Optimizing Area and Perimeter of Convex Sets for Fixed Circumradius and Inradius

By

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Abstract: We determine the planar convex sets which have maximum and minimum area or perimeter when the circumradius and the inradius are given. We use then these results to obtain complete systems of inequalities for the cases (A, R, r) and (p, R, r).

Keywords: Complete Systems of Inequalities, Extremum Problems, Area, Perimeter, Circumradius, Inradius.

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1. Introduction

Let K be a convex set in the plane. Associated with K are a number of well-known functionals: the area $A = A(K)$, the perimeter $p = p(K)$, the diameter $D = D(K)$, the minimal width $\omega = \omega(K)$, the inradius $r = r(K)$ and the circumradius $R = R(K)$. For many years mathematicians have been interested in inequalities involving these functionals; and moreover, in many cases the question arises for which convex sets the equality sign is attained, this is, to determine the extremal sets.

Each new inequality obtained is interesting on its own, but it is also possible to ask if a collection of inequalities involving several geometric magnitudes is large enough to determine the existence of the figure. Such a collection is called a *complete system of inequalities*: a system of inequalities relating all the geometric characteristics such that for any set of numbers satisfying those conditions, a planar figure with these values of the characteristics exists in the given class.

In 1961, Santaló [6] studied complete systems of inequalities concerning triples of the six classic geometric measures: he asked for a characterization of the set of all points in E^3 of the form $(a_1(K),\ a_2(K),\ a_3(K))$, where a_i, $i = 1, 2, 3$ represent three of the six classic geometric quantities, as K ranges over the family of all compact convex sets in E^2. Following an approach by Blaschke [1], Santaló proposed mapping the family of compact planar convex sets into a compact region of the unit square $[0, 1] \times [0, 1] \subset E^2$, which is called the *Santaló Diagram* (see Section 5). The extremal sets for the considered inequalities were mapped into the boundary points of this diagram.

The solution is trivial for subsets consisting of a single quantity. Suitable known inequalities form a complete system of inequalities for each pair of these quantities, as observed by Santaló [6], who also provided the solutions for (A, p, ω), (A, p, r), (A, p, R), (A, D, ω), (p, D, ω), and (D, r, R). He left the remaining cases as open problems. Recently, in [3], [4] and [5], the cases (D, ω, R), (ω, R, r), (D, ω, r), (A, D, R) and (p, D, R) have been settled by the second author and Segura Gomis.

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In this paper, we derive four new inequalities relating the area or the perimeter with the circumradius and the inradius of a planar convex set (Theorems 1 and 2). More precisely, we determine the sets with maximum and minimum area or perimeter for fixed circumradius and inradius. Then, we will use these results to obtain the complete systems of inequalities for the cases (A, R, r) and (p, R, r), determining their corresponding Santaló Diagrams.

2. Results

For the sake of brevity, let us denote by $B^2(\rho)$ the disc centered in the origin of coordinates O and with radius ρ.

For the area, the circumradius and the inradius of a planar convex set K, the well-known relationships between *pairs* of these geometric measures are (see, for instance, [2])

$A \leq \pi R^2$	Equality for the circle	(1)
$A \geq \pi r^2$	Equality for the circle	(2)
$r \leq R$	Equality for the circle	(3)

Now, for the perimeter, the circumradius and the inradius of K, the well-known relationships between *pairs* of these geometric measures are (3) and

$p \leq 2\pi R$	Equality for the circle	(4)
$p \geq 4R$	Equality for the line segment	(5)
$p \geq 2\pi r$	Equality for the circle	(6)

(see also [2]).

But in both cases (A, R, r) and (p, R, r), no inequality relating the three measures is known. We prove the following theorems.

Theorem 1. *Let K be a compact convex set in the euclidean plane E^2. Then,*

$$A \leq 2\left(r\sqrt{R^2-r^2} + R^2 \arcsin\frac{r}{R}\right) \tag{7}$$

and

$$p \leq 4\left(\sqrt{R^2-r^2} + R \arcsin\frac{r}{R}\right). \tag{8}$$

In both inequalities, equality holds when and only when K is the circular symmetric slice, i.e., the part of the disc $B^2(R)$ bounded by two parallel lines equidistant from the center O, at distance apart $2r$. We denote it by K^s.

Theorem 2. *Let K be a compact convex set in the euclidean plane E^2. Then,*

$$A \geq 2r\left(\sqrt{R^2-r^2} + r \arcsin\frac{r}{R}\right) \tag{9}$$

and

$$p \geq 4\left(\sqrt{R^2-r^2} + r \arcsin\frac{r}{R}\right). \tag{10}$$

In both inequalities, equality holds when and only when K is the symmetric cap-body generated by two points, i.e., the convex hull of the disc $B^2(r)$ and two centrally symmetric points at distance apart $2R$. We denote it by K_2^c.

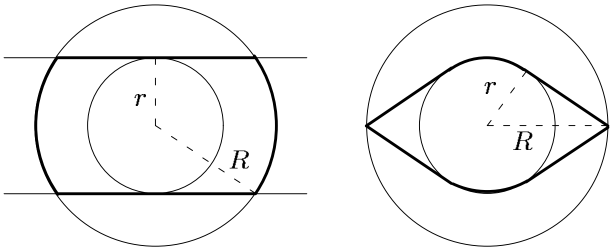

Figure 1. The extremal sets for Theorems 1 and 2

In the last section of this paper, we will use these inequalities to obtain the solutions for the Santaló problems $(A,\ R,\ r)$ and $(p,\ R,\ r)$, i.e., we will prove the following results.

Theorem 3. *Inequalities (1), (3), (7) and (9) form a complete system of inequalities for the case $(A,\ R,\ r)$.*

Theorem 4. *Inequalities (3), (4), (8) and (10) form a complete system of inequalities for the case $(p,\ R,\ r)$.*

3. Maximizing the Area and the Perimeter

In this section we are going to prove Theorem 1. We will do it in different steps by stating the following four preliminary lemmas.

Lemma 1. *For a triangle T with inradius r and for a $P \in T$, let x_1, x_2 and x_3 be the distances from P to each side of T. Then there exist $i, j \in \{1, 2, 3\}$ such that $x_i + x_j \leq 2r$.*

Proof. The three radius of the incircle which are perpendicular to the sides of the triangle, say r_1, r_2 and r_3, divide T in three regions with empty intersection (see Figure 2).

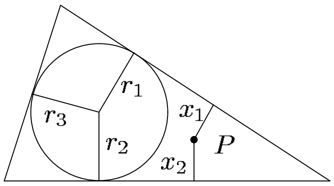

Figure 2.

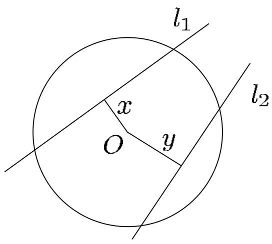

Figure 3.

If P lies within the region determined by r_i and r_j, then it is easy to see that $x_i + x_j \leq 2r$. □

Lemma 2. *Let l_1 and l_2 be two secant lines to the disc $B^2(R)$ such that the intersection point $l_1 \cap l_2$ lies outside the circle and the origin belongs to the strip determined by these two lines. Let K be the convex region bounded by l_1, l_2 and $B^2(R)$ (see Figure 3). Now, let us denote by x and y the distances from the origin O to l_1 and l_2, respectively. If $x + y \leq 2r$ then*

$$A(K) \leq 2\left(r\sqrt{R^2 - r^2} + R^2 \arcsin \frac{r}{R}\right)$$

and

$$p(K)\leq 4\left(\sqrt{R^2-r^2}+R\arcsin\frac{r}{R}\right),$$

with equality, in both inequalities, when and only when $x=y=r$.

Proof. If we compute the area and the perimeter of K in terms of x and y we obtain

$$A(K)=f(x,y)=x\sqrt{R^2-x^2}+y\sqrt{R^2-y^2}+R^2\left(\arcsin\frac{x}{R}+\arcsin\frac{y}{R}\right)$$

and

$$p(K)=g(x,y)=2\left[\sqrt{R^2-x^2}+\sqrt{R^2-y^2}+R\left(\arcsin\frac{x}{R}+\arcsin\frac{y}{R}\right)\right].$$

Then, it is easy to check that the first derivatives of f and g in each variable equal the positive functions

$$f_x=2\sqrt{R^2-x^2},\quad f_y=2\sqrt{R^2-y^2},\quad g_x=2\sqrt{\frac{R-x}{R+x}},\quad g_y=2\sqrt{\frac{R-y}{R+y}}.$$

So, $f(x,y)$ and $g(x,y)$ are increasing functions in each variable.

Now, let us suppose that $x+y<2r$. Then, there exists a positive real number x' such that $(x+x')+y=2r$, and both $f(x,y)<f(x+x',y)$ and $g(x,y)<g(x+x',y)$. So it suffices to prove the lemma in the case when $x+y=2r$. For we observe that in this case, the area and the perimeter of K are

$$f(x)=x\sqrt{R^2-x^2}+(2r-x)\sqrt{R^2-(2r-x)^2}+R^2\left(\arcsin\frac{x}{R}+\arcsin\frac{2r-x}{R}\right)$$

and

$$g(x)=2\left[\sqrt{R^2-x^2}+\sqrt{R^2-(2r-x)^2}+R\left(\arcsin\frac{x}{R}+\arcsin\frac{2r-x}{R}\right)\right],$$

respectively, where $0\leq x\leq 2r$. An elementary computation assures us that the first derivatives of both f and g,

$$f'(x)=2\left(\sqrt{R^2-x^2}-\sqrt{R^2-(2r-x)^2}\right),\ g'(x)=2\left(\sqrt{\tfrac{R-x}{R+x}}-\sqrt{\tfrac{R-(2r-x)}{R+(2r-x)}}\right)$$

vanish when and only when $x=r$. And also that $f''(r)<0$ and $g''(r)<0$. So, the absolute maximums of $f(x)$ and $g(x)$ are attained only when $x=r$; hence

$$A(K)\leq f(r)=2\left(r\sqrt{R^2-r^2}+R^2\arcsin\frac{r}{R}\right)$$

and

$$p(K)\leq g(r)=4\left(\sqrt{R^2-r^2}+R\arcsin\frac{r}{R}\right).$$

□

Lemma 3. *For two positive real numbers $r \le R$, let B_r be a disc with radius r contained in $B^2(R)$ and let s be the line that joins the centers of both discs (see Figure 4). We denote by l the tangent line to B_r which is perpendicular to s and whose distance from the origin is greater or equal than r. Finally, let l_1 and l_2 be two tangent lines to the disc B_r, symmetric with respect to s and such that the intersection point $l_1 \cap l$ (and so $l_2 \cap l$) lies outside $B^2(R)$. If K is the convex region bounded by l_1, l_2, l and $B^2(R)$, then*

$$A(K) \le 2\left(r\sqrt{R^2-r^2} + R^2 \arcsin\frac{r}{R}\right)$$

and

$$p(K) \le 4\left(\sqrt{R^2-r^2} + R \arcsin\frac{r}{R}\right).$$

In both inequalities, equality holds when and only when $B^2(R)$ and B_r are concentric, and $l_1 \equiv l_2$ is parallel to l; so, when $K = K^s$.

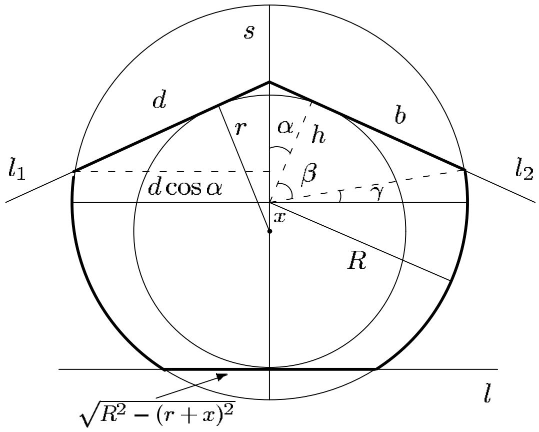

Figure 4.

Proof. Following the notation of Figure 4, it is easy to check that the following relations hold:

$$\begin{aligned} &h = r - x\cos\alpha, && b = \sqrt{R^2-(r-x\cos\alpha)^2},\\ &\gamma = \arcsin\frac{r-x\cos\alpha}{R} - \alpha, && \cos\beta = \frac{r-x\cos\alpha}{R},\\ &d = b + h\tan\alpha = \sqrt{R^2-(r-x\cos\alpha)^2} + (r-x\cos\alpha)\tan\alpha. \end{aligned} \tag{11}$$

Since x is the distance between the centers of both discs $B^2(R)$ and B_r, if we compute the area and the perimeter of K we obtain

$$A(K) = (r+x)\sqrt{R^2-(r+x)^2} + R^2\arcsin\frac{r+x}{R} + (r-x\cos\alpha)d + R^2\gamma,$$

$$p(K) = 2\left(\sqrt{R^2-(r+x)^2} + R\arcsin\frac{r+x}{R}\right) + 2d + 2R\gamma.$$

Using the relations of (11), $A(K)$ and $p(K)$ can be expressed as functions on the distance x and the angle α, say $A(K) = \widetilde{f}(x,\alpha)$ and $p(K) = \widetilde{g}(x,\alpha)$. We are going to see that $\widetilde{f}(x,\alpha)\le\widetilde{f}(0,\alpha)$ and $\widetilde{g}(x,\alpha)\le\widetilde{g}(0,\alpha)$, which means that the maximum for both the area and the perimeter is attained when $B_r=B^2(r)$, i.e., when $B^2(R)$ and B_r are concentric.

The case of the area is easy. The first derivative with respect to x of the function $\widetilde{f}(x,\alpha)$ takes the value

$$\widetilde{f}_x(x,\alpha)=2\left(\sqrt{R^2-(r+x)^2}-d\cos\alpha\right).$$

It is clear that $\sqrt{R^2-(r+x)^2}\le d\cos\alpha$ (see Figure 4), because the equality would be attained only when both discs are concentric and the lines l_1 and l are parallel. Hence, $\widetilde{f}_x(x,\alpha)\le 0$, and so, $\widetilde{f}$ is a decreasing function in x for each fixed value of α. Since $x\ge 0$, $\widetilde{f}(x,\alpha)\le\widetilde{f}(0,\alpha)$, as desired.

In the case of the perimeter, if we develop $\widetilde{g}(x,\alpha)$ by replacing d and γ by their corresponding values in (11), we obtain

$$\begin{aligned}\widetilde{g}(x,\alpha)=\;&2\left(\sqrt{R^2-(r+x)^2}+R\arcsin\frac{r+x}{R}\right)\\&+2\left(\sqrt{R^2-(r-x\cos\alpha)^2}+R\arcsin\frac{r-x\cos\alpha}{R}\right)\\&+2(r-x\cos\alpha)\tan\alpha-2R\alpha.\end{aligned}$$

Thus,

$$\widetilde{g}(x,\alpha)=g(r+x,r-x\cos\alpha)+2(r-x\cos\alpha)\tan\alpha-2R\alpha,$$

where g is the function defined in the proof of Lemma 2. From this proof it can be deduced that $g(u,v)\le g\left(\frac{u+v}{2},\frac{u+v}{2}\right)$. Therefore,

$$g(r+x,r-x\cos\alpha)\le 4\left(\sqrt{R^2-\left(r+x\tfrac{1-\cos\alpha}{2}\right)^2}+R\arcsin\frac{r+x\frac{1-\cos\alpha}{2}}{R}\right).$$

Let us denote by $h(x,\alpha)$ the function

$$\begin{aligned}h(x,\alpha)=\;&4\left(\sqrt{R^2-\left(r+x\tfrac{1-\cos\alpha}{2}\right)^2}+R\arcsin\frac{r+x\frac{1-\cos\alpha}{2}}{R}\right)\\&+2(r-x\cos\alpha)\tan\alpha-2R\alpha.\end{aligned}$$

Then, $\widetilde{g}(x,\alpha)\le h(x,\alpha)$. Now, if we compute the first derivative of $h(x,\alpha)$ with respect to x we obtain

$$\frac{1}{2}h_x(x,\alpha)=(1-\cos\alpha)\sqrt{\frac{R-\left(r+x\frac{1-\cos\alpha}{2}\right)}{R+\left(r+x\frac{1-\cos\alpha}{2}\right)}}-\sin\alpha.$$

Since the function $\sqrt{\frac{1-u}{1+u}}$ is decreasing, and it holds

$$\frac{r-x\cos\alpha}{R}\le\frac{r+x\frac{1-\cos\alpha}{2}}{R},$$

we have

$$\frac{1}{2}h_x(x,\alpha)\le(1-\cos\alpha)\sqrt{\frac{1-\frac{r-x\cos\alpha}{R}}{1+\frac{r-x\cos\alpha}{R}}}-\sin\alpha.$$

Finally, using (11), we obtain

$$\frac{1}{2}h_x(x,\alpha) \le (1-\cos\alpha)\sqrt{\frac{1-\cos\beta}{1+\cos\beta}} - \sin\alpha = (1-\cos\alpha)\tan\frac{\beta}{2} - \sin\alpha.$$

Because of the construction of K, $\alpha+\beta<\pi$; hence, $\tan\frac{\beta}{2} \le \tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1-\cos\alpha}$, and therefore $h_x(x,\alpha)\le 0$. Then, $h(x,\alpha)$ is a decreasing function in the variable x and $h(x,\alpha)\le h(0,\alpha)$.

But $h(0,\alpha)=\widetilde{g}(0,\alpha)$; so putting together the relations obtained for $\widetilde{g}$ and h, we finally have $\widetilde{g}(x,\alpha)\le h(x,\alpha)\le h(0,\alpha)=\widetilde{g}(0,\alpha)$, as desired.

Now, we know that both the area and the perimeter of K are maximum when the discs $B^2(R)$ and B_r are concentric. This fact implies that the distances from the origin O to the lines l, l_1 and l_2 are the same and equal to r. Again, if we denote by K' the convex region bounded by l, l_1 and $B^2(R)$, it holds $K\subset K'$. Therefore $A(K)\le A(K')$, $p(K)\le p(K')$, and Lemma 2 applied to this new set K' leads to the required result. The equality would be attained when $K=K'$ and, since r is the inradius of the extremal set, when l and l_1 are parallel; hence, equality holds only if $K=K^s$. □

The following lemma states an analogous result to the preceding one, but when the symmetry condition on l_1 and l_2 is replaced by a more general hypothesis.

Lemma 4. *For two positive real numbers $r\le R$, let B_r be a disc with radius r contained in $B^2(R)$ and let s be the line that joins the centers of both discs. We denote by l the tangent line to B_r which is perpendicular to s and whose distance from the origin is greater or equal than r. Finally, let l_1 and l_2 be two tangent lines to the disc B_r, such that s separates the contact points of l_1 and l_2 with B_r, and such that the intersection points $l_1\cap l$ and $l_2\cap l$ lie outside $B^2(R)$. Let us suppose also that the angle* $\operatorname{ang}(l_1,l_2)\ge\pi/2$. *If K is the convex region bounded by l_1, l_2, l and $B^2(R)$, then*

$$A(K)\le 2\left(r\sqrt{R^2-r^2}+R^2\arcsin\frac{r}{R}\right)$$

and

$$p(K)\le 4\left(\sqrt{R^2-r^2}+R\arcsin\frac{r}{R}\right).$$

In both inequalities, equality holds when and only when $B^2(R)$ and B_r are concentric, and $l_1\equiv l_2$ is parallel to l; so, when $K=K^s$.

Proof. Let K_1 and K_2 be the convex regions bounded by l_1, l, s and $B^2(R)$, and l_2, l, s and $B^2(R)$, respectively. Then, since s separates the contact points of l_1 and l_2 with B_r, $K_1\cup K_2$ is a disjoint union with $K\subset K_1\cup K_2$.

We distinguish two cases. If the intersection point $l_1\cap s$ lies within $B^2(R)$, we take the symmetral of K_1 about s, K_1^*, and the union $K_1\cup K_1^*$ (see Figure 5). Then, applying Lemma 3 to this set, we have

$$A(K_1)=A(K_1\cup K_1^*)/2\le r\sqrt{R^2-r^2}+R^2\arcsin(r/R).$$

Let us denote by $p'(K_1)$ the *relative perimeter* of the set K_1, i.e., the difference between the perimeter of K_1 and the length of the line segment $s \cap K_1$. Then, Lemma 3 also assures us that

$$p'(K_1) = p(K_1 \cup K_1^*)/2 \le 2\left(r\sqrt{R^2 - r^2} + R^2 \arcsin(r/R)\right).$$

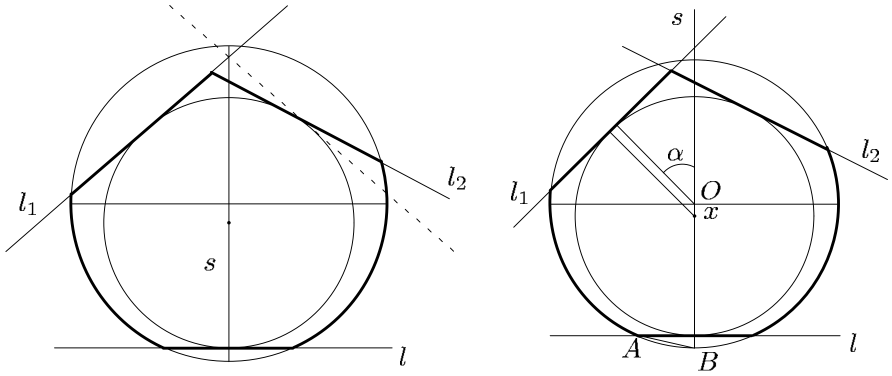

Figure 5.

Now, we suppose that l_1 intersects s outside $B^2(R)$. Let K_1' be the region bounded by l_1, s and $B^2(R)$, which contains K_1. The distance from the origin to the line l_1 equals $r - x\cos\alpha$ (see Figure 5). Since, by hypothesis, $\operatorname{ang}(l_1, l_2) \ge \pi/2$ and also s separates the contact points $l_1 \cap B_r$ and $l_2 \cap B_r$, it is clear that $\alpha \le \pi/2$. Hence, $r - x\cos\alpha \le r$, and then

$$\begin{aligned} A(K_1) \le A(K_1') &= (r - x\cos\alpha)\sqrt{R^2 - (r - x\cos\alpha)^2} + R^2 \arcsin\frac{r - x\cos\alpha}{R} \\ &\le r\sqrt{R^2 - r^2} + R^2 \arcsin\frac{r}{R}. \end{aligned}$$

For the perimeter, we note that the distance from the point labelled A in Figure 5 to s is always less or equal than the length of the arc $\overset{\frown}{AB}$, due to B lies on the line s. So,

$$\begin{aligned} p'(K_1) \le p'(K_1') &= 2\left(\sqrt{R^2 - (r - x\cos\alpha)^2} + R\arcsin\frac{r - x\cos\alpha}{R}\right) \\ &\le 2\left(\sqrt{R^2 - r^2} + R\arcsin\frac{r}{R}\right). \end{aligned}$$

Using the same argument for the set K_2, we can also see that

$$A(K_2) \le r\sqrt{R^2 - r^2} + R^2 \arcsin\frac{r}{R}, \quad p'(K_2) \le 2\left(\sqrt{R^2 - r^2} + R\arcsin\frac{r}{R}\right).$$

Therefore, we can conclude that

$$A(K) \le A(K_1 \cup K_2) = A(K_1) + A(K_2) \le 2\left(r\sqrt{R^2 - r^2} + R^2 \arcsin\frac{r}{R}\right),$$

and

$$p(K) \le p(K_1 \cup K_2) = p'(K_1) + p'(K_2) \le 4\left(\sqrt{R^2 - r^2} + R\arcsin\frac{r}{R}\right).$$

□

Now we can conclude the proof of Theorem 1.

Let $B^2(R)$ and B_r be the circumcircle and an incircle of K, respectively. It is known (see [2]) that the incircle B_r of K meets the boundary of K either in two diametrically opposite points, or in three points that form the vertices of an acute-angled triangle. In the first case, they will exist two parallel support lines to the set K, l_1 and l_2, throught those two points, at distance apart $2r$. Let K' denote the intersection of the strip determined by these lines with the disc $B^2(R)$. Then $K \subset K'$ and so,

$$A(K) \leq A(K') \text{ and } p(K) \leq p(K').$$

Since the sum of the distances from the origin O to both l_1 and l_2 is equal to $2r$, Lemma 2 leads to the result, with the equality only when both distances are equal to r and hence, only when $K = K^s$.

So, we can suppose that the incircle B_r meets the boundary of K in three points. Thus, there are three support lines to both B_r and K, l_1, l_2 and l', which form a triangle T that contains K. Moreover, $K \subset T \cap B^2(R)$. Let us denote by x_1, x_2 and x' the distances from the origin O to each of those lines l_1, l_2 and l', respectively. We have to distinguish two opposite cases:

CASE 1. We suppose that there exist two sides of the triangle, say l_1 and l_2, such that both the sum of the distances $x_1 + x_2 \leq 2r$ and the intersection point $l_1 \cap l_2$ lies outside the disc $B^2(R)$. Then, if K' is the region bounded by l_1, l_2 and $B^2(R)$, $K \subset K'$, and applying Lemma 2 we obtain the result.

CASE 2. Now we suppose that there are not two sides of the triangle such that both facts occur simultaneously, i.e., the intersection point of both sides lies outside the circumcircle and also the sum of the distances from O to those lines is not greater than $2r$.

Since Lemma 1 assures us that there exist two sides, say l_1 and l_2, with $x_1 + x_2 \leq 2r$, the above assumption guarantees that $l_1 \cap l_2 \in B^2(R)$. Therefore, the other two vertices $l_1 \cap l', l_2 \cap l' \notin \mathrm{int}(B^2(R))$ because R is the circumradius of K. But then, using again the assumption of this case, we have

$$x_1 + x' > 2r \qquad \text{and} \qquad x_2 + x' > 2r. \tag{12}$$

Let r_1 and r_2 denote the radii of B_r which are perpendicular to l_1 and l_2, respectively (see Figure 6), and let $\mathcal{Q}$ denote the kite determined by r_1, r_2, l_1 and l_2. We also represent by l the line tangent to B_r which is perpendicular to the one joining O and the center of B_r; so, l is the tangent line to B_r whose distance from the origin is the greatest possible.

Thus, the following properties hold:

(i) The conditions of (12) and Lemma 1 assure that O lies within the interior of the kite $\mathcal{Q}$.

(ii) Besides, since $l_1 \cap l_2 \in \mathrm{int}\, B^2(R)$, the triangle T contains a diameter of the circumcircle, and also the angle $\mathrm{ang}(l_1, l_2) \geq \pi/2$; otherwise, the circumradius of K would be strictly less than R.

Then, properties (i) and (ii) imply that the points $l_1 \cap l$ and $l_2 \cap l$ lie outside $B^2(R)$.

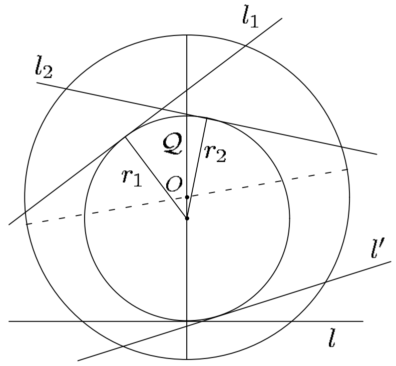

Figure 6

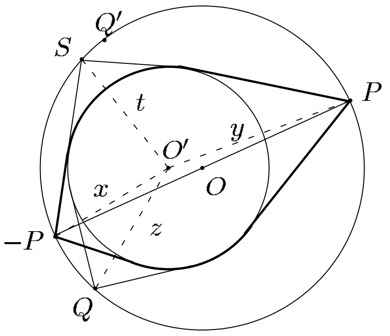

Figure 7

Let $\bar{K}$ be the region determined by the lines l_1, l_2, l and the disc $B^2(R)$. Lemma 4 assures us that

$$A(\bar{K}) \leq 2\left(r\sqrt{R^2-r^2} + R^2 \arcsin\frac{r}{R}\right), \quad p(\bar{K}) \leq 4\left(\sqrt{R^2-r^2} + R \arcsin\frac{r}{R}\right).$$

Now $K \subset T \cap B^2(R)$ yields that $A(K) \leq A(T \cap B^2(R))$ and $p(K) \leq p(T \cap B^2(R))$. And moreover; as the distance from O to l is greater or equal than the distance from this point to l', it is clear that $A(T \cap B^2(R)) \leq A(\bar{K})$ and $p(T \cap B^2(R)) \leq p(\bar{K})$. Therefore we have verified that

$$A(K) \leq A(T \cap B^2(R)) \leq A(\bar{K}) \leq 2\left(r\sqrt{R^2-r^2} + R^2 \arcsin\frac{r}{R}\right)$$

and

$$p(K) \leq p(T \cap B^2(R)) \leq p(\bar{K}) \leq 4\left(\sqrt{R^2-r^2} + R \arcsin\frac{r}{R}\right).$$

The equality will be attained, in both inequalities, only when $l' \equiv l$, and because of Lemma 4, only when $K = K^s$. This concludes the proof of Theorem 1.

4. Minimizing the Area and the Perimeter

Now we prove Theorem 2. Let $B^2(R)$ and B_r be the circumcircle and an incircle of K respectively, with centers O and O'. It is known (see [2]) that the circumcircle $B^2(R)$ either contains two points of the boundary of K, which are diametrically opposite, or it contains three points of K that form the vertices of an acute-angled triangle. We name these points P, Q and S (in the case of two symmetric points, then $Q = S = -P$). Then, the diameter that passes through P, i.e., the line $(-P)P$, separates Q from S (see Figure 7). Besides, the segment lines $\overline{PQ}$, $\overline{QS}$ and $\overline{SP}$ either intersect or have to be tangent to the incircle B_r; if not the inradius of K should be strictly greater than r.

Let us denote by x, y, z and t the distances from the center of the incircle, O', to each point $-P$, P, Q and S, respectively. Without loss of generality, we can suppose that $y \geq z \geq t$. It is clear that, on the boundary of the circumcircle $B^2(R)$, there exists another point Q', whose distance from O' is also z, the distance from O' to Q. Since $z \geq t$, this new point Q' lies on the boundary of $B^2(R)$ at the right hand side of S (in clockwise order). Hence, all the points of the arc QQ' have less distance to O' than Q. Thus, $x \leq z$.

Let K_3 be the convex hull of B_r and the three points P, Q and S. We also denote by K_2 the convex hull of B_r and the points $-P$ and P. Then, clearly $r(K_2) = r(K) = r(K_3)$ and $R(K_2) = R(K) = R(K_3)$. Besides, K contains K_3 and hence, $A(K) \geq A(K_3)$ and $p(K) \geq p(K_3)$. Now, since $-P$ is closer to O' than Q ($x \leq z$), it holds that $A(K_2) \leq A(K_3)$ and $p(K_2) \leq p(K_3)$. Therefore, $A(K) \geq A(K_2)$ and $p(K) \geq p(K_2)$.

If we compute the formula for the area of K_2 in terms of x and y we obtain

$$A(K_2) = r\left(\sqrt{x^2 - r^2} + \sqrt{y^2 - r^2} + r \arcsin\frac{r}{x} + r \arcsin\frac{r}{y}\right) = rf(x, y).$$

We are going to show that $f(x, y) \geq f\left(\frac{x+y}{2}, \frac{x+y}{2}\right)$. For it suffices to see that the function $f(x, y)$ has a relative minimum in the point $(a/2, a/2)$ under the condition $x + y = a$, which is an elementary exercise of calculus. Besides, this minimum is unique. So,

$$A(K_2) = rf(x, y) \geq rf\left(\tfrac{x+y}{2}, \tfrac{x+y}{2}\right) = 2r\left(\sqrt{\left(\tfrac{x+y}{2}\right)^2 - r^2} + r \arcsin\frac{2r}{x+y}\right).$$

Since $x + y \geq 2R$ and $\sqrt{u^2 - r^2} + r\arcsin(r/u)$ is an increasing function in the variable u, we can finally deduce inequality (9):

$$A(K) \geq A(K_2) \geq 2r\left(\sqrt{R^2 - r^2} + r \arcsin\frac{r}{R}\right).$$

On the other hand, a general cap-body (not necessarily symmetric) verifies the relation $2A = pr$ (see [2]). In particular, this relation holds for our set K_2. Therefore,

$$p(K) \geq p(K_2) = \frac{2}{r}A(K_2) \geq 4\left(\sqrt{R^2 - r^2} + r \arcsin\frac{r}{R}\right),$$

which proves inequality (10).

The equality, in both inequalities, will be attained when and only when $K = K_2$ and $x = y = R$. As $x = y = R$ is equivalent to the fact that $O' \equiv O$, the equality will hold only when $K = K_2^c$. This concludes the proof of Theorem 2.

5. Complete Systems of Inequalities for (A, R, r) and (p, R, r)

Let (a_1, a_2, a_3) be any triple of the considered measures. Using Blaschke-Santaló's approach, we can observe that the problem of finding a complete system of inequalities for (a_1, a_2, a_3) can be expressed by mapping each compact convex set K to a point $(x, y) \in [0, 1] \times [0, 1]$. In this diagram x and y represent particular functions of two of the measures a_1, a_2, and a_3, which are invariant under dilatations; each of those functions depends on the specific problem that is considered in each case.

Blaschke's convergence theorem states that any given infinite uniformly bounded family of compact convex sets contains a sequence that converges to a compact convex set. So, by Blaschke's theorem, the range of this map $\mathcal{D}$ is a closed subset of the square $[0, 1] \times [0, 1]$.

Each of the optimal inequalities relating a_1, a_2, and a_3 determines part of the boundary of $\mathcal{D}$. As a consequence, the set of inequalities determines the whole boundary of $\mathcal{D}$ if and only if these inequalities form a complete system; if some inequality is missing, some part of the boundary of $\mathcal{D}$ remains unknown. But moreover; the sets which are mapped into the boundary points of $\mathcal{D}$ are the extremal sets of each considered inequality (i.e. the convex sets that maximize or minimize a particular measure). So, from a geometric point of view, it is sufficient to find the extremal sets of the corresponding inequalities to close the diagram.

5.1. Proof of Theorem 3: The case (A, R, r). Let us make the following choice of coordinates:

$$x = \frac{A}{\pi R^2} \quad \text{and} \quad y = \frac{r}{R}.$$

Inequalities (1) and (3) assure us that $0 \le x \le 1$ and $0 \le y \le 1$. So, $(x, y) \in [0, 1] \times [0, 1]$.

If we rewrite inequality (7) in terms of the x and y coordinates, we obtain

$$\pi x \le 2\left(y\sqrt{1-y^2} + \arcsin y\right),$$

and the equality is attained only for the circular symmetric slices. Therefore, we have the lower part of the boundary of $\mathcal{D}$ determined by the curve $\pi x = 2(y\sqrt{1-y^2} + \arcsin y)$, from the point $O = (0, 0)$, which corresponds to the line segments to $B = (1, 1)$, that is, the discs. The circular symmetric slices are mapped to the points of the curve OB (see Figure 8).

Now we consider inequality (9), which is equivalent to

$$\pi x \ge 2y\left(\sqrt{1-y^2} + y \arcsin y\right).$$

The corresponding equation $\pi x = 2y(\sqrt{1-y^2}+y\arcsin y)$ completes the boundary of $\mathcal{D}$: this curve connects the point O with the point B , and the family of the cap-bodies is mapped to the points of this curve.

If we use inequality (2), we obtain a curve in which the equality holds just for the disc, that is, for the point $B = (1, 1)$. Therefore, such a bound is too wide because there are no other figures for which equality holds. It gives no information, and hence, can be removed. Thus, we have determined the boundary of the domain corresponding to the planar convex sets with given area, circumradius and inradius.

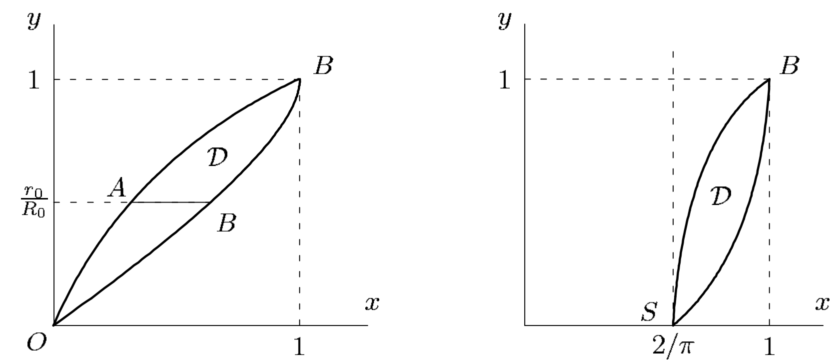

Figure 8. **Figure 9.**

Figures 8-9. Santaló Diagrams for the cases (A, R, r) and (p, R, r).

To complete this case, we must also show that $\mathcal{D}$ is simply connected, i.e., some convex set is mapped into each interior point in the diagram. To this end, let r_0 and R_0 be two positive real numbers with $r_0 \leq R_0$. We construct a continuous family of convex sets $\{K_\alpha : \alpha \in [0, \arcsin(r_0/R_0)]\}$ in the following way:

Let us consider the concentric discs $B^2(r_0)$ and $B^2(R_0)$ with radius r_0 and R_0, respectively. Now we take the sequence of 2-cap bodies

$$\operatorname{conv}\{B^2(r_0), (x_0, 0), (-x_0, 0)\}, \quad \text{where } x_0 \geq R_0,$$

whose limit when x_0 tends to infinity is the infinite strip with width $2r_0$. The intersection of each of these sets with the disc $B^2(R_0)$ (see Figure 10) forms a continuous family of sets K_α verifying $R(K_\alpha) = R_0$, $r(K_\alpha) = r_0$, and such that the area $A(K_\alpha)$ is a continuous and increasing function, where $\alpha \in [0, \arcsin(r_0/R_0)]$. This area is expressed by means of the linear function

$$A(K_\alpha) = 2\left(r_0\sqrt{R_0^2 - r_0^2} + r_0^2 \arcsin\frac{r_0}{R_0} + \alpha(R_0^2 - r_0^2)\right).$$

If $\alpha = 0$ we obtain the symmetric 2-cap body K_2^c, which will be mapped to the point labelled A in Figure 8. When $\alpha = \arcsin(r_0/R_0)$ we obtain the circular symmetric slice K^s, which corresponds to the point labelled B in the diagram. Since the circumradius and the inradius remain constant for the sets K_α of the family, and the area increases continuously, all these sets will be mapped to the points of the horizontal line segment $\overline{AB}$.

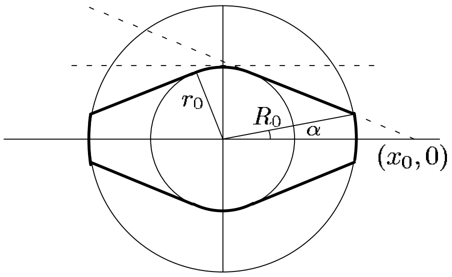

Figure 10. Continuous family of sets that fills $\mathcal{D}$.

This completes the proof of the case, and allows us to state Theorem 3.

5.2. PROOF OF THEOREM 4: THE CASE (p, R, r). Now, we make the choice of coordinates

$$x = \frac{p}{2\pi R} \quad \text{and} \quad y = \frac{r}{R}.$$

Inequalities (4) and (3) assure again that $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So, $(x, y) \in [0, 1] \times [0, 1]$.

Using inequality (8), we can translate it in terms of x and y, obtaining

$$\pi x \leq 2\left(\sqrt{1 - y^2} + \arcsin y\right).$$

Thus, we have the lower part of the boundary of $\mathcal{D}$ determined by the curve $\pi x = 2(\sqrt{1 - y^2} + \arcsin y)$, from the point $S = (2/\pi, 0)$, which corresponds to the line segments to $B = (1, 1)$, representing the discs. The circular symmetric slices are mapped to the points of the curve SB (see Figure 9).

Inequality (10) is equivalent to

$$\pi x \geq 2\left(\sqrt{1-y^2} + y \arcsin y\right),$$

and hence, the equation $\pi x = 2(\sqrt{1-y^2} + y \arcsin y)$ gives the upper part of the boundary of $\mathcal{D}$, joining S and B; the family of the cap-bodies is mapped to the points of this curve.

Note that again, inequalities (5) and (6) give no further information, and hence, they are superfluous for the complete system.

Following the same argument that we used in the precedent case, with the same family of sets, we can see that the domain $\mathcal{D}$ bounded by these two curves is simply connected. This finishes the proof of the case (p, R, r), and states the theorem.

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