Stability of some inequalities for three–polyhedra

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1 The isoperimetric inequality for three polyhedra

It is well–known that among convex bodies of given volume in the three space $\mathbb{E}^3$, the surface area is minimal for the Euclidean ball. Soon after verifying this result in the 19th century, attention turned towards the related question about polyhedra: Among the polyhedra with given volume and given number of faces find those which have minimal surface area. In this generality the question is very difficult, however already Steiner conjectured that if the number of faces is four, six, or twelve, then the solution of the problem is the regular tetrahedron, the cube and the regular dodecahedron, respectively. Using the symmetrization bearing his name today, Steiner himself conformed the conjecture for the tetrahedron. The case of the cube was settled by Steinitz [22], and Goldberg [14] proved that the regular dodecahedron is extremal among all polytopes with twelve faces. (A gap in Goldberg's argument was filled by Florian [11]). An elegant treatment of all three cases giving the first complete proof for the case of the dodecahedron was given by L. Fejes Tóth [8] based on his Spherical Moment Theorem.

In recent years the case of equality for many classical inequalities in convexity has got a more general treatment in a "stability version" of the respective result (see [15] for a survey). In this paper we prove stability versions for several known inequalities among others the isoperimetric inequality mentioned above.

We say that the convex bodies C and K are *ε–close* for $\varepsilon > 0$ if there exists a translate of $\frac{1}{1+\varepsilon}C$ that is contained in K, and a translate of $(1+\varepsilon)C$ that contains K. In this paper, $P_{(n)}$ always denotes a convex three–polyhedron that has at most n faces. In addition, for $n = 4$, 6, and 12 we denote by $P^*_{(n)}$ a regular tetrahedron, cube, and regular dodecahedron circumscribed about the unit ball B^3, respectively.

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THEOREM 1.1 *There exist positive absolute constants c and ε_0 such that if for $n = 4$, 6, or 12 and $0 < \varepsilon < \varepsilon_0$*

$$V(P_{(n)}) = V(P^*_{(n)}) \quad \textit{and} \quad S(P_{(n)}) \leq (1+\varepsilon)\, S(P^*_{(n)}),$$

then $P_{(n)}$ is $c\sqrt{\varepsilon}$–close to a congruent copy of $P^*_{(n)}$.

Remark: The estimate $c\sqrt{\varepsilon}$ is optimal. This is the easiest to see for the cube, if we choose $P_{(6)}$ to be rectangular box such that two orthogonal edges are of length $2(1+\sqrt{\varepsilon})^{-1}$, and the third edge is of length $2(1+\sqrt{\varepsilon})^2$.

The proof of the isoperimetric inequality for polyhedra rests on two results: the theorem of Lindelöf and the Spherical Moment Theorem. The theorem of Lindelöf says (see e.g. [13]) that if Q is a polyhedron circumscribed around some ball, and P is a polyhedron that has the same family of exterior normals of faces as Q, then

$$\frac{S(P)^3}{V(P)^2} \geq \frac{S(Q)^3}{V(Q)^2},$$

with equality holding if and only if P and Q are homothetic. We shall establish the following stability version of this result.

THEOREM 1.2 *Let Q be a simple polyhedron that is circumscribed around some ball. Then there exist positive constants γ and ε_0 depending on Q with the following property. If P is a polyhedron that has the same family of exterior normals of faces as Q and*

$$\frac{S(P)^3}{V(P)^2} \leq (1+\varepsilon)\, \frac{S(Q)^3}{V(Q)^2}$$

for $0 < \varepsilon < \varepsilon_0$, then P is $\gamma\sqrt{\varepsilon}$–close to a homothetic copy of Q. In addition, the same γ and ε_0 work if Q is deformed slightly while keeping its combinatorial type unchanged.

Remark: The order of the estimate $\sqrt{\varepsilon}$ is optimal, as it is shown by the example of a rectangular box.

The Spherical Moment Theorem estimates certain integrals over the faces of a tiling of S^2 for given $n \geq 4$. Before stating this important theorem we introduce some notations and notions. The spherical distance of two points p and q on S^2 is denoted by pq. If p and q are not antipodal, then the connecting spherical segment is denoted by pq, as well. If $a, b, p \in S^2$ are not contained in any great circle, then the angle enclosed by the spherical segments pa and pb is denoted by $\angle apb$. The directed arc of a great circle issuing from a point and ending in the antipodal point is called a *ray*. For $n \geq 4$ let $\bar{\Delta}_n = \bar{p}\bar{a}_n\bar{b}_n$ be a spherical triangle such that

$\angle\bar{p}\bar{a}_n\bar{b}_n = \angle\bar{p}\bar{b}_n\bar{a}_n\frac{\pi}{3}$ and $\angle\bar{a}_n\bar{p}\bar{b}_n = \frac{2\pi n}{6n-12}$. We note that the area of $\bar{\Delta}_n$ is $\frac{4\pi}{6n-12}$.

THEOREM (L. Fejes Tóth's Spherical Moment Theorem)
Let $f:[0,\pi]\to\mathbb{R}$ be a non–negative monotone decreasing function. If $D_1,\dots,D_n$ are Jordan measurable subsets that tile S^2, and $p_1,\dots,p_n\in S^2$ then

$$\sum_{i=1}^{n}\int_{D_i} f(xp_i)\,dx \le (6n-12)\int_{\bar{\Delta}_n} f(x\bar{p})\,dx.$$

If f is monotone increasing then the analogous statement holds, with the inequality sign reversed.

We note that equality holds if $n=4$, 6, or 12, $D_1,\dots,D_n$ are faces of a regular tetrahedral, cubical, or dodecahedral tiling and $p_1,\dots,p_n$ are the centers of the tiles. It is known that if f is strictly monotone and each D_i is the closure of its interior then equality occurs in the only in these cases. Besides the original argument by L. Fejes Tóth (see [8], [9] or [10]), additional proofs for the Spherical Moment Theorem are given in [6] and [12]).

We prove the stability version of the Spherical Moment Theorem only for tilings of S^2 by starshaped polygons. The spherical polygon D is *starshaped* with respect to p if $p\in \operatorname{int} D$, and if $x\in D$ then any point of the segment xp different from x lies in the interior of D. Let $D_1,\dots,D_n$, $n\ge 4$, be an edge to edge tiling of S^2 by polygons that are starshaped with respect to $p_1,\dots,p_n$, respectively. Any triangle whose vertices are some p_i, and two neighboring vertices of D_i, is called an *associated triangle* to the tiling, and we write $\Delta_1,\dots,\Delta_m$ to denote these triangles. Now $m\le 6n-12$ according to the Euler formula, and if $m<6n-12$ then we define the associated triangles $\Delta_{m+1},\dots,\Delta_{6n-12}$ to be the point p_1. For $\varepsilon>0$ and $i=1,\dots,6n-12$, we say that Δ_i is ε–close to $\bar{\Delta}_n$ if Δ_i is a triangle p_jab such that

$$\left|\frac{ab}{\bar{a}_n\bar{b}_n}-1\right|<\varepsilon,\quad \left|\frac{p_ja}{\bar{p}\bar{a}}-1\right|<\varepsilon,\quad \left|\frac{p_jb}{\bar{p}\bar{b}}-1\right|<\varepsilon.$$

We set $\bar{R}_n=\bar{p}\bar{a}_n$, and define $\bar{r}_n$ to be the spherical distance of $\bar{p}$ from the segment $\bar{a}_n\bar{b}_n$. If $n=4$, 6, or 12, then $\bar{r}_n$ and $\bar{R}_n$ are the in– and circumradius of the optimal tile for the Spherical Moment Theorem, respectively.

Now we are in the position to formulate a stability version of the Spherical Moment Theorem.

THEOREM 1.3 *Let $f:[0,\pi]\to\mathbb{R}$ be a non–negative monotone decreasing function, and let $D_1,\dots,D_n$ form a tiling of S^2 by polygons where every D_i is starshaped with respect to p_i. If at least k associated triangles are not ε–close to $\bar{\Delta}_n$ for $0<\varepsilon<\varepsilon_0$ then*

$$\sum_{i=1}^{n}\int_{D_i} f(xp_i)\,dx \le (6n-12)\int_{\bar{\Delta}_n} f(x\bar{p})\,dx - cm_n(f)\,\varepsilon^2\cdot\frac{k}{n}$$

where $m_n(f) = f(0.99\bar{R}_n) - f(1.01\bar{r}_n)$*, and* c *and* ε_0 *are positive absolute constants.*

If f *is monotone increasing then the analogous statement holds, with the inequality sign reversed.*

Naturally, the choice of $0.99\bar{R}_n$ and $1.01\bar{r}_n$ is rather arbitrary in Theorem 1.3. For any $0 < \mu < 7 - 4\sqrt{3}$, $m_n(f)$ can be replaced by $m_{n,\mu}(f) = f((1-\mu)\bar{R}_n) - f((1+\mu)\bar{r}_n)$, but c and ε_0 then depend on μ.

2 Polyhedral approximation

Now we discuss related problems for polyhedral approximation of the unit ball B^3 with respect to the volume, surface area and the mean width (see the papers [16] and [17] for comprehensive surveys on polyhedral approximation). Besides the volume $V(\cdot)$ and the surface area $S(\cdot)$ we shall consider the edge curvature or mean width of a polyhedron as well. For a convex body C, we write $h_C(\cdot)$ to denote the support function of C; namely, $h_C(u)$ is the maximum of $\langle x, u\rangle$ over all $x \in C$. Observe that if $u \in S^2$ then $h_C(u) + h_C(-u)$ is the length of the projection of C into some line parallel to u, which is called the width of C in the direction of u. Therefore it is natural to define the *mean width* by the formula

$$M(C) = \frac{1}{4\pi}\int_{S^2} h_C(u) + h_C(-u)\,du = \frac{1}{2\pi}\int_{S^2} h_C(u)\,du.$$

We recall that $P_{(n)}$ always denotes a polyhedron that has at most n faces. On the other hand, P_n will always denote a polyhedron that has at most n vertices. We search for the minimum of the volume $V(P_{(n)})$, or surface area $S(P_{(n)})$ if $P_{(n)}$ contains B^3, and the maximum of the mean width $M(P_n)$ if $P_n \subset B^3$. The extremal values readily occur when $P_{(n)}$ has n faces, and is circumscribed around B^3, and P_n has n vertices, and is inscribed into P_n. Moreover, we may assume that the faces of $P_{(n)}$ touch B^3 at $p_1, \ldots, p_n$, and P_n is the convex hull of $p_1, \ldots, p_n$. We write D_i to denote the Dirichlet–Voronoi cell corresponding to p_i on S^2. D_i is the set of points whose spherical distance from p_i is not larger than the spherical distance from any p_j, $j = 1, \ldots, n$. In particular, D_i is the radial projection onto S^2 of the face of $P_{(n)}$ containing p_i.

We have

$$V(P_{(n)}) = \frac{1}{3}\sum_{i=1}^{n}\int_{D_i}\frac{1}{\cos^3 xp_i}\,dx; \tag{1}$$

$$S(P_{(n)}) = \sum_{i=1}^{n}\int_{D_i}\frac{1}{\cos^3 xp_i}\,dx; \tag{2}$$

$$M(P_n) = \frac{1}{2\pi}\sum_{i=1}^{n}\int_{D_i}\cos xp_i\,dx. \tag{3}$$

Equations (1) and (2) are obviously equivalent. For (2), we observe that if $x \in D_i$, and an area element dx on S^2 is projected radially to the corresponding face of $P_{(n)}$ then the projection is of area $\frac{1}{\cos^3 xp_i}\, dx$. Finally, (3) is a consequence of the fact that if $x \in D_i$ then $h_{P_n}(x) = \cos xp_i$.

Let $n = 4$, 6, or 12. It follows by the Spherical Moment Theorem that if $P_{(n)}$ contains B^3 then $V(P_{(n)}) \geq V(P^*_{(n)})$. An analogous result holds for the surface area. Recall that $P^*_{(n)}$ ($n = 4$, 6, 12) denotes a regular tetrahedron, cube and regular dodecahedron circumscribed around B^3, respectively. We also introduce the notation P^*_n for a regular tetrahedron, octahedron, and icosahedron inscribed in B^3, respectively. Then we have for $n = 4$, 6, or 12 and $P_n \subset B^3$ $M(P_n) \leq M(P^*_n)$, a result of Linhart [19] which in view of (3) also follows from the Spherical Moment Theorem. The following theorem states a stability version of these results.

THEOREM 2.1 *There exist positive absolute constants c and ε_0 with the following properties:*

(i) If for $n = 4$, 6, or 12 $P_{(n)}$ contains B^3 and $V(P_{(n)}) \leq (1+\varepsilon)\, V(P^*_{(n)})$ for $0 < \varepsilon < \varepsilon_0$, then $P_{(n)}$ is $c\sqrt{\varepsilon}$–close to a congruent copy of $P^*_{(n)}$. The analogous statement holds for the surface area.

(ii) If $P_n \subset B^3$ and $M(P_n) \geq (1-\varepsilon)\, V(P^*_n)$ for $0 < \varepsilon < \varepsilon_0$ then P_n is $c\sqrt{\varepsilon}$–close to a congruent copy of P^*_n.

Finally, we discuss polyhedral approximation for large n. The Spherical Moment Theorem yields that

$$V(P_{(n)} \backslash B^3) \geq \frac{20\pi^2}{9\sqrt{3}} \cdot \frac{1}{n} + \frac{c_1}{n^2}; \tag{4}$$

$$S(P_{(n)}) - S(B^3) \geq \frac{20\pi^2}{3\sqrt{3}} \cdot \frac{1}{n} + \frac{c_2}{n^2}; \tag{5}$$

$$M(B^3) - M(P_n) \geq \frac{10\pi}{9\sqrt{3}} \cdot \frac{1}{n} + \frac{c_3}{n^2} \tag{6}$$

where c_1, c_2 and c_3 are positive absolute constants. On the other hand, it is not hard to show that there exist polyhedra $P_{(n)}$ circumscribed around B^3, and P_n inscribed into B^3 such that

$$V(P_{(n)} \backslash B^3) \leq \frac{20\pi^2}{9\sqrt{3}} \cdot \frac{1}{n} + O\left(\frac{1}{n^{\frac{4}{3}}}\right); \tag{7}$$

$$S(P_{(n)}) - S(B^3) \leq \frac{20\pi^2}{3\sqrt{3}} \cdot \frac{1}{n} + O\left(\frac{1}{n^{\frac{4}{3}}}\right); \tag{8}$$

$$M(B^3) - M(P_n) \leq \frac{10\pi}{9\sqrt{3}} \cdot \frac{1}{n} + O\left(\frac{1}{n^{\frac{4}{3}}}\right). \tag{9}$$

An efficient way of distributing the corresponding set of n points on S^2 is presented in [5]. Gruber [18] gives a slightly worse error term for approximation of any smooth convex bodies in $\mathbb{E}^3$.

Let $P_{(n)}$ be a polyhedron circumscribed around B^3 with n faces, and let P_n be a polyhedron inscribed in B^3 with n vertices where the points of tangency and the vertices, respectively, are chosen randomly from S^2 with uniform distribution. Then the expected values of $V(P_{(n)})$, $S(P_{(n)})$ and $M(P_n)$ have Taylor expansions of any length in terms of $\frac{1}{n}$ ([3], [20]). Since, for large n, random polyhedra often behave like the optimal ones, it was widely believed that the true order of the error term was $\frac{1}{n^2}$. Here we show that the order of the error term in (7), (8) and (9) is larger than $\frac{1}{n^2}$.

THEOREM 2.2 *There exists a function $f(n)$ tending to infinity such that if $P_n \subset B^3 \subset P_{(n)}$, then*

$$\begin{aligned} V(P_{(n)}\backslash B^3) &\geq \tfrac{20\pi^2}{9\sqrt{3}} \cdot \tfrac{1}{n} + \frac{f(n)}{n^2}; \\ S(P_{(n)}) - S(B^3) &\geq \tfrac{20\pi^2}{3\sqrt{3}} \cdot \tfrac{1}{n} + \frac{f(n)}{n^2}; \\ M(B^3) - M(P_n) &\geq \tfrac{10\pi}{9\sqrt{3}} \cdot \tfrac{1}{n} + \frac{f(n)}{n^2}. \end{aligned}$$

3 The proof of the Lindelöf theorem and its stability

The proof of the theorem of Lindelöf, as well as its stability version, rest on the Brunn–Minkowski theory. The comprehensive monograph by Schneider [21] is a standard reference, but the classical book of Bonnesen and Fenchel [1] contains everything what we need.

For planar convex domains, we write $A(\cdot)$ to denote the area. The mixed area $A(M,N)$ of two convex domains M and N is defined by the relation

$$A(\lambda\, M + \mu\, N) = \lambda^2 A(M) + 2\lambda\mu A(M,N) + \mu^2 A(N).$$

The mixed area is non–negative, and satisfies the Bonnesen inequality (see [15])

$$A(M,N)^2 - A(M)\, A(N) \geq \tfrac{1}{4}\, A(M)^2 \cdot \left(R_M(N) - r_M(N)\right)^2. \tag{10}$$

Here $R_M(N)$ and $r_M(N)$ denote the smallest R, and the largest r, respectively, such that a translate of RM contains N, and a translate of rM is contained in N. In particular, we deduce the following special form of the Brunn–Minkowski inequality:

$$A\left(\tfrac{1}{2}\, M + \tfrac{1}{2}\, N\right) \geq \tfrac{1}{4} \cdot \left(\sqrt{A(M)} + \sqrt{A(N)}\right)^2. \tag{11}$$

Next let P be a polyhedron, and let C be a convex body in $\mathbb{E}^3$. Then for $\lambda, \mu \geq 0$, we have

$$V(\lambda\, P + \mu\, C) = \lambda^3 V(P) + 3\lambda^2\mu V(P,P,C) + 3\lambda\mu^2 V(P,C,C) + \mu^3 V(C)$$

where the non–negative coefficients $V(P, P, C)$ and $V(P, C, C)$ are called mixed volumes. If $F_1, \dots, F_n$ denote the faces of P, and u_i denotes the exterior unit normal vector to F_i then

$$V(P, P, C) = \frac{1}{3} \lim_{\mu \to 0} \frac{V(P + \mu C) - V(P)}{\mu} = \frac{1}{3} \sum_{i=1}^{n} A(F_i) \cdot h_C(u_i). \tag{12}$$

In particular, $V(P, P, P) = V(P)$ and $V(P, P, B^3) = \frac{1}{3} S(P)$. According to the Brunn–Minkowski inequality, the function

$$\varphi_{P,C}(\lambda) = V((1 - \lambda) P + \lambda C)^{\frac{1}{3}} \tag{13}$$

is a concave function of $0 \leq \lambda \leq 1$, and $\varphi_{P,C}(\lambda)$ is linear if and only if P and C are homotetic. The Brunn–Minkowski inequality yields the Minkowski inequality, which states that

$$V(P, P, C) \geq V(P)^{\frac{2}{3}} V(C)^{\frac{1}{3}} \tag{14}$$

where equality holds if and only if P and C are homotetic.

Before proving Theorem 1.2, let us show how the theory above leads to the Lindelöf theorem. For a three polyhedron P, let Q be the polyhedron that is circumscribed around the unit ball, and has the same family of exterior normals of faces as P. Then $S(P) = 3V(P, P, Q)$ and $S(Q) = 3V(Q)$, and hence the Minkowski inequality for $V(P, P, Q)$ readily yields

$$\frac{S(P)^3}{V(P)^2} \geq \frac{S(Q)^3}{V(Q)^2} \tag{15}$$

where equality holds if and only if P is circumscribed around some ball.

There exist various stability versions of the Minkowski inequality. One, due to Diskant [4] (see H. Groemer [15] p. 137), says that if

$$V(P, P, C) \leq (1 + \varepsilon) V(P)^{\frac{2}{3}} V(C)^{\frac{1}{3}} \tag{16}$$

for small $\varepsilon > 0$ then C is $c\sqrt[3]{\varepsilon}$–close λP, where $\lambda = \sqrt[3]{\frac{V(C)}{V(P)}}$ and c is a positive absolute constant. This result yields a stability version of the Lindelöf theorem where the conclusion is that P and some homotetic copy of Q are $c_0 \sqrt[3]{\varepsilon}$–close ($c_0$ is a positive absolute constant). Our aim is to verify a bound of smaller order, thus we improve on the stability estimate for the Minkowski inequality for certain polyhedra:

LEMMA 3.1 *For any simple polyhedron Q in $\mathbb{E}^3$, there exist positive constants α and ν_0 depending on Q with the following properties: If P is a polyhedron that has the same family of exterior normals of faces as Q, and P is ν_0–close to* *some homotetic copy of Q but not ν–close to any homotetic copy of Q for some $0<\nu<\nu_0$ then*

$$V(P,P,Q)\geq(1+\alpha\,\nu^2)\,V(P)^{\frac{2}{3}}V(Q)^{\frac{1}{3}}.$$

In addition, the same α and ν_0 work if Q is deformed slightly while keeping its combinatorial type unchanged.

Proof: The symbols $\alpha_0,\alpha_1,\ldots$ always denote some positive constants depending on Q.

We choose ν_0 such that for any polyhedron P that has the same family of exterior normals of faces as Q, and is ν_0–close to Q, the following properties hold:

(i) If H is a plane which is parallel to some face of P, intersects the interior of P, and contains some vertex of P, then both parts of P cut off by H have volume at least $\alpha_0 V(P)$;

(ii) If P is not ν–close to Q for some $0<\nu<\nu_0$ then P has a face that is not $\alpha_1\nu$–close to the corresponding face of Q.

Let P be a polyhedron that has the same family of exterior normals of faces as Q and suppose that P is ν_0–close to Q but not ν–close to Q for some $0<\nu<\nu_0$. We may assume that $V(P)=V(Q)=1$. Since $\varphi_{P,Q}(\lambda)$ is concave according to the Brunn–Minkowski inequality (see (13)), it is sufficient to show that

$$V(\tfrac{1}{2}\,P+\tfrac{1}{2}\,Q)\geq 1+\alpha_2\,\nu^2. \tag{17}$$

Let F and G be faces of P and Q that share a common exterior unit normal u, and are not $\alpha_1\nu$–close to eachother. We may asume that $h_P(u)=h_Q(u)=0$. In what follows, we mimic the original argument of H. Brunn for his theorem (see [1] p. 94–96). For $0<v<1$, we define $f(v)$ and $g(v)$ in the following way: the parts of P and of Q that are contained in the halfspaces with equation $\langle x,u\rangle\leq f(v)$ and $\langle x,u\rangle\leq g(v)$, respectively, are of volume v. In addition, we define M_v and N_v to be the sections of P and Q by the planes with equations $\langle x,u\rangle=f(v)$ and $\langle x,u\rangle=g(v)$, respectively. It follows that

$$f'(v)=\frac{1}{A(M_v)}\quad\text{and}\quad g'(v)=\frac{1}{A(N_v)}.$$

Now the section of $\frac{1}{2}(P+Q)$ by the plane with equation $\langle x,u\rangle=\frac{1}{2}(f(v)+g(v))$ contains $\frac{1}{2}(M_v+N_v)$, which in turn yields

$$V\left(\tfrac{1}{2}\,(P+Q)\right)\geq\int_0^1 A\left(\tfrac{1}{2}\,(M_v+N_v)\right)\cdot\frac{1}{2}\left(\frac{1}{A(M_v)}+\frac{1}{A(N_v)}\right)\,dv. \tag{18}$$

Let $h(v)$ denote the intengrand on the right hand side. We note that for any $a, b > 0$,

$$\tfrac{1}{4}\left(\sqrt{a}+\sqrt{b}\right)^2 \cdot \tfrac{1}{2}\left(\tfrac{1}{a}+\tfrac{1}{b}\right) \geq \frac{(\sqrt{a}+\sqrt{b})^2}{4\sqrt{a}\sqrt{b}} = 1 + \frac{1}{4}\left(\sqrt[4]{\frac{a}{b}} - \sqrt[4]{\frac{b}{a}}\right)^2, \qquad (19)$$

and hence $h(v) \geq 1$ for any $v \in (0,1)$. Since F and G are not $\alpha_1\nu$–close, we deduce for suitable α_2 that either M_v and N_v are not $\alpha_2\nu$–close for any $\frac{\alpha_0}{5} < v < \frac{2\alpha_0}{5}$, or M_v and N_v are not $\alpha_2\nu$–close for any $\frac{3\alpha_0}{5} < v < \frac{4\alpha_0}{5}$. We present the argument if M_v and N_v are not $\alpha_2\nu$–close for any $\frac{\alpha_0}{5} < v < \frac{2\alpha_0}{5}$, and the other case follows analogously. We claim that if $\frac{\alpha_0}{5} < v < \frac{2\alpha_0}{5}$ then

$$h(v) > 1 + \alpha_3\nu^2. \qquad (20)$$

First we assume that either $\frac{A(M_v)}{A(N_v)} > 1 + \frac{\alpha_2\nu}{9}$ or $\frac{A(N_v)}{A(M_v)} > 1 + \frac{\alpha_2\nu}{9}$. Then the planar Brunn–Minkowski inequality (11) and the estimate (19) yield the claim (20). Thus let

$$\frac{A(M_v)}{A(N_v)}, \frac{A(N_v)}{A(M_v)} \leq 1 + \frac{\alpha_2\nu}{9}.$$

In this case, M_v is not $\frac{\alpha_2\nu}{9}$–close to any homotetic image of N_v, and we conclude (20) by the Bonnesen inequality (10). Finally, the claim (20) readily yields Lemma 3.1.

Q.E.D.

In order to verify Theorem 1.2, we may assume that Q is a simple polyhedron that is circumscribed around the unit ball, and P has the same family of exterior normals of faces as Q. If $\frac{S(P)^3}{V(P)^2} < (1+\varepsilon)\frac{S(Q)^3}{V(Q)^2}$ then $V(P,P,Q)^3 < (1+\varepsilon)V(P)^2V(Q)$. Since $V(P,P,Q)^3 = V(P)^2V(Q)$ yields that P and Q are translates, we conclude Theorem 1.2 by Lemma 3.1.

Q.E.D.

4 Simplifying the statement of the stability theorem for moments

Since we work mostly on S^2, segment, distance and angle always mean spherical segment, spherical distance and spherical angle, unless otherwise stated. We write $|\cdot|$ to denote spherical area.

The proof of Theorem 1.3 is based on the argument presented in [6]. We modify the argument similarly as it was done in [7] for the Euclidean case.

It is sufficient to consider the case when f is monotone decreasing. For $0 < \varrho < \pi$, let us define the step function $\gamma_\varrho : [0,\pi] \to \mathbb{R}$ by

$$\gamma_\varrho(x) = \begin{cases} 1 & \text{if } x \leq \varrho, \\ 0 & \text{otherwise.} \end{cases}$$

Knowing the Moment Theorem for step functions readily yields its validity for any monotone function. A simple argument ([7], Section 2.4) shows that in turn, Theorem 1.3 is a consequence of the Moment Theorem and the following statement:

LEMMA 4.1 *Let $1.01\bar{r}_n \leq \varrho < 0.99\bar{R}_n$, and let $D_1, \dots, D_n$ form a tiling of S^2 by polygons where every D_i is starshaped with respect to p_i. If at least k associated triangles are not ε–close to $\bar{\Delta}_n$ for $0 < \varepsilon < \varepsilon_0$ then*

$$\sum_{i=1}^{n} \int_{D_i} \gamma_\varrho(xp_i)\, dx \leq (6n-12) \int_{\bar{\Delta}_n} \gamma_\varrho(x\bar{p})\, dx - \frac{c\varepsilon^2 k}{n}$$

where c and ε_0 are positive absolute constants.

We write Lemma 4.1 into a "local form" (see Lemma 4.2), and verify Lemma 4.2 in the next section.

First of all, possibly by splitting vertices, we may assume that any vertex of the tiling $D_1, \dots, D_n$ is contained in exactly three tiles. Thus the number of associated triangles is $6n-12$, and we write $\Delta_1, \dots, \Delta_{6n-12}$ to denote these triangles. For $p \in S^2$ and $0 < r < \frac{\pi}{2}$, we write $C(p,r)$ to denote the circular disc of centre p and of radius r. If Δ_i is contained in D_j then let S_i be the sector of $C(p_j, \varrho)$ cut off by the two spherical rays emanating from p_j that contain the two corresponding sides of Δ_i. Then we define

$$F_i = \Delta_i \cap S_i = \Delta_i \cap C(p_j, \varrho).$$

In addition, let $\bar{S}$ be the sector of $C(\bar{p}, \varrho)$ determined by $\bar{\Delta}_n$, and let $\bar{F} = \bar{\Delta} \cap \bar{S}$. It follows that the inequality of Lemma 4.1 is equivalent with

$$\sum_{i=1}^{6n-12} |F_i| \leq (6n-12)|\bar{F}| - \frac{c\varepsilon^2 k}{n}. \tag{21}$$

We write $\bar{d}$ to denote one of the intersection points of the segment $\bar{a}\bar{b}$ and $\partial C(\bar{p}, \varrho)$, $\bar{m}$ to denote the midpoint of $\bar{a}\bar{b}$ and define

$$\eta = \frac{\sin \bar{m}\bar{d}}{\sin \bar{m}\bar{a}} \quad \text{and} \quad \vartheta = 1 - \eta \cdot \frac{|C(\bar{p}, \bar{R}_n)|}{|C(\bar{p}, \varrho)|}$$

For a triangle $\Delta = apb$, we consider the corresponding sets $S = C(p, \varrho) \cap \angle apb$ and $F = \Delta \cap S$, and define the quantity

$$w(\Delta) = |F| - \eta\, |\Delta| - \vartheta\, |S|.$$

We note that (see [6])

$$w(\bar{\Delta}_n) > 0 \quad \text{and} \quad w(\Delta) \leq w(\bar{\Delta}_n). \tag{22}$$

Now $w(\Delta)$ was defined in a way to ensure that

$$\sum_{i=1}^{6n-12} |F_i| = \sum_{i=1}^{6n-12} \eta\,|\Delta_i| + \sum_{i=1}^{6n-12} \vartheta\,|S_i| + \sum_{i=1}^{6n-12} w(\Delta_i)$$

where

$$\sum_{i=1}^{6n-12} |\Delta_i| = 4\pi \quad \text{and} \quad \sum_{i=1}^{6n-12} |S_i| = n \cdot |C(\bar{p},\varrho)|.$$

Thus the definition of $\bar{\Delta}_n$ yields that (21) is equivalent with

$$\sum_{i=1}^{6n-12} w(\Delta_i) \leq (6n-12)w(\bar{\Delta}_n) - \frac{c\varepsilon^2 k}{n}.$$

Therefore (22) yields the Spherical Moment Theorem, and in addition, Lemma 4.1 is a consequence of the following statement:

LEMMA 4.2 *If $\Delta = apb$ is not ε–close to $\bar{\Delta}_n$ for $0 < \varepsilon < \varepsilon_0$ then*

$$w(\Delta) \leq w(\bar{\Delta}_n) - \frac{c\varepsilon^2}{n}$$

where c and ε_0 are positive absolute constants.

5 The proof of Lemma 4.2

Let $\Delta = apb$ be the triangle in Lemma 4.2, where we may assume that $p = \bar{p}$. In order to simplify formulae, we write $C(r)$ to denote the circular disc $C(\bar{p}, r)$, and $\bar{\Delta}, \bar{R}, \ldots$ to denote $\bar{\Delta}_n, \bar{R}_n, \ldots$. The symbols $c_0, c_1, \ldots$ always denote positive abosolute constants.

As the argument in [6] shows, we may assume the following conditions about $\Delta = a\bar{p}b$:

$$\begin{array}{l} \text{neither } a \text{ nor } b \text{ is contained in } \operatorname{int} C(\varrho); \\ ab \cap C(\varrho) \neq \emptyset; \\ \text{if } ab \cap \operatorname{int} C(\varrho) = \emptyset \text{ then } ab \text{ touches } C(\varrho). \end{array} \tag{23}$$

We introduce two operations that transform Δ in a way that $w(\Delta)$ increases. If $a \in \partial C(\bar{R})$ then we simply define $a' = a$, and similarly for b.

(*) If $a \notin C(\bar{R})$ or $b \notin C(\bar{R})$.

We only describe the operation for a. We write a' and a'' to denote the points of $\partial C(\bar{R})$ such that $a' \in ab$ and $a'' \in a\bar{p}$, respectively. In case $ab \cap \partial C(\bar{R})$ consists of two points then a' is the one closer to a. In addition, let $a_1 \neq a$ be a point of

aa', and let a_2 be the point of intersection of $a_1\bar{p}$ and $\partial C(\bar{R})$. Then the triangle $\Delta^* = a_1\bar{p}b$, and the region Ω_* that is bounded by the segments $a''a$, aa_1, a_1a_2, and by the shorter arc of $\partial C(\bar{R})$ connecting a_2 and a'' satisfy

$$w(\Delta^*) - w(\Delta) = \eta \cdot |\Omega_*|. \tag{24}$$

(**) If $a \in \operatorname{int} C(\bar{R})$ or $b \in \operatorname{int} C(\bar{R})$.

We only describe the operation for a. In this case, we write a' and a'' to denote the points of $\partial C(\bar{R})$ such that ba' and $\bar{p}a''$, respectively, contain a. In addition, let $a_1 \neq a$ be a point of aa', and let a_2 be the point of $\partial C(\bar{R})$ such that $\bar{p}a_2$, respectively, contain a_1. Then the triangle $\Delta^{**} = a_1\bar{p}b$, and the region Ω_{**} that is bounded by the segments $a''a$, aa_1, a_1a_2, and by the shorter arc of $\partial C(\bar{R})$ connecting a_2 and a'' satisfy

$$w(\Delta^{**}) - w(\Delta) = \eta \cdot |\Omega_{**}|. \tag{25}$$

The operations (*) and (**) yield that $w(\Delta) \leq w(\Delta')$ for the triangle $\Delta' = a'\bar{p}b'$. We define $\omega(x) = w(\Delta')$ where x is the distance of $\bar{p}$ from the segment ab.

PROPOSITION 5.1

(i) $\omega(x)$ is strictly monotone increasing (decreasing) for $x < \bar{r}$ $(x > \bar{r})$;

(ii) $\omega(x) \leq \omega(\bar{r}) - \tilde{c}(x-\bar{r})^2$ if $|x - \bar{r}| \leq \tilde{\varepsilon}\bar{r}$ where $\tilde{c}$ and $\tilde{\varepsilon}$ are positive constants.

Proof: We fix a spherical ray s emanating from $\bar{p}$, and write m_t to denote the point of s of distance t from $\bar{p}$. For $0 < t < r < \frac{\pi}{2}$, let e_t and f_t be the endpoints of the secant of $C(r)$ that is orthogonal to s in m_t, and let $l_r(t) = e_tm_t$. We note that

$$\begin{aligned} \sin l_r(t) &= \sqrt{1 - \frac{\cos^2 r}{\cos^2 t}}; \\ \eta &= \frac{\sin l_\varrho(\bar{r})}{\sin l_{\bar{R}}(\bar{r})}. \end{aligned}$$

Let us consider the area of the region $Q_{t,t'}$ that is bounded by the segments e_tf_t and $e_{t'}f_{t'}$, and by the arcs $e_te_{t'}$ and $f_tf_{t'}$ of $\partial C(r)$. If $t' - t$ is small then a simple argument shows that, asymptotically,

$$|Q_{t,t'}| \sim 2\sin l_r(t) \cdot |t' - t|. \tag{26}$$

It is not hard to see that

$$\omega'(x) = 2\sin l_\varrho(x) - 2\eta \sin l_{\bar{R}}(x).$$

Now (i) follows if $\frac{\sin l_\varrho(x)}{\sin l_{\bar{R}}(x)}$ is a strictly decreasing function of x on $(0,\varrho)$. Since $\frac{\sin^2 l_\varrho(x)}{\sin^2 l_{\bar{R}}(x)} = \frac{\cos^2 x - \cos^2 \varrho}{\cos^2 x - \cos^2 \bar{R}}$ is readily strictly decreasing, we conclude (i).

Turning to (ii), we assume that $|x-\bar{r}| \leq \tilde{\varepsilon}\bar{r}$ where $\tilde{\varepsilon} = 0.001$. We observe that

$$\omega'(x) = \left(\sin^2 l_\varrho(x) - \eta^2 \sin^2 l_{\bar{R}}(x)\right) \cdot \frac{2}{\sin l_\varrho(x) + \eta \sin l_{\bar{R}}(x))}.$$

Now elementary calculations yield that

$$\begin{aligned} \sin^2 l_\varrho(x) - \eta^2 \sin^2 l_{\bar{R}}(x) \quad &= \quad \frac{(\cos x - \cos \bar{r})(\cos \varrho - \cos \bar{R})}{\cos \bar{r} - \cos \bar{R}} \times \\ & \quad\quad \frac{(\cos x + \cos \bar{r})(\cos \varrho + \cos \bar{R})}{(\cos \bar{r} + \cos \bar{R}) \cos^2 x} \end{aligned}$$

where the second term on the right hand side is larger than a positive constant. Since $\cos\alpha - \cos\beta = 2\sin\frac{\beta-\alpha}{2}\sin\frac{\beta+\alpha}{2}$, it follows that

$$\omega'(x) < -c_0|x-\bar{r}|.$$

In turn, we conclude (ii). Q.E.D.

Since $\varrho \geq 1.01\bar{r}$, there exists c_1 such that

$$\eta > c_1.$$

If $|\frac{x}{\bar{r}} - 1| > \frac{\varepsilon}{100}$ then we use that $w(\Delta) \leq w(\Delta')$, hence Proposition 5.1 yields Lemma 4.2. Thus let $|\frac{x}{\bar{r}} - 1| \leq \frac{\varepsilon}{100}$. Since Δ is not ε–close to $\bar{\Delta}$, we may assume that say $|\frac{a\bar{p}}{\bar{a}\bar{p}} - 1| \geq \varepsilon$, and according to operations (*) and (**), we may actually assume that $|\frac{a\bar{p}}{\bar{a}\bar{p}} - 1| = \varepsilon$. It follows that $aa' \geq c_2\varepsilon\bar{r}$. Since $\angle\bar{p}\bar{a}\bar{b} = \frac{\pi}{3}$, we deduce that the angles $\angle\bar{p}a\bar{b}$ and $\angle\bar{p}\bar{a'}\bar{b}$ are both at least $\frac{\pi}{12}$ and at most $\frac{5\pi}{12}$. Therefore Ω_* contains a triangle $aa'a_0$ where $\angle aa'a_0 = \angle a'aa_0 = \frac{\pi}{24}$, and a congruent triangle is contained in Ω_{**}, as well. Since the area of the triangle $aa'a_0$ is at least $c_3\varepsilon^2\bar{r}^2$, we deduce that

$$w(\Delta') - w(\Delta) \geq \tfrac{1}{2}\, c_1 c_3 \varepsilon^2 \bar{r}^2 > \frac{c_4\varepsilon^2}{n}.$$

On the other hand, $w(\bar{\Delta}) \geq w(\Delta')$ according to Proposition 5.1, which completes the proof of Lemma 4.2. Q.E.D.

6 Arguments for Theorem 2.1 and Theorem 1.1

Theorem 2.1 directly follows from Theorem 1.3 and from formulae (1), (2) and (3).

We recall that $P_{(n)}$ denotes any polyhedron that has at most twelve faces, and for any $P_{(n)}$, we write $Q_{(n)}$ to denote the polyhedron circumscribed around B^3 that has the same exterior normals of faces as $P_{(n)}$ (naturally, $Q_{(n)}$ varies with $P_{(n)}$). First we choose $\tilde{\varepsilon}_0$ for $P^*_{(n)}$ in Theorem 1.2, and $\tilde{c}$ for $P^*_{(n)}$ in Theorem 2.1. Now let c be the maximum of $\tilde{c}$, and the constant γ in Theorem 1.2 for $P^*_{(n)}$. We may choose a positive number $\varepsilon_0 < \frac{1}{4}\,\tilde{\varepsilon}_0$ such that if $Q_{(n)}$ is $4c\sqrt{\varepsilon_0}$–close to $P^*_{(n)}$ then γ (hence c, as well) works in Theorem 1.2 for $Q_{(n)}$.

Now let $V(P_{(n)}) = V(P^*_{(n)})$ and $S(P_{(n)}) \leq (1+\varepsilon)\,S(P^*_{(n)})$ for $0 < \varepsilon < \varepsilon_0$. It follows by the Lindelöf theorem that

$$S(Q_{(n)}) = 9\cdot\frac{S(Q_{(n)})^3}{V(Q_{(n)})^2} \leq 9\cdot\frac{S(P_{(n)})^3}{V(P_{(n)})^2} \leq 9\cdot\frac{(1+\varepsilon)^3\,S(P^*_{(n)})^3}{V(P^*_{(n)})^2} < (1+4\varepsilon)\,S(P^*_{(n)}).$$

Thus $Q_{(n)}$ is $4c\sqrt{\varepsilon}$–close to some congruent copy of $P^*_{(n)}$ according to Theorem 2.1. On the other hand,

$$\begin{aligned}\frac{S(P_{(n)})^3}{V(P_{(n)})^2} &\leq \frac{(1+\varepsilon)^3\,S(P^*_{(n)})^3}{V(P^*_{(n)})^2} < 9(1+4\varepsilon)\,S(P^*_{(n)}) \leq 9(1+4\varepsilon)\,S(Q_{(n)})\\ &= (1+4\varepsilon)\,\frac{S(Q_{(n)})^3}{V(Q_{(n)})^2}.\end{aligned}$$

Therefore Theorem 1.2 yields Theorem 1.1.

7 The proof of Theorem 2.2

The main idea is to estimate how many out of the n triangles in a triangulation of S^2 can be ε–close to the regular spherical triangle $\bar{T}_n$ of area $\frac{4\pi}{n}$. If $\bar{s}_n$ denotes the length of the sides of $\bar{T}_n$ then we say that a spherical triangle T is ε–close to $\bar{T}_n$ if $\left|\frac{\bar{s}_n}{s} - 1\right| < \varepsilon$.

LEMMA 7.1 *Let Σ_n be a triangulation of S^2 with n triangles. Then there exist an $\varepsilon > 0$, and a function $g(n)$ that tends to infinity with n such that Σ_n has at least $g(n)$ triangles, which are not ε–close to $\bar{T}_n$.*

Proof: The proof is indirect, thus we suppose that for any $\varepsilon > 0$, $\Sigma_{n'}$ has at most $K(\varepsilon)$ triangles that are not ε–close to $\bar{T}_{n'}$. where $K(\varepsilon)$ is independent of n'. Let X be the union of these at most $K(\varepsilon)$ triangles. Any connected component of X is bounded by at most $3K(\varepsilon)$ segments whose length is at most $(1+\varepsilon)s_{n'}$, thus the diameter of each triangle in $\Sigma_{n'}$ is at most $6K(\varepsilon)s_{n'}$.

Now we can apply the argument in [2] (from the third paragraph of p. 305 to the end of p. 307) in order to complete the proof of Lemma 7.1. Q.E.D.

We present the argument for Theorem 2.2 only in case of the volume. The case of the surface and the mean width is completely analogous. We suppose that there exists a sequence $P_{(n)}$ of polyhedra that are circumscribed around the unit ball B^3, and

$$V(P_{(n')}) \leq \tfrac{20\pi^2}{9\sqrt{3}} \cdot \tfrac{1}{n'} + \tfrac{K}{n'^2} \tag{27}$$

holds along a subsequence $P_{(n')}$ where K is a constant. We may assume that three faces meet at each vertex of $P_{(n')}$, and write $p_1, \ldots, p_{n'}$ to denote the points where the faces of $P_{(n')}$ touch S^2.

We consider two cell decompositions of S^2. The first is provided by the Dirichlet–Voronoi cells $D_1, \ldots, D_{n'}$, and let $\Delta_1, \ldots, \Delta_{6n'-12}$ be the associated triangles. The other cell decomposition is the so–called Delone triangulation. We observe that the faces of the convex hull $P_{n'}$ of the points $p_1, \ldots, p_{n'}$ are triangles, and define $\Sigma_{n'}$ to be the triangulation of S^2 that results from projecting the faces of $P_{n'}$ radiallyon S^2. Then the Delone triangulation $\Sigma_{n'}$, and the Dirichlet–Voronoi cell decomposition are duals. According to Lemma 7.1, there exists an $\varepsilon > 0$, and a function $\varphi(n')$ tending to infinity with n' such that at least $\varphi(n')$ triangles in $\Sigma_{n'}$ are not 9ε–close to $\bar{T}_{n'}$. If v is a vertex for the Dirichlet–Voronoi cell decomposition, and each associated triangle that contains v is ε–close to $\bar{\Delta}_{n'}$ then the Delone triangle that belongs to v is 9ε–close to $\bar{T}_{n'}$. Therefore at least $\frac{1}{2}\varphi(n')$ associated triangles are not ε–close to $\bar{\Delta}_{n'}$. Thus (1) and Theorem 1.3 yield Theorem 2.2.

Q.E.D.

References

[1] T. Bonnesen and W. Fenchel: *Theory of Convex bodies.* BCS Assoc., Moscow (Idaho), 1987. Translated from German: Theorie der konvexen Körper, Springer, 1934.

[2] K. Böröczky, Jr.: *About the error term for best approximation with respect to the Hausdorff related metrics.* Disc. Comp. Geom., 25:293–309, 2001.

[3] K. Böröczky, Jr. and M. Reitzner: *Approximation of Smooth Convex Bodies by Random Circumscribed Polytopes.* (in preparation)

[4] V.I. Diskant. *Refinements of an isoperimetric inequality and stability theorems in the theory of convex bodies.* (Russian) Trudy Inst. Mat. (Novosibirsk) 14: Sovrem. Probl. Geom. Analiz., 98–132, 1989.

[5] G. Fejes Tóth: *The thinnest packing of the sphere by circles.* MSc. thesis, Roland Eötvös University, Budapest, 1970, (in Hungarian).

[6] G. Fejes Tóth: *Sum of moments of convex polygons*, Acta Math. Acad. Sci. Hungar. 24:417–421, 1973.

[7] G. Fejes Tóth: *A stability criterion to the moment theorem*, Stud. Math. Acad. Sci. Hung., 2001.

[8] L. Fejes Tóth. *The isepiphan problem for n-hedra.* Amer. J. Math. 70:174–180, 1948.

[9] L. Fejes Tóth. *Regular Figures.* Pergamon Press, 1964.

[10] L. Fejes Tóth: *Lagerungen in der Ebene, auf der Kugel und im Raum.* Springer-Verlag, Berlin, 2nd edition, 1972.

[11] A. Florian: *Eine Ungleichung über konvexe Polyeder* Monatsh. Math. 60:130–156, 1956.

[12] A. Florian: *Integrale auf konvexen Mozaiken.* Period. Math. Hungar., 6:23–38, 1975.

[13] A. Florian: *Extremum problems for convex discs and polyhedra.* In: *Handbook of Convex Geometry*, P.M. Gruber and J.M. Wills (eds), North Holland, 177–221, 1993.

[14] M. Goldberg: *The isoperimewtric problem for polyhedra.* Tohoku Math. J., 40:226–236, 1935.

[15] H. Groemer. *Stability of geometric inequalities.* In: Handbook of Convex Geometry, P.M. Gruber and J.M. Wills (eds), North Holland, 125–150, 1993.

[16] P.M. Gruber: *Aspects of approximation of convex bodies.* In: Handbook of convex geometry A, 319–345. North-Holland, Amsterdam, 1993.

[17] P.M. Gruber: *Comparisons of best and random approximation of convex bodies by polytopes.* Rend. Circ. Mat. Palermo, 50:189–216, 1997.

[18] P.M. Gruber: *Error of asymptotic formulae for volume approximation of convex bodies in $\mathbb{E}^3$.* (submitted)

[19] J. Linhart: *Kantenkrümmung und Umkugelradius konvexer Polyeder* Acta Math. Acad. Sci. Hungar. 34:1–2, 1979.

[20] M. Reizner: *Stochastical approximation of smooth convex bodies.* Trans. AMS, (accepted).

[21] R. Schneider: *Convex bodies: the Brunn–Minkowski theory.* Cambridge Univ. Press, Cambridge, 1993.

[22] E. Steinitz: *Über isoperimetrische Probleme bei konvexen Polyedern, I, II* J. reine angew. Math. 158:129–153, 1927, 159:133–143, 1928.

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