner [21] proved that there exists some positive c_1 such that at least $2^{c_1 n}$ points can arranged on S^{d-1} such that the distance between any two is at least $(1+\varepsilon)\alpha$.

Now let K be any convex body in $\mathbb{R}^n$, n large, and set $C = K - K$. We use the notation of Lemma 2 for C and ε. Denote by C' and by K' the projections of C and K into H, respectively. We may assume by affine invariance that

$$B^d \subset L \cap C' \subset (1+\varepsilon) \cdot B^d$$

where B^d is the unit ball in L. Choose $x_1, \ldots, x_m \in S^{d-1}$ such that the distance between any two is at least $(1+\varepsilon)\alpha$ and $m \ge 2^{c_1 d}$. For any x_i, let $y_i = t_i x_i$ be the point of $\partial K' - \alpha K'$ for the suitable $t_i \ge 1$, and hence

$$\|y_i - y_j\|_{C'} \ge \frac{1}{1+\varepsilon} \cdot \|y_i - y_j\| \ge \frac{1}{1+\varepsilon} \cdot \|x_i - x_j\| \ge \alpha$$

for any $i \ne j$. We deduce that if $z_i \in \partial K - \alpha K$, $i = 1, \ldots, m$, projects onto y_i then $\|z_i - z_j\|_{DK} \ge 2\alpha$ for any $i \ne j$. Therefore choosing $c_2 = c_1 \cdot c(\varepsilon)$, we deduce that

Theorem 2 *If $\alpha < \sqrt{2}$ then there exists a positive constant c_2 (depending only on α) such that*

$$N_\alpha(K) \ge 2^{c_2 n} 4$$

holds for any convex body K in $\mathbb{R}^n$.

Finally, we show by the example of Reuleaux polygons that $N_\alpha(K)$ can be both smaller or larger than $N_\alpha(DK)$.

Example: *The case of the Reuleaux polygons.*

Let P be the regular m–gon with diameter 2 where m is odd. Add the shorter circular arcs with radius two connecting neighbouring vertices, and this way we have constructed a Reuleaux polygon, which is a convex domain $K = K_m$ of constant width 2. In particular, DK is the unit disc, and $\|\cdot\|_{DK}$ is the Euclidean norm.

Choose $\alpha_m > 0$ so that m non–overlapping circles of radius α_m can be placed around the unit circle in a way that each touches the unit circle and its two neighbours. Then

$$\alpha_m = \frac{\sin\frac{\pi}{m}}{1 - \sin\frac{\pi}{m}}.$$

Assume that that the origin is the center of the circle around P, and denote the vertices of P by $v_1, \ldots, v_m$. We consider $\lambda v_i + \alpha_m K$, $i = 1, \ldots, m$, where $\lambda > 1$

is chosen so that $\lambda v_i + \alpha_m C$ touches K at v_i. Some elementary calculation shows that $\lambda \|v_i - v_j\| > 2\alpha$. Therefore there exists some $\varepsilon_m > 0$ such that $N_\alpha(K) \geq m$ holds for

$$\alpha = \frac{\sin\frac{\pi}{m}}{1-\sin\frac{\pi}{m}} + \varepsilon_m.$$

On the other hand, $N_\alpha(DK) \leq m-1$, and hence

$$N_\alpha(K_m) > N_\alpha(DK_m) \quad \text{if} \quad \alpha = \frac{\sin\frac{\pi}{m}}{1-\sin\frac{\pi}{m}} + \varepsilon_m, \quad m \text{ odd}.$$

Next consider the case if α is very small. We claim that if l is large then

$$N_\alpha(K_m) < N_\alpha(DK_m) \quad \text{if} \quad \alpha = \frac{\sin\frac{\pi}{l}}{1-\sin\frac{\pi}{l}}. \tag{2}$$

The value of α is chosen so that that the regular l–gon with side length 2α has circumradius $1+\alpha$, and hence $N_\alpha(DK) = l$.

The boundary of $K - \alpha K$ consists of m circular arcs with radius 2 and with length $\frac{2\pi}{m}$, and m circular arcs with radius 2α near the vertices of K, which have length $2\alpha \cdot \frac{\pi}{m}$. Assume that there exists l points $x_1, \ldots, x_l$ on $\partial(K - \alpha K)$ such that the distance between any two x_i's is at least 2α. To each vertex of C there correspond a pair of consecutive x_i's such that the shorter arc σ of $\partial(K - \alpha K)$ connecting them contains at last half of the cicular arc with radius 2α. Therefore there exists a $\tau > 0$, depending on l, such that the length of σ is at least $(2+\tau)\cdot\alpha$. We deduce that

$$P(K - \alpha K) \geq (l-m)\cdot 2\alpha + m\cdot(2+\tau)\,\alpha \geq l\cdot 2\alpha + m\,\tau\cdot\alpha.$$

On the other hand, the definition of α yields that

$$P(K - \alpha K) = P((1+\alpha)DK) = l\cdot 2\alpha + O\left(\alpha^2\right),$$

which is absurd for large n. In turn, the claim (2) follows.

Problem 1: If $\alpha < \sqrt{2}+1$ then does $N_\alpha(K) > 2^{c_3 n}$ for any convex body K in $\mathbb{R}^d$ where $c_3 > 0$ depends only on α?

3 Asymptotics for the kissing number

Let K be a convex body in $\mathbb{R}^n$. When integrating on ∂K, we always do it with respect to the Hausdorff $(n-1)$–measure (see Falconer [7] or Rogers [16]), normalizing it so that it coincides with the Lebesgue $(n-1)$–measure for subsets of $\mathbb{R}^{n-1}$. The $(n-1)$–measure of a set σ is denoted by $|\sigma|$. Note that if the Lipschitz condition

$$\|f(x) - f(y)\| \leq c \cdot \|x - y\|$$

holds for the map f of σ then

$$|f(\sigma)| \leq c^{n-1} \cdot |\sigma|. \tag{3}$$

When considering a subset σ of ∂K, we speak about open and closed sets, meaning the topology on ∂K. In particular, σ is called Jordan measurable if the $(n-1)$–measure of its boundary is zero.

A very useful tool when subdividing the boundary of C is the Vitali covering theorem (see Falconer [7]). We need only the following version:

Lemma 3 *Let σ be an open, Jordan measurable subset of the boundary of the convex body K, and consider a family $\{U_x\}_{x\in\sigma}$ of open subsets of σ where $x \in U_x$ for $x \in \sigma$. covering. Then for any $\varepsilon > 0$ there exist finitely many closed, Jordan measurable and pairwise disjoint subsets $\{\sigma_i\}$ of σ such that any σ_i is contained in some U_x, and the part of σ not covered by $\{\sigma_i\}$ has $(d-1)$–measure at most ε.*

Remark: In order to obtain a Vitali class for σ in the sense of Falconer [7], one considers for each $x \in \sigma$ the family of closures of the open, Jordan measurable neighbourhoods of x whose closures contained in σ.

If C is an $(n-1)$–dimensional convex body then denote by $\delta(C)$ the maximal density of an infinite packing of translates of C (see Rogers [15]). We have

$$2^{-(n-1)} \leq \delta(C) \leq 1. \tag{4}$$

For any convex body M in $\mathbb{R}^{n-1}$, the maximal number $m(\alpha)$ of non–overlapping translates of $\alpha\, C$ contained in M satisfies

$$m(\alpha) \sim \frac{\delta(C)}{|C|} \cdot |M| \cdot \frac{1}{\alpha^{n-1}} \tag{5}$$

as α tends to zero.

Recall that $N_\alpha(K)$ is the maximal number m of points $x_1, \ldots, x_m$ on the boundary of $K - \alpha K$ such that $x_1 + \alpha\, DK, \ldots, x_m + \alpha\, DK$ is a packing. The main result of this section is

Theorem 3 *Let K be a convex body in $\mathbb{R}^n$, and choose an exterior unit normal $u(x)$ at each $x \in \partial K$. Then as α tends to zero, we have*

$$N_\alpha(K) \sim \int_{\partial K} \frac{\delta\left(u(x)^\perp \cap DK\right)}{|u(x)^\perp \cap DK|}\, dx \cdot \frac{1}{\alpha^{n-1}}.$$

Remark: Assume K is a polytope. Then the number of translates of αK around the $(n-2)$–faces of K is $O\left(\alpha^{-(n-2)}\right)$. Therefore in this case, Theorem 3 follows directly from (5) (see L. Fejes Tóth [10]).

Let $\varepsilon > 0$. The statement we actually prove is the following one: There exists $\alpha_0 > 0$ such that if $0 < \alpha < \alpha_0$ then

$$N_\alpha(K) = (1 + O(\varepsilon)) \cdot \int_{\partial K} \frac{\delta\left(u(x)^\perp \cap DK\right)}{|u(x)^\perp \cap DK|}\, dx \cdot \frac{1}{\alpha^{n-1}}, \tag{6}$$

where the implied constant in $O(\cdot)$ depends only on n (α_0 does depend also on K and ε). Observe that (6) yields Theorem 3.

Large part of the proof for (6) consists of constructing flat patches on the boundary of K.

Since both $N_\alpha(K)$ and the integral expression above are affine invariant, we may assume by the John's theorem [14] that

$$B^n \subset K \subset n \cdot B^n. \tag{7}$$

In particular, if u is an exterior unit normal at an $x \in \partial K$ then

$$\langle u, x\rangle \geq \tfrac{1}{n} \cdot \|x\|. \tag{8}$$

Elementary considerations yield that if $\langle u, v\rangle \geq \sqrt{1-h^2}$ and $\langle v, w\rangle = 0$ hold for $0 < h < 1$ and for the unit vectors u, v, w then

$$|\langle u, w\rangle| \leq h. \tag{9}$$

So fix a $0 < h < \varepsilon$ satisfying the condition (i) below. Note that (ii) is the consequence of (i), (8) and (9).

(i) For any $u \in S^{n-1}$, $x \in \partial K$ and $y \in u^\perp$, if

$$|\langle u, x\rangle| \leq h \cdot \|x\| \quad \text{and} \quad |\langle u, x-y\rangle| \geq \tfrac{1}{2n} \cdot \|x-y\|$$

then we have

$$(1+\varepsilon)^{-1} \leq \|y\|_{DK} \leq 1+\varepsilon.$$

(ii) For any unit vectors u and v, if $\langle u, v\rangle \geq \sqrt{1-h^2}$ then there exists a congruent copy of $u^\perp \cap DK$ which is $O(\varepsilon)$–close to $v^\perp \cap DK$.

Let us turn to the boundary of ∂K. Denote by σ_h the set of points $x \in \partial K$ such that there exist two exterior unit normals u and v satisfying $\langle u, v\rangle \leq \sqrt{1-\frac{1}{2}h^2}$. Then σ_h is closed, and has zero $(n-1)$–measure (for the later property, see say [18]). Therefore each $x \in \partial K \backslash \sigma_h$ has an open neighbourhood U_x such that $U_x \cap \sigma_h = \emptyset$, and if u and v are exterior unit normals at some points of U_x then $\langle u, v\rangle \geq \sqrt{1-h^2}$. We deduce by Lemma 3 that there exist closed, Jordan measurable and pairwise disjoint subsets $\sigma_1, \ldots, \sigma_k$ of ∂K with non–empty interior such that if u and v are exterior unit normals at some points of σ_i then $\langle u, v\rangle \geq \sqrt{1-h^2}$, and the part of ∂K not covered by $\sigma_1, \ldots, \sigma_k$ has $(n-1)$–measure at most ε.

Choose $\varrho > 0$ with the property that the set of points $x \in \partial K$ whose distance from $\Sigma = \partial K \backslash (\sigma_1 \cup \ldots \cup \sigma_k)$ is at most 4ϱ, has measure at most 2ε.

Denote by $\pi_{\partial K}$ the radial projection into ∂K. For each σ_i, let P_i be a possibly unbounded polyhedron such that each facet of P_i touches K at some point of of σ_i, and for any $x \in \sigma_i$ the ray $\operatorname{pos}\{x\}$ intersects the boundary of P_i in some point y which satisfies

$$\|x\| \leq \|y\| \leq (1+\varepsilon) \cdot \|x\|. \tag{10}$$

Subdivide ∂P_i into a family of $(d-1)$–polytopes $\mathcal{F}_i$ such that the radial projection of σ_i into ∂P_i meets only finitely many elements of σ_i. In addition, we assume that if F is one of these finitely many polytopes, then the radial projection Π of F into ∂K has diameter at most ϱ, and

$$\langle u, x\rangle \geq \tfrac{1}{2n} \cdot \|x\| \tag{11}$$

holds for the exterior unit normal u of F and for any $x \in \Pi$. Let $\{F_1, \ldots F_l\}$ be the subfamily of the the union of the $\mathcal{F}_i$'s, characterized by the fact that if $F_j \in \mathcal{F}_i$ then the projection Π_j of F_j into ∂K has distance at least ϱ from Σ.

Now for each F_j, choose a $(d-1)$–polytope F'_j such that $F'_j \subset \operatorname{int} F_j$, and $|F_j| < (1+\varepsilon) \cdot |F'_j|$. Denote the radial projection of F'_j into ∂K by Π'_j, and the the exterior unit vector for F_j by u_j.

We deduce by (4) and (7) that the area and the density of the central sections of DK are bounded by positive constants depending on n. Therefore

$$\sum_j \int_{\Pi'_j} \frac{\delta\left(u(x)^\perp \cap DK\right)}{|u(x)^\perp \cap DK|}\, dx = (1+O(\varepsilon)) \cdot \int_{\partial K} \frac{\delta\left(u(x)^\perp \cap DK\right)}{|u(x)^\perp \cap DK|}\, dx. \tag{12}$$

On the other hand, if u is an exterior unit nurmal at a point $x \in \Pi_j$ then $\langle u_j, u\rangle \geq \sqrt{1-h^2}$. We deduce from (ii) and (10) that

$$\int_{\Pi'_j} \frac{\delta\left(u(x)^\perp \cap DK\right)}{|u(x)^\perp \cap DK|}\, dx = (1+O(\varepsilon)) \cdot \frac{\delta(u_j^\perp \cap DK)}{|u_j^\perp \cap DK|} \cdot |F'_j|. \tag{13}$$

Let $\Pi_j(\alpha)$ be the family of points on the boundary of $K - \alpha K$ which are contained in $x - \alpha K$ for some $x \in \Pi_j$, and define $\Pi'_j(\alpha)$ analogously. Observe that if α is small then the radial projection of F'_j into $\partial(K - \alpha K)$ is contained in $\Pi'_j(\alpha)$, while the radial projection of $\Pi'_j(\alpha)$ into $\operatorname{aff} F_j$ is contained in F_j.

Now if $F_j \in \mathcal{F}_i$ then $\operatorname{aff} F_j$ touches σ_i. We deduce from the definition of σ_i that

$$\langle u, u_j\rangle \geq \sqrt{1-h^2}$$

holds for any unit vector u which is the exterior normal at some $x \in \Pi_j(\alpha)$.

Consider two points $y,z \in \Pi_j(\alpha)$, which are either contained in $\Pi'_j(\alpha)$ or in the radial projection of F'_j. In both cases, the 2–cone $\mathrm{pos}\{y,z\}$ intersects the boundary of $K-\alpha K$ in an arc, which is actually contained in $\Pi_j(\alpha)$. Assume that p is a farthest point of this arc from aff $\{y,z\}$, and hence there exists an exterior unit normal v at p satisfying $\langle v, z-y\rangle = 0$. We deduce from (9) that

$$|\langle u_j, z-y\rangle| \leq h \cdot \|z-y\|.$$

Denoting by y' and z' the radial projection of y and z into F_j, respectively, the properties (i) and (10) yield that

$$(1+\varepsilon)^{-2} \cdot \|y-z\|_{DK} \leq \|y'-z'\|_{DK} \leq (1+\varepsilon)^2 \cdot \|y-z\|_{DK}. \tag{14}$$

First we prove that

$$N_\alpha(K) \geq (1+O(\varepsilon)) \cdot \int_{\partial K} \frac{\delta\left(u(x)^\perp \cap DK\right)}{|u(x)^\perp \cap DK|}\, dx \cdot \frac{1}{\alpha^{n-1}}. \tag{15}$$

For small α, place $m_j(\alpha)$ non–overlapping translates $x'_t + (1+\varepsilon)^2 \cdot DK$, $t = 1,\ldots,m_j(\alpha)$, into F'_j with

$$m_j(\alpha) \geq (1+O(\varepsilon)) \cdot \frac{\delta(u_j^\perp \cap DK)}{|u_j^\perp \cap DK|} \cdot |F'_j| \cdot \frac{1}{\alpha^{n-1}},$$

which is possible by (5). Denote by x_t the radial projection of x'_t into $\Pi_j(\alpha)$, and hence $\|x_t - x_s\|_{DK} \geq 2\alpha$ for any $t \neq s$ by (14). Therefore (12) and (13) yield (15).

Next, let us verify

$$N_\alpha(K) \leq (1+O(\varepsilon)) \cdot \int_{\partial K} \frac{\delta\left(u(x)^\perp \cap DK\right)}{|u(x)^\perp \cap DK|}\, dx \cdot \frac{1}{\alpha^{n-1}}. \tag{16}$$

In this case, we use a basic fact on curvature measures (see Schneider [18]): if $\mathcal{N}(\Sigma, r)$ denotes the the set of points $y \in \mathbb{R}^n$ such that the closest point x of K to y lies in Σ and $d(x,y) \leq r$ then

$$V\left(\mathcal{N}(\Sigma, r)\right) \sim |\Sigma| \cdot r$$

holds as r tends to zero.

Assume that $x_1,\ldots,x_m \in \partial(K-\alpha K)$ satisfy $\|x_t - x_s\|_{DK} \geq 2\alpha$ for any $t \neq s$. Call an x_t bad if it is not contained in any of the $\Pi'_j(\alpha)$'s. Observe that if α is small and x_t is bad then for any $y \in x_t + \alpha_D K$, the closest point of K to y lies in Σ. In particular, if α is small then the number m_0 of bad points satisfies

$$m_0 \leq \frac{V\left(\mathcal{N}(\Sigma, 2\alpha)\right)}{V\left(\alpha\, DK\right)} = O(\varepsilon) \cdot \frac{1}{\alpha^{n-1}}. \tag{17}$$

Now denote by $m_j(\alpha)$ the number of x_t's in $\Pi'_j(\alpha)$. If $x_s, x_t \in \Pi'_j(\alpha)$ and $s \neq t$ then their radial projections x'_s and x'_t into F_j satisfy

$$\|x_t - x_s\|_{DK} \geq (1+\varepsilon)^{-2} \cdot 2\alpha$$

by (14). Therefore (5) yields for small α that

$$m_j(\alpha) \leq (1+O(\varepsilon)) \cdot \frac{\delta(u_j^{\perp} \cap DK)}{|u_j^{\perp} \cap DK|} \cdot |F_j| \cdot \frac{1}{\alpha^{n-1}}.$$

We conclude (16) by (12) and (13), which finally completes the proof of Theorem 3. Q.E.D.

Define

$$\Omega(K) = \int_{\partial K} \frac{\delta\left(u(x)^{\perp} \cap DK\right)}{|u(x)^{\perp} \cap DK|}\, dx.$$

$\Omega(K)$ is very closely related to Buseman's notion of surface area (see [20]) of a convex body M with respect to DK; namely,

$$S_{DK}(M) = \frac{1}{|B^{n-1}|} \cdot \int_{\partial M} \frac{1}{|u(x)^{\perp} \cap DK|}\, dx.$$

In particular, $S_{B^n}(\cdot)$ is the ordinary surface area $S(\cdot)$. Note that there exists an o–symmetric convex body I^*DK such that $S_{DK}(M) = V(M, I^*DK; 1)$.

We use Buseman's notion of surface area to determine the maximum of $\Omega(K)$. First, (1) yields that $S_{DK}(M) \leq S_{DK}(DM)$. On the other hand, $S_{DK}(DK)$ is maximized if and only if DK is a paralellotope (see [20]). Since this yields that K is also a parallelotope, we conclude that $\Omega(K)$ is maximized by paralellotopes, as it is expected by Theorem 1.

Remark: Let $n \geq 3$, and consider a convex body K of constant width 2 in $\mathbb{R}^n$, different from the unit ball. Then DK is the unit ball, and

$$\Omega(K) = \frac{\delta(B^{d-1})}{|B^{d-1}|} \cdot S(K),$$

where $S(K)$ is the ordinary surface area. Since K is not centrally symmetric, it has an $(n-1)$–dimensional projection which is not centrally symmetric. We deduce from the Cauchy formula and the Brunn–Minkowski inequality for the projections (see [2]) that $S(K) < S(DK)$. In particular, Theorem 3 yields that

$$N_\alpha(K) < N_\alpha(DK)$$

holds when α is small.

4 The Blocking number

The definitions of the kissing number and the blocking number yield that

$$B_\alpha(K) \le N_\alpha(K)$$

for every convex body K and for any positive α. Naturally,

$$B_1(K) = B_1(D(K)).$$

Similarly as $N_\alpha(K)$ is related to packing in $\mathbb{R}^{n-1}$ for small α, the blocking number $B_\alpha(K)$ is related to covering in $\mathbb{R}^{n-1}$. If C is an $(n-1)$–dimensional o–symmetric convex body then denote by $\vartheta(C)$ the minimal density of an infinite covering of translates of C (see Rogers [15]). Unfortunately, we have to work with a more complicated notion; namely, let $\theta(C)$ be the infimum of the density of the arrangements $\{x_i + C\}$ such that $\{x_i + \operatorname{int} C\}$ cover $\mathbb{R}^{n-1}$, while $\{x_i + \frac{1}{2}C\}$ is a packing. In other words, $\frac{1}{2^n}\theta(C)$ is the infimum of the density of maximal translative packings of $\frac{1}{2}C$. Readily $\theta(C) \ge \vartheta(C)$. Observe that $\vartheta(C)$ is readily continuous in C, but we do not know this about $\theta(C)$. In the plane, a loosest covering by translates of C can be obtained by hexagonal lattice coverings (see [9]). Therefore there exists a loosest covering $\{x_i + C\}$ of the plane such that $\{x_i + (\frac{1}{2} + \varepsilon)\, C\}$ form a packing where $\varepsilon > 0$ depends on C. In particular, we can apply the argument for Theorem 3 in $\mathbb{R}^3$, and we deduce

Theorem 4 *Let K be a convex body in $\mathbb{R}^3$, and choose an exterior unit normal $u(x)$ at each $x \in \partial K$. Then as α tends to zero, we have*

$$B_\alpha(K) \sim \int_{\partial K} \frac{\vartheta\left(u(x)^\perp \cap DK\right)}{4 \cdot |u(x)^\perp \cap DK|}\, dx \cdot \frac{1}{\alpha^{n-1}}.$$

The technical difficulties do not arise in the case of polytopes. Threfore if P is a polytope in $\mathbb{R}^n$, $n \ge 3$, with facets $F_1, \ldots, F_m$ and facet normals $u_1, \ldots, u_m$ then

$$B_\alpha(P) \sim \sum_{F_i} \frac{\theta(u_i^\perp \cap DP)}{2^{n-1} \cdot |u_i^\perp \cap DP|} \cdot |F_i| \cdot \frac{1}{\alpha^{n-1}}. \tag{18}$$

In the case of the tetrahedron, simple calculations show that $B_\alpha(T) < B_\alpha(DT)$ if α is small.

Problem 2: Can you give for any $n \ge 3$ a convex body K in $\mathbb{R}^n$ such that $B_\alpha(K) \ne B_\alpha(DK)$ holds for some α? Note that in $\mathbb{R}^2$, if K is a regular Reuleaux polygon then $B_\alpha(K) < B_\alpha(D(K))$ holds for infinitely many small α.

Clearly the unit cube I_n is a very special case. Since $B_\alpha(I_n)$ changes rather regularly, we deal with this case in full detail.

Theorem 5 *If* $0 < \alpha \leq 1/2$, *then*

$$B_\alpha(I_n) = \begin{cases} 2(k+1)^n - 2k^n & \textit{if } \frac{1}{2k+1} < \alpha \leq \frac{1}{2k}, \\ 2n(k+1)^{n-1} & \textit{if } \frac{1}{2k+2} < \alpha \leq \frac{1}{2k+1}. \end{cases}$$

In order to prove Theorem 4 we study $B'_\alpha(I_n)$, the similar number without the nonoverlapping restriction. Clearly we have

$$B_\alpha(I_n) \leq B_\beta(I_n) \tag{19}$$

and

$$B'_\alpha(I_n) \leq B'_\beta(I_n) \tag{20}$$

for $0 < \beta \leq \alpha$, and

$$B'_\alpha(I_n) \leq B_\alpha(I_n). \tag{21}$$

In addition, according to an observation of Zong [22] (see also [23]), to prove $\alpha I_n + X$ is a blocking configuration it is sufficient to prove

$$\partial((1+\alpha)I_n) \subset \bigcup_{\mathbf{x}\in X} (\mathrm{int}(2\alpha I_n) + \mathbf{x}).$$

Thus, we have

$$B'_\alpha(I_n) = G_\alpha(I_n), \tag{22}$$

where $G_\alpha(I_n)$ indicates the smallest number of translates $\mathrm{int}(2\alpha I_n)+\mathbf{x}$, $\mathbf{x} \in \partial((1+\alpha)I_n)$, which can cover $\partial((1+\alpha)I_n)$. It is clear that, for $0 < \beta \leq \alpha$,

$$G_\alpha(I_n) \leq G_\beta(I_n). \tag{23}$$

Now we introduce three technical lemmas.

Lemma 4 *Let* α *be a positive number such that* $\alpha \leq 1/2$, *and let* m *be the smallest integer such that* $m\alpha > 1+\alpha$, *i.e.* $\alpha > 1/(m-1)$. *Then,*

$$G_\alpha(I_n) = G_\beta(I_n)$$

for $\alpha \geq \beta > 1/(m-1)$.

Proof: Let X be a set of points such that

$$\partial((1+\alpha)I_n) \subset \bigcup_{\mathbf{x}\in X} (\mathrm{int}(2\alpha I_n) + \mathbf{x}),$$

and let ϵ be a small positive number. Without loss of generality, we assume that X belong to the union of the interiors of the $2n$ facets of $(1+\alpha)I_n$. Denote by F the interior of the facet $\{\mathbf{x} \in (1+\alpha)I_n : \ x_n = (1+\alpha)/2\}$, and write

$$\Phi = \partial((1+\alpha)I_n) \setminus \{F \cup \{-F\}\}$$

and

$$X^* = \{\mathbf{x} \in X : \ \Phi \cap \mathrm{int}(2\alpha I_n + \mathbf{x}) \neq \emptyset\}.$$

Furthermore, we write

$$I(\mu, \nu) = \{\mathbf{x} : \ |x_i| \leq \mu \ for \ 1 \leq i \leq n-1; \ -\mu \leq x_n \leq \nu\},$$

$$\mathbf{u} = (0, 0, \ldots, 0, -1),$$

and enumerate the points of $X^* \setminus F$ as $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_l$ (where $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n})$) such that

$$x_{i+1,n} \leq x_{i,n}$$

for all $i = 1, 2, \ldots, l-1$. Now we introduce an inductive process to adjust the points of $X^* \setminus F$.

First, let r_1 be the maximum of the numbers r such that

$$\Phi \subset \mathrm{int}(I(\alpha, \alpha - \epsilon) + \mathbf{x}_1 + r\mathbf{u}) \cup \bigcup_{\mathbf{x} \in X^* \setminus \{\mathbf{x}_1\}} (\mathrm{int}(2\alpha I_n) + \mathbf{x}).$$

Then, replace $\mathbf{x}_1$ by $\mathbf{x}_1 + r_1\mathbf{u}$. Assume that the first $i-1$ points of $X^* \setminus F$ have been adjusted, and assume that r_i is the maximum of the numbers r such that

$$\Phi \subset \mathrm{int}(I(\alpha, \alpha - \epsilon) + \mathbf{x}_i + r\mathbf{u}) \cup \bigcup_{\mathbf{x} \in X^* \setminus \{\mathbf{x}_i\}} (\mathrm{int}(2\alpha I_n) + \mathbf{x}),$$

where X^* is the updated set for the first $i-1$ steps. Then replace $\mathbf{x}_i$ by $\mathbf{x}_i + r_i\mathbf{u}$. After l steps we obtain a new set X^*. This process produces many chains

$$\mathbf{x}_{j_1}, \mathbf{x}_{j_2}, \ldots, \mathbf{x}_{j_{f(j)}}$$

in X^* such that $x_{j_1,n} = (1+\alpha)/2$ or $(1-\alpha)/2 + \epsilon$,

$$x_{j_{i-1},n} - x_{j_i,n} = 2\alpha - \epsilon$$

for $2 \leq i \leq f(j)$, and

$$(\mathrm{int}(2\alpha I_n) + \mathbf{x}_{j_{f(j)}}) \cap \{-F\} \neq \emptyset.$$

In addition, every point $\mathbf{x} \in X^*$ with $\{-F\} \cap \mathrm{int}(2\alpha I_n + \mathbf{x}) \neq \emptyset$ is the last point of some of these chains. Thus, for the new X^*,

$$\bigcup_{\mathbf{x} \in X^*} \mathrm{int}(2\alpha I_n + \mathbf{x})$$

covers the corresponding Φ of

$$P=\Big\{\mathbf{x}:\ |x_i|\le\tfrac{1+\alpha}{2}\ for\ 1\le i\le n-1;$$
$$\tfrac{1+\alpha}{2}-m\alpha+l\epsilon\le x_n\le\tfrac{1+\alpha}{2}\Big\}.$$

This rectangular parallelepiped has height $m\alpha-l\epsilon$ in the direction of $\mathbf{u}$. We define

$$x'_{i,n}=\begin{cases}\frac{1+\beta}{2} & \text{if } \mathbf{x}_i\in X\cap F,\\ -\frac{1+\beta}{2} & \text{if } \mathbf{x}_i\in X\cap\{-F\},\\ \frac{1+\beta}{2}+\frac{\beta}{\alpha}(x_{i,n}-\frac{1+\beta}{2}) & \text{if } \mathbf{x}_i\in X\setminus\{F\cup\{-F\}\},\end{cases}$$

and the corresponding set

$$X'=\{(x'_{i,1},x'_{i,2},\ldots,x'_{i,n}):\ x'_{i,j}=x_{i,j}\ for\ 1\le j\le n-1\}.$$

It can be verified that

$$\partial((1+\beta)I_n)\subset\mathrm{int}(I(\alpha,\beta)+X'.$$

Repeating this process with respect to all coordinates, proves Lemma 4. Q.E.D.

Lemma 5 *Let k be a positive integer, then*

$$G_\alpha(I_n)\ge\begin{cases}2(k+1)^n-2k^n & if\ \alpha=\frac{1}{2k},\\ 2n(k+1)^{n-1} & if\ \alpha=\frac{1}{2k+1},\end{cases}$$

Proof: We deal with the two cases by different methods.

Case 1. $\alpha=1/2k$. In this case we proceed to choose a centrally symmetric set $Y_n\subset\partial((1+\alpha)I_n)$ such that

$$\mathrm{card}\{Y_n\}=2(k+1)^n-2k^n$$

and, for any $\mathbf{x}\in\partial((1+\alpha)I_n)$, $\mathrm{int}(2\alpha I_n)+\mathbf{x}$ contains at most one point of Y_n. For this purpose we apply induction on dimension. As usual, we use $\mathbf{e}_i$ to indicate the i-th normalized base.

When $n=2$, writing $\mathbf{v}=-((1+\alpha)/2,(1+\alpha)/2)$,

$$Y_2^*=\{\mathbf{v}+2j\alpha\mathbf{e}_1:\ 0\le j\le k\},$$

$$Y_2^\star=\{\mathbf{v}+2j\alpha\mathbf{e}_2:\ 1\le j\le k\},$$

and

$$Y_2=Y_2^*\cup Y_2^\star\cup\{-Y_2^*\}\cup\{-Y_2^\star\},$$

it can be verified that Y_2 satisfies the conditions.

Assuming that the assertion is true in E^{n-1}, we proceed to prove it for E^n. In E^n, we take

$$\mathbf{v} = -\left(\tfrac{1+\alpha}{2}, \tfrac{1+\alpha}{2}, \ldots, \tfrac{1+\alpha}{2}, 0\right),$$

$$F = \{\mathbf{x} \in (1+\alpha)I_n : \ x_n = 0\},$$

and let Y_{n-1} be the set corresponding to F. Then, divide Y_{n-1} into two disjoint sets Y^*_{n-1} and $Y^\star_{n-1}$ such that the first belongs to the facets of F which contain $\mathbf{v}$, the second belongs to the facets of F which contain $-\mathbf{v}$, and

$$Y^*_{n-1} = -Y^\star_{n-1}.$$

Now, we define

$$Y^*_n = \left\{\mathbf{y} + \left(\tfrac{\alpha-1}{2} + 2j\alpha\right)\mathbf{e}_n : \ 0 \le j \le k-1, \ \mathbf{y} \in Y^*_{n-1}\right\},$$

$$Y'_n = \left\{\mathbf{y} : \ y_n = -\tfrac{1+\alpha}{2}, \ y_i = \tfrac{1+\alpha}{2} - 2j\alpha, \ 0 \le j \le k \ for \ i \ne n\right\},$$

and

$$Y_n = Y^*_n \cup Y'_n \cup \{-Y^*_n\} \cup \{-Y'_{n-1}\}.$$

It can be verified that, for any point $\mathbf{x} \in \partial((1+\alpha)I_n)$, the set $\mathrm{int}(2\alpha I_n) + \mathbf{x}$ contains at most one point of Y_n,

$$\mathrm{card}\{Y^*_n \cup \{-Y^*_n\}\} = 2k\left((k+1)^{n-1} - k^{n-1}\right),$$

$$\mathrm{card}\{Y'_n\} = (k+1)^{n-1},$$

and therefore

$$\begin{aligned}\mathrm{card}\{Y_n\} &= 2(k+1)^{n-1} + 2k\left((k+1)^{n-1} - k^{n-1}\right)\\ &= 2(k+1)^n - 2k^n.\end{aligned}$$

Case 1 follows.

Case 2. $\alpha = 1/(2k+1)$. For convenient, we write

$$F_i = \left\{\mathbf{x} : \ x_i = \tfrac{1+\alpha}{2}, \ |x_j| \le \tfrac{1-\alpha}{2}, \ j \ne i\right\},$$

$$X_i = \{\mathbf{x} \in X : \ (\mathrm{int}(2\alpha I_n) + \mathbf{x}) \cap F_i \ne \emptyset\}$$

and

$$X_{n+i} = \{\mathbf{x} \in X : \ (\mathrm{int}(2\alpha I_n) + \mathbf{x}) \cap \{-F_i\} \ne \emptyset\}.$$

It can be verified that

$$X_i \cap X_j = \emptyset$$

for $1 \le i < j \le 2n$, and

$$\mathrm{card}\{X_j\} \ge \left\lfloor \tfrac{1-\alpha}{2\alpha} + 1 \right\rfloor^{n-1} = (k+1)^{n-1}.$$

Thus, we have

$$\mathrm{card}\{X\} = \sum_{i=1}^{2n} \mathrm{card}\{X_i\} \ge 2n(k+1)^{n-1},$$

which proves the second case.

This proves Lemma 5. Q.E.D.

Lemma 6

$$B_\alpha(I_n) \le \begin{cases} 2(k+1)^n - 2k^n & \text{if } \frac{1}{2k+1} < \alpha \le \frac{1}{2k}, \\ 2n(k+1)^{n-1} & \text{if } \frac{1}{2k+2} < \alpha \le \frac{1}{2k+1}. \end{cases}$$

Proof: We deal with the two cases with different methods.

Case 1. $1/(2k+1) < \alpha \le 1/2k$. In this case we apply induction on the dimensions. Clearly, the assertion is true when $n = 1$. Assume it is true for $n-1$, and $\alpha I_{n-1} + X_{n-1}$ is an optimal blocking configuration. Let ϵ be a small positive number, and define

$$X_n^* = \bigcup_{j=0}^{k-1} \left(X_{n-1} + \left(\tfrac{1-3\alpha+2\epsilon}{2} - j\tfrac{1-3\alpha+2\epsilon}{k-1} \right) \mathbf{e}_n \right),$$

$$X_n^\star = \left\{ \mathbf{x} : \ x_n = \tfrac{1+\alpha}{2}, \ x_i = \tfrac{(k-2j)(2\alpha-\epsilon)}{2} \ for\ 0 \le j \le k \right\},$$

and

$$X_n = X_n^* \cup X_n^\star \cup \{-X_n^\star\}.$$

It can be verified that $\alpha I_n + X_n$ is a blocking configuration and

$$\mathrm{card}\{X_n\} = 2(k+1)^n - 2k^n.$$

Thus, in this case,

$$b_\alpha(I_n) \le 2(k+1)^n - 2k^n.$$

Case 2. $1/(2k+2) < \alpha \le 1/(2k+1)$. We proceed to show that there is a centrally symmetric set X of $2n(k+1)^{n-1}$ points such that $\alpha I_n + X$ is a packing, $\mathrm{int}(I_n) \cap (\alpha I_n + X) = \emptyset$, and $\alpha I_n + X$ can block any other translate of αI_n from touching I_n. It is clear that the assertion is true when $n = 1$. Assuming that it is true in E^{n-1}, we consider E^n.

In E^n we write

$$v = \left(\frac{1+\alpha}{2}, \frac{1+\alpha}{2}, \ldots, \frac{1+\alpha}{2}, 0 \right), \tag{24}$$

$$I_{n-1} = \{\mathbf{x} \in I_n : \ x_n = 0\},$$

and let X_{n-1} be a corresponding optimal set. Similar to the first case of Lemma 5, we divide X_{n-1} into X^*_{n-1} and $X^\star_{n-1}$ corresponding to $\mathbf{v}$ and $-\mathbf{v}$, respectively. Thus, let ϵ be a small positive number and define

$$X^*_n = \left\{\mathbf{x} + \left[j\tfrac{1-2\alpha+\epsilon}{k} - \tfrac{1}{2}(1-\alpha+\epsilon)\right]\mathbf{e}_n : \ \mathbf{x} \in X^*_{n-1}, \ 0 \le j \le k\right\},$$

$$X'_n = \left\{\mathbf{x} : \ x_n = \tfrac{1+\alpha}{2}, \ x_i = j\tfrac{1-2\alpha+\epsilon}{k} - \tfrac{1}{2}(1-3\alpha+\epsilon), \ 0 \le j \le k\right\},$$

and

$$X_n = X^*_n \cup X'_n \cup \{-X^*_n\} \cup \{-X'_n\},$$

it can be verified that X_n satisfies the requirement. Thus, in this case, we have

$$\begin{aligned} B_\alpha(I_n) &\le \operatorname{card}\{X_n\} = 2(\operatorname{card}\{X^*_n\} + \operatorname{card}\{X'_n\}) \\ &= 2n(k+1)^{n-1}. \end{aligned}$$

This estimate finally completes the proof of Lemma 6. Q.E.D.

Now Theorem 5 follows from (19)-(23), and the three lemmas.

Corollary 1

$$B_\alpha(I_n) = B'_\alpha(I_n) = G_\alpha(I_n).$$

Let $G'_\alpha(I_n)$ be the analogue of $G_\alpha(I_n)$, without the restriction $\mathbf{x} \in \partial((1+\alpha)I_n)$. In other words, $G'_\alpha(I_n)$ is the smallest number of translates αI_n which can block any other from touching I_n. Similar to Lemma 1, one can prove that, as a function of α, $G'_\alpha(I_n)$ changes its values only at the rational numbers of $1/2k$ type. More explicitly, we have the following result about $G'_\alpha(I_n)$. Then the proof of Theorem 5 shows that if $1/2(k+1) < \alpha \le 1/2k$ then

$$G'_\alpha(I_n) = (k+1)^n - (k-1)^n. \tag{25}$$

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