Transfinite constructions in $V = L$

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2-point sets

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Theorem. (Bouhjar, Dijkstra, and van Mill) It cannot be F_σ!

Inductive proof

Standard proof of the existence:

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Question

The freedom of choice is very large. Could it be done in a "nice" way?

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Irregularity properties

- $Con(\omega_1 < 2^\omega$+exists an A $\mathbf{\Pi}^1_1$, such that $|A| = \omega_1)$
- $\Rightarrow$ $Con(\exists$ an uncountable coanalytic set without a perfect subset)

Miller's theorem

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- Hamel basis
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Method

Miller's method is frequently needed, but he does not give a general condition. The proof is hard, uses effective descriptive set theory and model theory.

General method

$x \leq_T y$

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Cofinality in the Turing degrees

Definition. A set $X \subset \mathbb{R}$ is *cofinal in the Turing degrees* if $(\forall z \in \mathbb{R})(\exists y \in X)(y \leq_T x)$.

General method

Compatibility

Definition. Let $F \subset \mathbb{R}^{\leq\omega} \times \mathbb{R} \times \mathbb{R}$, and $X \subset \mathbb{R}$. We say that X is compatible with F if there exist enumerations $P = \{p_\alpha : \alpha < \omega_1\}$, $X = \{x_\alpha : \alpha < \omega_1\}$ and for every $\alpha < \omega_1$ a sequence $A_\alpha \in \mathbb{R}^{\leq\omega}$ that is an enumeration of $\{x_\beta : \beta < \alpha\}$ in type $\leq \omega$ such that $(\forall \alpha < \omega_1)(x_\alpha \in F_{(A_\alpha, p_\alpha)})$ holds.

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Theorem 1. (V=L) Suppose that $F \subset \mathbb{R}^{\leq\omega} \times \mathbb{R} \times \mathbb{R}$ is a coanalytic set and for all $p \in \mathbb{R}$, $A \in \mathbb{R}^{\leq\omega}$ the section $F_{(A,p)}$ is cofinal in the Turing degrees.

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Effectiveness

$\Sigma_1^0(y)$, $\Pi_1^0(y)$

Definition. Let $\{I_n : n \in \omega\}$ be a recursive enumeration of the open intervals with rational endpoints. An open set G is called *recursive in* y, iff $\{n \in \omega : I_n \subset G\}$ (as an element of 2^ω) $\leq_T y$. (denoted by $\Sigma_1^0(y)$).

$$\Pi_1^0(y) = \{G^c : G \in \Sigma_1^0(y)\}$$

We can define these classes similarly for subsets of ω, $\omega \times \mathbb{R}$, $\mathbb{R}^2$ etc. using a recursive enumeration of $\{n\}$, $\{n\} \times I_m$, $I_n \times I_m$ etc.

The lightface classes

Let us define for $n \geq 2$

$$\Sigma^0_n(y) = \{proj_{\mathbb{R}}(A) : A \subset \mathbb{R} \times \omega, A \in \Pi^0_{n-1}(y)\},$$

$$\Pi^0_n(y) = \{A^c : A \in \Sigma^0_n(y)\}.$$

Furthermore:

$$\Sigma^1_1(y) = \{proj_{\mathbb{R}}(A) : A \subset \mathbb{R} \times \mathbb{R}, A \in \Pi^0_2(y)\},$$

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$$\Delta^1_1(y) = \Sigma^1_1(y) \cap \Pi^1_1(y).$$

For $x, y \subset \omega$ $x \in \Delta^1_1(y)$ is denoted by $x \leq_h y$.

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Lightface and boldface

$\boldsymbol{\Sigma}^{\mathbf{i}}_{\mathbf{j}} = \cup_{y \in \mathbb{R}} \Sigma^i_j(y)$

Cofinality in the hyperdegrees

Definition. A set $X \subset \mathbb{R}$ is called *cofinal in hyperdegrees* if $(\forall z \in \mathbb{R})(\exists y \in X)(z \leq_h y)$.

Strengthenigns

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Stronger version

Theorem 2. (V=L) Let $t \in \mathbb{R}$, $F \subset \mathbb{R}^{\leq\omega} \times \mathbb{R} \times \mathbb{R}$ be a $\Pi^1_1(t)$ set. Assume that for every $(A, p) \in \mathbb{R}^{\leq\omega} \times \mathbb{R}$ the section $F_{(A,p)}$ is cofinal in the hyperdegrees.

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Remark

The previous theorem holds true replacing $\mathbb{R}$ with $\mathbb{R}^n$, ω^ω or 2^ω.

A tipical application

Miller's results

Theorem 1. implies Miller's results: consistent existence of coanalytic MAD family, 2-point set and Hamel basis.

2-point set

Recall Theorem 1.

(V=L) If F is $\mathbf{\Pi}^1_1$ and every section $F_{(A,p)}$ is cofinal in the Turing degrees then there exists a Π^1_1 set X and enumerations $\mathbb{R} = \{p_\alpha : \alpha < \omega_1\}$, $X = \{x_\alpha : \alpha < \omega_1\}$, A_α of $\{x_\beta : \beta < \alpha\}$, such that $(\forall \alpha < \omega_1)(x_\alpha \in F_{(A_\alpha, p_\alpha)})$.

Proof

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Theorem 1 $\implies$ X is Π^1_1

Existence

(CH) There exists an uncountable $X \subset \mathbb{R}^2$ intersecting every C^1 curve in countably many points.

Consequences: C^1 curves

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Consistent nonexistence

Theorem (J. Hart, K. Kunen) (PFA) For every uncountable $X \subset \mathbb{R}^2$ there exists a C^1 curve intersecting it in uncountably many points.

Consequences: C^1 curves

General version

Theorem.(V=L) Suppose that $G \subset \mathbb{R} \times \mathbb{R}^n$ is a Borel set and for every countable $A \subset \mathbb{R}$ the complement of the set $\cup_{p \in A} G_p$ is cofinal in the Turing degrees. Then there exists an uncountable coanalytic set $X \subset \mathbb{R}^n$ which intersects every G_p in a countable set.

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Coanalytic in V=L

Theorem 1. implies that under (V=L) there exists an uncountable coanalytic $X \subset \mathbb{R}^2$ set intersecting every C^1 curve in countably many points.

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Remark

In almost every cases there are no $\mathbf{\Sigma}^1_1$ sets.

Thank you!