Haar null sets without G_δ hulls

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Abstract

Let G be an abelian Polish group, e.g. a separable Banach space. A subset $X \subset G$ is called *Haar null (in the sense of Christensen)* if there exists a Borel set $B \supset X$ and a Borel probability measure μ on G such that $\mu(B+g) = 0$ for every $g \in G$. The term *shy* is also commonly used for Haar null, and co-Haar null sets are often called *prevalent*.

Answering an old question of Mycielski we show that if G is *not* locally compact then there exists a Borel Haar null set that is not contained in any G_δ Haar null set. We also show that G_δ can be replaced by any other class of the Borel hierarchy. Moreover, we also produce a coanalytic generalised Haar null set without a Borel Haar null hull. Actually, the results readily generalise to all Polish groups that admit a two-sided invariant metric.

1 Introduction

Throughout the paper, let G be an abelian Polish group, that is, an abelian topological group that is separable and admits a complete metric (the group operation will be denoted by $+$ and the neutral element by 0). It is a well-known result of Birkhoff and Kakutani that any metrisable group admits a left invariant metric [2, 1.1.1], which is clearly two-sided invariant for abelian groups. Moreover, it is also well-known that a two-sided invariant metric on a Polish group is complete [2, 1.2.2]. Hence from now on let d be a fixed complete invariant metric on G. For the ease of notation we will restrict our attention to abelian groups, but we remark that all our results easily generalise to all Polish groups admitting a two-sided invariant metric.

If G is locally compact than there exists a Haar measure on G, that is, a regular invariant Borel measure that is finite for compact sets and positive for non-empty open sets. This measure, which is unique up to a positive multiplicative constant, plays a fundamental role in the study of locally compact groups. Unfortunately, it is known that non-locally compact Polish groups admit no Haar measure. However, the notion of a Haar nullset has a very well-behaved generalisation. The following definition was invented by Christensen [3], and later independently by Hunt, Sauer and Yorke [5].

Definition 1.1 *A set $X \subset G$ is called* Haar null *if there exists a Borel set $B \supset X$ and a Borel probability measure μ on G such that $\mu(B+g) = 0$ for every $g \in G$.*

Note that the term *shy* is also commonly used for Haar null, and co-Haar null sets are often called *prevalent*. Numerous authors actually use the following slightly weaker definition, in which B is only required to be universally measurable. (A set is called *universally measurable* if it is measurable with respect to every Borel probability measure.)

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Definition 1.2 *A set $X \subset G$ is called* generalised Haar null *if there exists a universally measurable set $B \supset X$ and a Borel probability measure μ on G such that $\mu(B+g)=0$ for every $g \in G$.*

In almost all applications X is actually Borel, so it does not matter which of the above two definitions we use. However, in the present paper this subtle difference will play some role.

Christensen showed that the Haar null and generalised Haar null sets form σ-ideals, and also that in locally compact groups a set is Haar null iff it is generalised Haar null iff it is of measure zero with respect to the Haar measure. During the last two decades Christensen's notion has been useful in studying exceptional sets in diverse areas such as analysis, functional analysis, dynamical systems, geometric measure theory, group theory, and descriptive set theory.

Therefore it is very important to understand the fundamental properties of this σ-ideal, such as the Fubini properties, ccc-ness, and all other similarities and differences between the locally compact and the general case.

One such example is the following very natural question, raised first by Mycielski [8, Problem P_1] more than 20 years ago, and also discussed in [7], [1] and [10].

Question 1.3 *Let G be a Polish group. Can every Haar null/generalised Haar null set be covered by a G_δ Haar null set?*

It is easy to see using the regularity of Haar measure that the answer to both versions of the question is in the affirmative if G is locally compact. However, Dougherty [7, p.86] showed that under the Continuum Hypothesis or Martin's Axiom the answer to the easier question (generalised Haar null sets) is in the negative in every non-locally compact Polish group with a two-sided invariant metric. Later Banakh [1] proved the same under slightly different set-theoretical assumptions. Dougherty uses transfinite induction, and Banakh's proof is basically an existence proof using that the so called cofinality of the σ-ideal of generalised Haar null sets is greater than the continuum in some models, hence these examples are very far from being Borel. Actually, they both produced generalised Haar null sets that cannot be covered by Borel Haar null sets.

The main goal of the present paper is to answer the stronger question by constructing a rather concrete Borel Haar null example without resorting to extra set theoretical assumptions.

Theorem 1.4 *If G is a non-locally compact abelian Polish group then there exists a Borel Haar null set $B \subset G$ that cannot be covered by a G_δ Haar null set.*

Actually, the proof will immediately yield that G_δ can be replaced by any other class of the Borel hierarchy. As usual, $\mathbf{\Pi}^0_\xi$ stands for the ξ'th multiplicative class of the Borel hierarchy.

Theorem 1.5 *If G is a non-locally compact abelian Polish group and $1 \le \xi < \omega_1$ then there exists a Borel Haar null set $B \subset G$ that cannot be covered by a $\mathbf{\Pi}^0_\xi$ Haar null set.*

With very similar methods we can actually improve upon the above mentioned results of Dougherty and Banakh in another direction, as well. As mentioned above, they proved that consistently there exists a universally measurable generalised Haar null set that cannot be covered by a Borel Haar null set. Recall that a set is *analytic* if it is the continuous image of a Borel set, and *coanalytic* if its complement is analytic. Analytic and coanalytic sets are known to be universally measurable. Since Solecki [10] proved that every analytic generalised Haar null set is contained in a Borel Haar null set, the following result is optimal.

Theorem 1.6 *If G is a non-locally compact abelian Polish group then there exists a coanalytic generalised Haar null set $P \subset G$ that cannot be covered by a Borel Haar null set.*

Remark 1.7 *We close this section by remarking that in both versions of the above definition we may require that the measure μ, which we will often refer to as a* witness measure*, has compact support. This is because in a Polish space for every Borel probability measure there exists a compact set with positive measure [6, 17.11], and then restricting the measure to this set and normalising yields a witness with a compact support.*

2 Notation and basic facts

The following notions and facts can all be found in [6].

Let $\mathcal{F}(G)$ denote the family of closed subsets of G equipped with the so called Effros Borel structure. Let $\mathcal{K}(G)$ be the family of compact subsets of G equipped with the Hausdorff metric. Then $\mathcal{K}(G)$ is a Borel subset of $\mathcal{F}(G)$ and the inherited Borel structure on $\mathcal{K}(G)$ coincides with the one given by the Hausdorff metric.

Let us denote by $\mathcal{P}(G)$ the set of Borel probability measures on G, where by Borel probability measure we mean the completion of a probability measure defined on the Borel sets. These measures form a Polish space equipped with the weak*-topology. For $\mu \in \mathcal{P}(G)$ we denote by $\operatorname{supp}(\mu)$ the support of μ, i.e. the minimal closed subset F of G so that $\mu(F) = 1$. Let $\mathcal{P}_c(G) = \{\mu \in \mathcal{P}(G) : \operatorname{supp}(\mu) \text{ is compact}\}$.

$\mathbf{\Pi}^0_\xi$ stands for the ξ'th multiplicative level of the Borel hierarchy, $\mathbf{\Delta}^1_1$, $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ denote the classes of Borel, analytic and coanalytic sets, respectively. For a Polish space X, $\mathbf{\Pi}^0_\xi(X)$, $\mathbf{\Delta}^1_1(X)$ etc. denote the collections of subsets of X in the appropriate classes. Symbols Γ and Λ will denote one of the above mentioned classes, and $\check{\Lambda} = \{A^c : A \in \Lambda\}$.

For a set $H \subset X \times Y$ we define its x-section as $H_x = \{y \in Y : (x, y) \in H\}$, and similarly if $H \subset X \times Y \times Z$ then $H_{x,y} = \{z \in Z : (x, y, z) \in H\}$, etc. For a function $f : X \times Y \to Z$ the x-section is the function $f_x : Y \to Z$ defined by $f_x(y) = f(x, y)$. We will sometimes also write $f_x = f(x, \cdot)$.

For $A, B \subset G$ let $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ and $A + B = \{a + b : a \in A, b \in B\}$. Let us denote by $B(g, r)$ and $\bar{B}(g, r)$ the open and closed ball centered at g of radius r.

3 The proofs

3.1 A function with a surprisingly thick graph

Throughout the proofs, let $\Gamma = \mathbf{\Delta}^1_1$ and $\Lambda = \mathbf{\Pi}^0_\xi$ for some $1 \le \xi < \omega_1$, or let $\Gamma = \mathbf{\Pi}^1_1$ and $\Lambda = \mathbf{\Delta}^1_1$.

The following result will be the starting point of our constructions. For a fixed measure μ statement 2. below describes the following strange phenomenon: There exists a Borel graph of a function in a product space such that every G_δ cover of the graph has a vertical section of positive measure.

Theorem 3.1 *Let $\Gamma = \mathbf{\Delta}^1_1$ and $\Lambda = \mathbf{\Pi}^0_\xi$ for some $1 \le \xi < \omega_1$, or let $\Gamma = \mathbf{\Pi}^1_1$ and $\Lambda = \mathbf{\Delta}^1_1$. Then there exists a partial function $f : \mathcal{P}_c(G) \times 2^\omega \to G$ with* $\operatorname{graph}(f) \in \Gamma$ *satisfying the following properties: $\forall \mu \in \mathcal{P}_c(G)$*

1. $(\forall x \in 2^\omega)\,[(\mu, x) \in \operatorname{dom}(f) \Rightarrow f(\mu, x) \in \operatorname{supp}(\mu)]$,
2. $(\forall S \in \Lambda(2^\omega \times G))\,[(\operatorname{graph}(f_\mu) \subset S \Rightarrow (\exists x \in 2^\omega)(\mu(S_x) > 0)]$.

Before the proof we need several technical lemmas.

Lemma 3.2 *$\mathcal{P}_c(G)$ is a Borel subset of $\mathcal{P}(G)$.*

PROOF. The map $\mu \mapsto \operatorname{supp}(\mu)$ between $\mathcal{P}(G)$ and $\mathcal{F}(G)$ is Borel (see [6, 17.38]) and $\mathcal{P}_c(G)$ is the preimage of $\mathcal{K}(G)$ under this map. $\square$

Lemma 3.3 *Let X be a Polish space and $C \subset \mathcal{P}_c(G) \times X \times G$ with $C \in \Gamma$. Then $\{(\mu, x) : \mu(C_{\mu,x}) > 0\} \in \Gamma$.*

PROOF. Let first $\Gamma = \mathbf{\Delta}^1_1$. If Y is a Borel space and $C \subset Y \times G$ is a Borel set then the map $\varphi \colon Y \times \mathcal{P}_c(G) \to [0, 1]$ defined by $\varphi(y, \mu) = \mu(C_y)$ is Borel ([6, 17.25]). Using this for $Y = \mathcal{P}_c(G) \times X$ we obtain that the map $\psi \colon \mathcal{P}_c(G) \times X \to [0, 1]$ given by $\psi(\mu, x) = \varphi((\mu, x), \mu) = \mu(C_{\mu,x})$ is also Borel. Then $\{(\mu, x) : \mu(C_{\mu,x}) > 0\} = \psi^{-1}((0, 1])$, hence Borel.

For $\Gamma = \mathbf{\Pi}^1_1$ this is simply a special case of [6, 36.23]. $\square$

Lemma 3.4 *The set $\{(\mu, g) : g \in \operatorname{supp}(\mu)\} \subset \mathcal{P}_c(G) \times G$ is Borel.*

PROOF. As mentioned above, the map $\mu \mapsto \operatorname{supp}(\mu)$ is Borel between $\mathcal{P}(G)$ and $\mathcal{F}(G)$, hence its restriction to $\mathcal{P}_c(G)$ is also Borel.

Let $E = \{(K, g) : K \in \mathcal{K}(G),\ g \in K\}$, which clearly is a closed subset of $\mathcal{K}(G) \times G$. If we denote by $\Psi : \mathcal{P}_c(G) \times G \to \mathcal{K}(G) \times G$ the Borel map defined by $(\mu, g) \mapsto (\operatorname{supp}(\mu), g)$ then we obtain that $\{(\mu, g) : g \in \operatorname{supp}(\mu)\} = \Psi^{-1}(E)$ is Borel. $\square$

Let us now prove Theorem 3.1.

PROOF. Let $U \in \Gamma(2^\omega \times 2^\omega \times G)$ be universal for the $\check{\Lambda}$ subsets of $2^\omega \times G$, that is, for every $A \in \check{\Lambda}(2^\omega \times G)$ there exists an $x \in 2^\omega$ such that $U_x = A$ (for the existence of such a set see [6, 22.3, 26.1]). Notice that $\check{\Lambda} \subset \Gamma$. Let

$$U' = \mathcal{P}_c(G) \times U.$$

Define

$$U'' = \{(\mu, x, g) \in \mathcal{P}_c(G) \times 2^\omega \times G : (\mu, x, x, g) \in U' \text{ and } \mu(U'_{\mu,x,x}) > 0\},$$

then $U'' \in \Gamma$ using that the map $(\mu, x, g) \mapsto (\mu, x, x, g)$ is continuous and by Lemma 3.3. Let

$$U''' = \{(\mu, x, g) \in U'' : g \in \operatorname{supp}(\mu)\},$$

then $U''' \in \Gamma$ by Lemma 3.4. Clearly,

$$U'''_{\mu,x} = \begin{cases} U'_{\mu,x,x} \cap \operatorname{supp}(\mu) & \text{if } \mu(U'_{\mu,x,x}) > 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since for all (μ, x) the section $U'''_{\mu,x}$ is either empty or has positive μ measure, by the 'large section uniformisation theorem' [6, 18.6] and the coanalytic uniformisation theorem [6, 36.14] there exists a partial function f with $\operatorname{graph}(f) \in \Gamma$ such that $\operatorname{dom}(f) = \{(\mu, x) \in \mathcal{P}_c(G) \times 2^\omega : \mu(U'_{\mu,x,x}) > 0\}$ and $\operatorname{graph}(f) \subset U'''$.

We claim that this f has all the required properties.

First, by the definition of U''', clearly $f(\mu, x) \in \operatorname{supp}(\mu)$ holds whenever $(\mu, x) \in \operatorname{dom}(f)$, hence Property 1. of Theorem 3.1 holds.

Let us now prove Property 2. Assume towards a contradiction that there exists $\mu \in \mathcal{P}_c(G)$ and $S \in \Lambda(2^\omega \times G)$ such that $\operatorname{graph}(f_\mu) \subset S$ and $\mu(S_x) = 0$ for every $x \in 2^\omega$. Define $B = (2^\omega \times G) \setminus S$. By the universality of U there exists $x \in 2^\omega$ such that $U_x = U'_{\mu,x} = B$. Now, for every $y \in 2^\omega$ the section B_y is of positive (actually full) μ measure, in particular $\mu(U'_{\mu,x,x}) > 0$, and therefore $(\mu, x) \in \mathrm{dom}(f)$ and

$$f(\mu, x) \in U'''_{\mu,x} \subset U''_{\mu,x} = U'_{\mu,x,x} = B_x.$$

However, $f(\mu, x) \in S_x = G \setminus B_x$, a contradiction. □

3.2 Translating the compact sets apart

This section heavily builds on ideas of Solecki [9], [10]. The main point is that if G is non-locally compact then one can apply a translation (chosen in a Borel way) to every compact subset of G so that the resulting translates are disjoint. (For technical reasons we will need to consider continuum many copies of each compact set and also to 'blow them up' by a fixed compact set C.)

Proposition 3.5 *Let $C \in \mathcal{K}(G)$ be fixed. Then there exists a Borel map $t : \mathcal{K}(G) \times 2^\omega \times 2^\omega \to G$ so that*

1. *if $(K, x, y) \neq (K', x', y')$ are elements of $\mathcal{K}(G) \times 2^\omega \times 2^\omega$ then*

$$(K - C + t(K, x, y)) \cap (K' - C' + t(K', x', y')) = \emptyset$$

2. *for every $K \in \mathcal{K}(G)$ and $y \in 2^\omega$ the map $t(K, \cdot, y)$ is continuous.*

Proof. We use Solecki's arguments [9], [10], which he used for different purposes, with some modifications. However, for the sake of completeness, we repeat large parts of his proofs.

Fix an increasing sequence of finite sets $Q_k \subset G$ with $0 \in Q_0$ such that $\cup_{k\in\omega} Q_k$ is dense in G.

Lemma 3.6 *For every $\varepsilon > 0$ there exists $\delta > 0$ and a sequence $\{g_k\}_{k\in\omega} \subset B(0, \varepsilon)$ such that for every distinct $k, k' \in \omega$*

$$d(Q_k + g_k, Q_{k'} + g_{k'}) \geq \delta.$$

Proof. Since G is not locally compact, there exists $\delta > 0$ and a countably infinite set $S \subset B(0, \varepsilon)$ such $d(s, s') \geq 2\delta$ for every distinct $s, s' \in S$.

Now we define g_k inductively as follows. Suppose that we are done for $i < k$. If for every $s \in S$ there are $a \in Q_k$, $i < k$ and $b \in Q_i$ with $d(a + s, b + g_i) < \delta$ then there is a pair s, s' of distinct members of S with the same a, i and b. But then

$$d(s, s') = d(a + s, a + s') \leq d(a + s, b + g_i) + d(b + g_i, a + s') < 2\delta,$$

a contradiction. Hence we can let $g_k = s$ for an appropriate $s \in S$. □

It is easy to see that using the previous lemma repeatedly we can inductively fix ε_n, $\delta_n < \varepsilon_n$ and sequences $\{g^n_k\}_{k\in\omega}$ such that for every $n \in \omega$

- $\{g^n_k\}_{k\in\omega} \subset B(0, \varepsilon_n)$,
- $d(Q_k + g^n_k, Q_{k'} + g^n_{k'}) \geq 2\delta_n$ for every distinct $k, k' \in \omega$,
- $\sum_{m>n} \varepsilon_m < \frac{\delta_n}{3}$.

Note that the second property implies that for every $n \in \omega$ the function $k \mapsto g^n_k$ is injective. Note also that $\varepsilon_n \to 0$ and hence $\delta_n \to 0$, moreover, $\sum \delta_n$ is also convergent.

Let us also fix a Borel injection $c : \mathcal{K}(G) \times 2^\omega \times 2^\omega \to \omega^\omega$ such that for each K and y the map $c(K, \cdot, y)$ is continuous. (E.g. fix a Borel injection $c_1 : \mathcal{K}(G) \to 2^\omega$ and continuous injection $c_2 : 2^\omega \times 2^\omega \times 2^\omega \to \omega^\omega$ and let $c(K, x, y) = c_2(c_1(K), x, y)$.)

Our goal now is to define $t(K,x,y)$, so let us fix a triple (K,x,y). First we define a sequence $\{h_n = h_n(K,x,y)\}_{n\in\omega}$ with $h_n \in \{g^n_k\}_{k\in\omega}$ as follows. Suppose that we are given h_i for $i<n$. By the density of $\cup_k Q_k$ we have $G = \cup_k(Q_k + B(0,\delta_n/2))$. Since $K-C$ is compact, there exists a minimal index $k_n(K,x,y)$ so that

$$K - C + \sum_{i<n} h_i \subset Q_{k_n(K,x,y)} + B(0,\delta_n/2).$$

Fix an injective map $\phi : \omega\times\omega \to \omega$ with $\phi(i,j) \geq i$ for every $i\in\omega$ and let

$$h_n = g^n_{\phi(k_n(K,x,y),c(K,x,y)(n))} \tag{1}$$

and

$$t(K,x,y) = \sum_{n\in\omega} h_n. \tag{2}$$

We claim that this function has the required properties.

First, it is well defined, that is, the sum is convergent since $h_n \in B(0,\varepsilon_n)$, and hence for all $n\in\omega$

$$\sum_{m>n} h_m \in \bar{B}(0,\delta_n/3). \tag{3}$$

In order to prove 1. of the Proposition, let us now fix $(K,x,y) \neq (K',x',y')$. Then there exists an $n\in\omega$ such that $c(K,x,y)(n) \neq c(K',x',y')(n)$. By the injectivity of ϕ and of the sequence $k \mapsto g^n_k$ and also by (1) we obtain that $h_n(K,x,y) \neq h_n(K',x',y')$. Denote by h_i and h'_i the elements $h_i(K,x,y)$ and $h_i(K',x',y')$, respectively. Set

$$k = \phi(k_n(K,x,y), c(K,x,y)(n)) \text{ and } k' = \phi(k_n(K',x',y'), c(K',x',y')(n)).$$

The condition $\phi(i,j)\geq i$ implies $k \geq k_n(K,x,y)$, hence $Q_k \supset Q_{k_n(K,x,y)}$ and similarly $k' \geq k_n(K',x',y')$, so $Q_{k'} \supset Q_{k_n(K',x',y')}$. Therefore, by the definition of k_n,

$$K - C + \sum_{i<n} h_i \in Q_k + B(0,\delta_n/2) \text{ and } K' - C + \sum_{i<n} h'_i \in Q_{k'} + B(0,\delta_n/2),$$

hence

$$K - C + \sum_{i\leq n} h_i \in Q_k + h_n + B(0,\delta_n/2) \text{ and } K' - C + \sum_{i\leq n} h'_i \in Q_{k'} + h'_n + B(0,\delta_n/2).$$

Thus, using the triangle inequality and the second property of the g^n_k we obtain

$$d(K - C + \sum_{i\leq n} h_i, K' - C + \sum_{i\leq n} h'_i) \geq d(Q_k + h_n, Q_{k'} + h'_n) - 2\cdot\frac{\delta_n}{2} =$$

$$= d(Q_k + g^n_k, Q_{k'} + g^n_{k'}) - \delta_n \geq 2\delta_n - \delta_n = \delta_n.$$

From this, using (3), we obtain $d(K-C+t(K,x,y), K'-C+t(K',x',y')) \geq \delta_n - 2\frac{\delta_n}{3} = \frac{\delta_n}{3} > 0$, which proves 1.

What remains to show is that t is a Borel map and for every K and y the map $t(K,\cdot,y)$ is continuous. But (3) shows that the series defining t in (2) is uniformly convergent, so the next lemma finishes the proof.

Lemma 3.7 *For every $n\in\omega$ the map h_n is Borel and for every K and y the map $h_n(K,\cdot,y)$ is continuous.*

PROOF. We will actually prove more by induction on n. Define $f_n\colon \mathcal{K}(G)\times 2^\omega \times 2^\omega \to \mathcal{K}(G)$ by

$$f_n(K,x,y) = K - C + \sum_{i<n} h_i(K,x,y). \tag{4}$$

We claim that the maps f_n, k_n and h_n are Borel and for every K and y the maps $f_n(K,\cdot,y)$, $k_n(K,\cdot,y)$ and $h_n(K,\cdot,y)$ are locally constant.

Note that if a function takes its values from a discrete set than locally constant is equivalent to continuous.

First we prove that the maps are Borel. Suppose that we are done for $i<n$. Let us check that f_n is Borel. Put $\eta : (K,x,y)\mapsto (K, \sum_{i<n} h_i(K,x,y))$ and $\psi : (K,g)\mapsto K-C+g$, then $f_n = \psi\circ\eta$. Moreover, η is Borel by induction, and ψ is easily seen to be continuous, hence f_n is Borel.

Next we show that k_n is Borel. Since $\operatorname{ran}(k_n)\subset\omega$, we need to check that for every fixed $m\in\omega$ the set $B=\{(K,x,y)\colon k_n(K,x,y)=m\}$ is Borel. By the definition of $k_n(K,x,y)$, clearly

$$B=\{(K,x,y)\colon f_n(K,x,y)\subset U \text{ and } f_n(K,x,y)\not\subset V\},$$

where $U = Q_m + B(0,\delta_n/2)$ and $V = Q_{m-1}+B(0,\delta_n/2)$ are fixed open sets.

Set $\mathcal{U}_W = \{L\in\mathcal{K}(G) : L\subset W\}$, which is open in $\mathcal{K}(G)$ for every open set $W\subset G$. Then clearly

$$B = f_n^{-1}(\mathcal{U}_U)\setminus f_n^{-1}(\mathcal{U}_V),$$

hence Borel.

Since the functions $k\mapsto g_k^n$ and ϕ defined on countable sets are clearly Borel, the Borelness of k_n and c imply by (1) that h_n is also Borel.

In order to prove that f_n, k_n and h_n are locally constant in the second variable, fix K and y and suppose that we are done for $i<n$. Then (4) shows that f_n is locally constant in the second variable by induction. This easily implies using the definition of k_n that k_n is also is locally constant in the second variable. But from this, and from the fact that $c(K,\cdot,y)(n)\colon 2^\omega\to\omega$ is continuous, hence locally constant, it is also clear using (1) that h_n is also locally constant in the second variable, which finishes the proof of the Lemma. □

Therefore the proof of the Proposition is also complete. □

4 Putting the ingredients together

Now we are ready to prove our main results, which are summarised in the following theorem.

Theorem 4.1 *Let $\Gamma = \mathbf{\Delta}^1_1$ and $\Lambda = \mathbf{\Pi}^0_\xi$ for some $1\le\xi<\omega_1$, or let $\Gamma=\mathbf{\Pi}^1_1$ and $\Lambda = \mathbf{\Delta}^1_1$. If G is a non-locally compact abelian Polish group then there exists a (generalised, in the case of $\Gamma=\mathbf{\Pi}^1_1$) Haar null set $E\in\Gamma(G)$ that is not contained in any Haar null set $H\in\Lambda(G)$.*

PROOF. Let f be given by Theorem 3.1.

Denote the Borel map $\mu\mapsto\operatorname{supp}(\mu)$ by $\operatorname{supp} : \mathcal{P}_c(G)\to\mathcal{K}(G)$. Let us also fix a Borel bijection $c:\mathcal{P}_c(G)\to 2^\omega$ (which we think of as a coding map) and a continuous probability measure ν on G with compact support C containing 0 (compactly supported continuous measures exist on every Polish space without isolated points, since such spaces contain copies of 2^ω). Let $t:\mathcal{K}(G)\times 2^\omega\times 2^\omega\to G$ be the map from Proposition 3.5 with the C fixed above, and define the map $\Psi\colon \mathcal{P}_c(G)\times 2^\omega\times G\to G$ by

$$\Psi(\mu,x,g) = g + t(\operatorname{supp}(\mu),x,c(\mu)). \tag{5}$$

Finally, define $E = \Psi(\text{graph}(f))$.

Claim 4.2 $E \in \Gamma$.

Proof. Ψ is clearly a Borel map. We claim that it is injective on $D = \{(\mu, x, g) : \mu \in \mathcal{P}_c(G), g \in \text{supp}(\mu)\}$, which is Borel by Lemma 3.2 and Lemma 3.4. Let $(\mu, x, g) \neq (\mu', x', g')$ be elements of D, we need to check that Ψ takes distinct values on them. The case $(\mu, x) = (\mu', x')$ is obvious, while the case $(\mu, x) \neq (\mu', x')$ follows from Property 1. in Proposition 3.5, since $\Psi(\mu, x, g) \in \text{supp}(\mu) - C + t(\text{supp}(\mu), x, c(\mu))$ (recall that $g \in \text{supp}(\mu)$ and $0 \in C$). Therefore Ψ is a Borel isomorphism on D. By $\text{graph}(f) \subset D$ this implies that $E = \Psi(\text{graph}(f))$ is in Γ (for $\Gamma = \mathbf{\Delta}^1_1$ see [6, 15.4], for $\Gamma = \mathbf{\Pi}^1_1$ notice that by [6, 25.A] a Borel isomorphism takes analytic sets to analytic sets, hence coanalytic sets to coanalytic sets). □

Claim 4.3 *E is Haar null (generalised Haar null in the case of $\Gamma = \mathbf{\Pi}^1_1$).*

Proof. We prove that ν is witnessing this fact. Actually, we prove more: $|C \cap (E+g)| \leq 1$ for every $g \in G$, or equivalently $|(C+g) \cap E| \leq 1$ for every $g \in G$. So let us fix $g \in G$.

$$E = \Psi(\text{graph}(f)) = \{\Psi(\mu, x, f(\mu, x)) : (\mu, x) \in \text{dom}(f)\} =$$

$$\{f(\mu, x) + t(\text{supp}(\mu), x, c(\mu)) : (\mu, x) \in \text{dom}(f)\},$$

hence the elements of E are of the form $g^{\mu,x} = f(\mu, x) + t(\text{supp}(\mu), x, c(\mu))$. This element $g^{\mu,x}$ is clearly in $A^{\mu,x} = \text{supp}(\mu) + t(\text{supp}(\mu), x, c(\mu))$ by Property 1. of Theorem 3.1, and the sets $A^{\mu,x}$ form a pairwise disjoint family as (μ, x) ranges over $\text{dom}(f)$, by Property 2. of Proposition 3.5. Hence it suffices to show that $C+g$ can intersect at most one $A^{\mu,x}$. But it can actually intersect at most one set of the form $K + t(K, x, y)$, since otherwise g would be in the intersection of two distinct sets of the form $K - C + t(K, x, y)$, contradicting Property 2. of Proposition 3.5. □

Claim 4.4 *There is no Haar null set $H \in \Lambda$ containing E.*

Suppose that $H \in \Lambda$ is such a set. Then by Remark 1.7 there exists a probability measure μ with compact support witnessing this fact. The section map $\Psi_\mu = \Psi(\mu, \cdot, \cdot)$ is continuous by (5) and Property 2. of Proposition 3.5. Now let $S = \Psi_\mu^{-1}(H)$, then $S \in \Lambda(2^\omega \times G)$.

It is easy to check that $\text{graph}(f_\mu) \subset S$, and therefore, using Theorem 3.1, there exists $x \in 2^\omega$ such that $\mu(S_x) > 0$. By the definition of S we have that $\Psi(\mu, x, S_x) \subset \Psi_\mu(S) \subset H$. But $\Psi(\mu, x, \cdot) : G \to G$ is a translation, so a translate of H contains S_x, which is of positive μ measure, contradicting that H is Haar null with witness μ. □

This concludes the proof. □

5 Questions

Question 5.1 *Let G be a non-locally compact abelian Polish group. Does there exist an F_σ Haar null set that cannot be covered by a G_δ Haar null set?*

Interestingly, our proof does not give any information about the Borel class of our example.

Question 5.2 *What is the least complexity of such a set? And in general, what is the least complexity of a Haar null set that cannot be covered by a $\mathbf{\Pi}^0_\xi$ Haar null set?*

Remark 5.3 *We remark that it is not hard to show that in abelian Polish groups every σ-compact Haar null set can be covered by a G_δ Haar null set.*

Question 5.4 *Do the results of the paper hold in all (not necessarily abelian) non-locally compact Polish groups?*

Question 5.5 *Does there exist a Polish group with a countable subset that cannot be covered by a G_δ Haar null set?*

In view of the above remark, the group in the last question cannot be abelian. Of course, it also cannot be locally compact. How about e.g. an arbitrary countable dense subset of $Homeo[0,1]$? For some related results see [4].

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