Random multisets and bootstrap means

Sándor Csörgő, Benedek Valkó and Wei Biao Wu

Abstract. First the turning points $\alpha = 1, 2$ and 3 in the asymptotic behavior, as $n \to \infty$, of the number of elements in the random integer-set $\mathcal{I}_n(\alpha) = \{\lfloor \lfloor n^\alpha \rfloor U_{n,1} \rfloor, \ldots, \lfloor \lfloor n^\alpha \rfloor U_{n,n} \rfloor\}$ are delineated, where $\{U_{n,1}, \ldots, U_{n,n}\}_{n=1}^\infty$ is an array of independent Uniform$(0,1)$ random variables and $\alpha > 0$ is a fixed parameter. Then, proving some conjectures made in [6], for $\alpha > 2$ the distribution of the number $N_n(\alpha)$ of elements in the random multiset $\mathcal{I}_n(\alpha) \cap [\cup_{j=n+1}^\infty \mathcal{I}_j(\alpha)]$ is approximated in the variation distance by suitable binomial and Poisson distributions, with specified rates of approximation depending on α, and an almost sure bound of the order of $n^{3-\alpha}$ is obtained for $N_n(\alpha)$. Finally, these results are used to extend necessary conditions concerning the strong convergence of some randomly rarefied and bootstrap means.

1. Random sets and multisets of integers

We begin with some notation and conventions from [6]. Given a finite or infinite multiset $\mathcal{I}$ of nonnegative integers, let $S[\mathcal{I}]$ be the base set of all the different elements in $\mathcal{I}$, i.e. the set that keeps exactly one of each kind of elements in $\mathcal{I}$. For $k \in \mathcal{I}$, let $\#(k, \mathcal{I})$ denote the multiplicity of k, so that $\#(k, S[\mathcal{I}]) = 1$ for every $k \in \mathcal{I}$ and $|\mathcal{I}| := \sum_{k \in S[\mathcal{I}]} \#(k, \mathcal{I})$ is the total number of elements in $\mathcal{I}$, including all multiplicities. Given the multisets $\mathcal{I}_1, \mathcal{I}_2, \ldots$ of nonnegative integers, it is natural to extend the usual set-theoretic notions and operations the following way: $\mathcal{I}_1 \subset \mathcal{I}_2$ if and only if $S[\mathcal{I}_1] \subset S[\mathcal{I}_2]$ and $\#(k, \mathcal{I}_1) \le \#(k, \mathcal{I}_2)$ for every $k \in S[\mathcal{I}_1]$, so that $\mathcal{I}_1 = \mathcal{I}_2$ if and only if $S[\mathcal{I}_1] = S[\mathcal{I}_2]$ and $\#(k, \mathcal{I}_1) = \#(k, \mathcal{I}_2)$ for every $k \in S[\mathcal{I}_1]$; for a finite or infinite set $H \subset \mathbb{N} := \{1, 2, \ldots\}$, the union $\cup_{m \in H} \mathcal{I}_m$ is a multiset with base set $\cup_{m \in H} S[\mathcal{I}_m]$ and multiplicity $\max_{m \in H} \#(k, \mathcal{I}_m)$ for each $k \in \cup_{m \in H} S[\mathcal{I}_m]$, while the intersection $\cap_{m \in H} \mathcal{I}_m$ is a multiset with base set $\cap_{m \in H} S[\mathcal{I}_m]$ and multiplicity $\min_{m \in H} \#(k, \mathcal{I}_m)$ for each $k \in \cap_{m \in H} S[\mathcal{I}_m]$; finally, if $\mathcal{I}_1$ and $\mathcal{I}_2$ are both finite, the difference $\mathcal{I}_1 \setminus \mathcal{I}_2$ may be formed when $\mathcal{I}_2 \subset \mathcal{I}_1$, in which case $S[\mathcal{I}_1 \setminus \mathcal{I}_2] := \{k \in \mathcal{I}_1 : \#(k, \mathcal{I}_1) > \#(k, \mathcal{I}_2)\}$ and the multiplicity of $k \in S[\mathcal{I}_1 \setminus \mathcal{I}_2]$ in $\mathcal{I}_1 \setminus \mathcal{I}_2$ is taken to be $\#(k, \mathcal{I}_1) - \#(k, \mathcal{I}_2)$.

Let $\{U_{n,1}, \ldots, U_{n,n}\}_{n=1}^\infty$ be a triangular array of independent Uniform$(0,1)$ random variables; to avoid any misunderstanding we underline that the Uniform$(0,1)$

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variables in the array are not just row-wise independent, but in their whole totality. For some real number $\alpha > 0$ and all $n \in \mathbb{N}$, consider the random multisets

$$\mathcal{I}_n(\alpha) := \{\lfloor \lfloor n^\alpha \rfloor U_{n,1} \rfloor, \ldots, \lfloor \lfloor n^\alpha \rfloor U_{n,n} \rfloor\},$$

where $\lfloor x \rfloor := \max\{l \in \mathbb{Z} : l \leq x\}$ is the usual integer part of $x \in \mathbb{R}$. It was demonstrated in [6] that the asymptotic behavior of the random cardinality

$$N_n(\alpha) := \big| \mathcal{I}_n(\alpha) \cap \big(\cup_{j=n+1}^{\infty} \mathcal{I}_j(\alpha) \big) \big| \tag{1}$$

plays an important role in deriving necessary conditions for the complete and almost sure convergence of bootstrap means and of some associated means of randomly rarefied sequences, considered again here in Section 3, both constructed from a sequence of independent and identically distributed random variables.

In particular, it was proved in [6] that if $\alpha > 3$, then there is a finite constant $K(\alpha) > 0$ such that $\sum_{n=1}^{\infty} \mathbb{P}\big\{N_n(\alpha) \geq K(\alpha)\big\} < \infty$. Subsequently, Problem 11 in the 1999 Schweitzer Competition (a yearly mathematics competition for college students in Hungary), posed by the first author here, called for a proof of what is equivalent to the statement that the sequence $\{N_n(\alpha)\}_{n=1}^{\infty}$ is almost surely bounded if and only if $\alpha > 3$. (The variant in the competition was about the cardinality $\big| \mathcal{J}_n(\alpha) \cap \big(\cup_{j=n+1}^{\infty} \mathcal{J}_j(\alpha)\big) \big|$, where $\mathcal{J}_n(\alpha)$ is the *set* $S[\{\lfloor n^\alpha U_{n,1} \rfloor, \ldots, \lfloor n^\alpha U_{n,n} \rfloor\}]$ rather than a multiset, with the inner integer parts dropped.) One of the two complete solutions, due to Gábor Braun, was analytic in nature and will be published in [16]. The other solution, due to the second author here, was done by a mixture of methods, one ingredient of which was a Poisson approximation similar to that in the present paper. Since for every $n \in \mathbb{N}$ and $m \in \{0, 1, \ldots, \lfloor n^\alpha \rfloor - 1\}$,

$$\mathbb{P}\left\{ m \notin \bigcup_{j=n+1}^{\infty} \mathcal{I}_j(\alpha) \right\} = \prod_{j=n+1}^{\infty} \prod_{k=1}^{j} \mathbb{P}\left\{ U_{j,k} \notin \left[\frac{m}{\lfloor j^\alpha \rfloor}, \frac{m+1}{\lfloor j^\alpha \rfloor} \right) \right\} = \prod_{j=n+1}^{\infty} \left(1 - \frac{1}{\lfloor j^\alpha \rfloor} \right)^j$$

for all $\alpha > 0$ and, since $\log(1 + x) \leq x$ for all $x > -1$,

$$0 \leq \prod_{j=n+1}^{\infty} \left(1 - \frac{1}{\lfloor j^\alpha \rfloor} \right)^j = \exp\left\{ \sum_{j=n+1}^{\infty} j \log\left(1 - \frac{1}{\lfloor j^\alpha \rfloor} \right) \right\} \leq \exp\left\{ - \sum_{j=n+1}^{\infty} \frac{1}{j^{\alpha-1}} \right\},$$

as both Braun and Valkó [16] point out, the probability in question is zero for every $n \in \mathbb{N}$, whenever $\alpha \in (0, 2]$. Hence $\mathcal{I}_n(\alpha) \subset \{0, 1, \ldots, \lfloor n^\alpha \rfloor - 1\} \subset \cup_{j=n+1}^{\infty} \mathcal{I}_j(\alpha)$, or what is the same, $\mathcal{I}_n(\alpha) = \mathcal{I}_n(\alpha) \cap \big(\cup_{j=n+1}^{\infty} \mathcal{I}_j(\alpha) \big)$, almost surely, that is, $\mathbb{P}\{N_n(\alpha) = |\mathcal{I}_n(\alpha)| = n\} = 1$ for all $n \in \mathbb{N}$ if $\alpha \in (0, 2]$.

Therefore, with the main motivation from [6], the problem concerning $N_n(\alpha)$ is nontrivial only for $\alpha > 2$. Since, as noted in the proof of Theorem 2.1 in [6],

$$\begin{aligned} \mathbb{P}\big\{ \lfloor \lfloor n^\alpha \rfloor U_{n,1} \rfloor, \ldots, \lfloor \lfloor n^\alpha \rfloor U_{n,n} \rfloor \text{ are different} \big\} &= \mathbb{P}\big\{ \big| S[\mathcal{I}_n(\alpha)] \big| = n \big\} \\ &= \prod_{j=1}^{n-1} \left(1 - \frac{j}{\lfloor n^\alpha \rfloor} \right) \end{aligned}$$

for all $\alpha \geq 1$ (which also follows as the special case $k = n$ of Theorem 1 below), understanding throughout all asymptotic relationships as $n \to \infty$ unless otherwise specified, we also see that $\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = n\} \to 1$ if and only if $\alpha > 2$ as well.

Thus for $\alpha \in (0, 2]$ the meaningful related quantity to look at is $|S[\mathcal{I}_n(\alpha)]|$, the number of elements in the ordinary random *set* $S[\lfloor \lfloor n^\alpha \rfloor U_{n,1} \rfloor, \ldots, \lfloor \lfloor n^\alpha \rfloor U_{n,n} \rfloor]$, which, letting $I(A)$ stand for the indicator function of the event A, is given by

$$\big|S[\mathcal{I}_n(\alpha)]\big| = \sum_{m=0}^{\lfloor n^\alpha \rfloor - 1} I_m(n, \alpha), \quad \text{where} \quad I_m(n, \alpha) = I(m \in \mathcal{I}_n(\alpha)).$$

We write $a_n \sim b_n$ for the asymptotic equality of the deterministic sequences $\{a_n\}$ and $\{b_n\}$, satisfied if $a_n / b_n \to 1$. Besides the usual notation $\binom{n}{k}$ for binomial coefficients, $\left\{ {n \atop k} \right\}$ will denote a Stirling number of the second kind, $k, n \in \{0, 1, 2, \ldots\}$, $k \leq n$, the number of ways a set with n elements can be partitioned into k nonempty subsets. The last statement of the following theorem, which is a consequence of the first statement, will also be important in Section 4.

Theorem 1. *For every $\alpha > 0$,*

$$\mathbb{P}\Big\{\big|S[\mathcal{I}_n(\alpha)]\big| = k\Big\} = \binom{\lfloor n^\alpha \rfloor}{k} \frac{1}{\lfloor n^\alpha \rfloor^n} \sum_{l=1}^{k} (-1)^{k-l} \binom{k}{l} l^n = \frac{\left\{ {n \atop k} \right\}}{\lfloor n^\alpha \rfloor^{n-k}} \prod_{j=1}^{k-1} \left(1 - \frac{j}{\lfloor n^\alpha \rfloor}\right)$$

for all possible values $k = 1, 2, \ldots, \min(n, \lfloor n^\alpha \rfloor)$. Furthermore, the mean

$$\mu_n(\alpha) := \mathbb{E}\Big(\big|S[\mathcal{I}_n(\alpha)]\big|\Big) = \lfloor n^\alpha \rfloor \left[1 - \left(1 - \frac{1}{\lfloor n^\alpha \rfloor}\right)^n\right] \sim \begin{cases} \lfloor n^\alpha \rfloor, & \text{if } \alpha < 1, \\ (1 - \frac{1}{e}) n, & \text{if } \alpha = 1, \\ n, & \text{if } \alpha > 1, \end{cases}$$

and the variance $\sigma_n^2(\alpha) := \operatorname{Var}\big(\big|S[\mathcal{I}_n(\alpha)]\big|\big)$ is

$$\sigma_n^2(\alpha) = \lfloor n^\alpha \rfloor (\lfloor n^\alpha \rfloor - 1) \left(1 - \frac{2}{\lfloor n^\alpha \rfloor}\right)^n + \lfloor n^\alpha \rfloor \left(1 - \frac{1}{\lfloor n^\alpha \rfloor}\right)^n - \lfloor n^\alpha \rfloor^2 \left(1 - \frac{1}{\lfloor n^\alpha \rfloor}\right)^{2n}$$
$$\begin{cases} = O\big(n^\alpha e^{-n^{1-\alpha}}\big), & \text{if } \alpha < 1, \\ \sim \frac{e-2}{e^2}\, n, & \text{if } \alpha = 1, \\ \sim \frac{1}{2}\, n^{2-\alpha}, & \text{if } \alpha > 1. \end{cases}$$

Also, if $1 \leq \alpha < 2$, then

$$\mathbb{P}\left\{ \frac{\big|S[\mathcal{I}_n(\alpha)]\big| - \mu_n(\alpha)}{\sigma_n(\alpha)} \leq x \right\} \to \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R},$$

while if $\alpha = 2$, *then*

$$\mathbb{P}\Big\{n - \big|S[\mathcal{I}_n(2)]\big| = k\Big\} \to \frac{(1/2)^k}{k!}\, e^{-1/2} \quad \textit{for each fixed } k = 0, 1, 2, \ldots,$$

and finally if $\alpha > 2$, *then*

$$\mathbb{P}\Big\{\big|S[\mathcal{I}_n(\alpha)]\big| \leq n - m\Big\} \leq \frac{2(1+e)}{3\, n^{(\alpha-2)m}}\,, \quad m = 1, 2, \ldots, n-1,$$

for all $n \geq 2$.

The proofs of all three theorems in this section are in Section 2. Except for the case $\alpha = 2$, the exact distribution in Theorem 1 appears too complicated for more refined direct analysis beyond the last statement, while the asymptotic formulae for the mean and the variance allow a couple of quick inferences. For $\alpha < 1$ the Chebyshev inequality and the Borel–Cantelli lemma imply that $|S[\mathcal{I}_n(\alpha)]| = \lfloor n^\alpha \rfloor$ almost surely for all n large enough, while for $\alpha > 2$ it obviously follows that $0 \leq n - |S[\mathcal{I}_n(\alpha)]| \to 0$ in probability, implying that $\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = n\} \to 1$. Furthermore, if $\alpha > 2$ then it follows from the last statement that

$$\sum_{n=1}^{\infty} \mathbb{P}\Big\{\big|S[\mathcal{I}_n(\alpha)]\big| \leq n - m_\alpha\Big\} < \infty\,, \tag{2}$$

so that $|S[\mathcal{I}_n(\alpha)]| \in \{n, n-1, \ldots, n - m_\alpha + 1\}$ almost surely for all n large enough, where m_α is the smallest integer m for which $(\alpha - 2)m > 1$. Note that $m_\alpha \geq 2$ if $\alpha \in (2, 3]$, while $m_\alpha = 1$ if $\alpha > 3$, so that in the latter case $|S[\mathcal{I}_n(\alpha)]| = n$ almost surely for all n large enough.

For all $\alpha > 0$ the terms $I_0(n, \alpha), \ldots, I_{\lfloor n^\alpha \rfloor - 1}(n, \alpha)$ in the sum representation of $|S[\mathcal{I}_n(\alpha)]|$ above are identically distributed (in fact, exchangeable) Bernoulli variables. Even though they turn out to be asymptotically pairwise uncorrelated, in the derivation of the variance formula in Theorem 1, their intricate dependence on one another makes the problem of asymptotic distribution for their sum nontrivial for all $\alpha \in [1, 2]$. On the other hand, $|S[\mathcal{I}_n(\alpha)]|$ equals in distribution to the number of occupied boxes when each of n balls is thrown equally likely into any one of $\lfloor n^\alpha \rfloor$ boxes. This is because for each $k \in \{1, 2, \ldots, n\}$ the variable $\lfloor \lfloor n^\alpha \rfloor U_{n,k} \rfloor$ may be viewed as telling which of the boxes numbered $0, 1, \ldots, \lfloor n^\alpha \rfloor - 1$ the k-th ball goes into, or, equivalently, one observes for each $k \in \{1, 2, \ldots, n\}$ the "ball" $U_{n,k}$ going into one of the "boxes" $[0, 1/\lfloor n^\alpha \rfloor), [1/\lfloor n^\alpha \rfloor, 2/\lfloor n^\alpha \rfloor), \ldots, [(\lfloor n^\alpha \rfloor - 1)/\lfloor n^\alpha \rfloor, 1)$. Of course, this is true in general: with $m \in \mathbb{N}$ boxes and $n \in \mathbb{N}$ balls, if $U_1, U_2, \ldots, U_n$ are independent Uniform$(0,1)$ random variables, then $|\{\lfloor mU_1 \rfloor, \lfloor mU_2 \rfloor, \ldots, \lfloor mU_n \rfloor\}|$ has the distribution of the number of occupied boxes. Thus the exact formula in Theorem 1 is not new, for example it follows from a special case of the formula in [10], Problem 22, p. 112, for the number of empty boxes. We include a short proof nevertheless, because the formula is crucial for the last statement of the theorem, which is needed in Section 4 as mentioned, and because, using the balls-and-boxes identification in a reversed fashion, the proof provides and alternative approach to exact probabilities in classical occupancy problems. For $\alpha = 1$ the asymptotic normality statement in the theorem follows from a classical result of Weiss [17] for the number of empty boxes, while the full generality for all $\alpha \in [1, 2)$ follows from Rényi's [15] equally classical generalization of Weiss's result. The statement for $\alpha = 2$, that $n - |S[\mathcal{I}_n(2)]|$ has an asymptotic Poisson distribution with mean $1/2$, follows from a result of Békéssy [4], also for the number of empty boxes, and all these results concerning the asymptotic normal and Poisson distributions of the number of empty boxes constitute an essential part of Chapter 1 in [13], where the original proofs are nicely streamlined. None of these proofs uses the exact distribution, only recursions for that, and we cannot offer new proofs for asymptotic normality. However, we find it interesting to point out in the proof of Theorem 1 that the asymptotic Poisson distribution for $\alpha = 2$, and similarly Békéssy's [4] limiting Poisson distribution for a more general case concerning the distribution of the number of empty boxes, follows directly from the exact formula by means of a much older result of Károly Jordan [12]. This is one more reason to include our proof of the formula.

Returning to our main theme here, consider an integer $d \in \mathbb{N}$, and for a d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$, given on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let $\mathcal{Q}_{\mathbf{X}}(B) = \mathbb{P}\{\mathbf{X} \in B\}$ denote its distribution as B runs through the σ-algebra $\mathcal{B}^d$ of Borel-measurable subsets of the d-dimensional Euclidean space $\mathbb{R}^d$. With analogous notation for a random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)$, let $d_{\mathrm{TV}}(\mathcal{Q}_{\mathbf{X}}, \mathcal{Q}_{\mathbf{Y}}) := \sup_{B \in \mathcal{B}^d} |\mathcal{Q}_{\mathbf{X}}(B) - \mathcal{Q}_{\mathbf{Y}}(B)|$ stand for the total variation distance between the distributions $\mathcal{Q}_{\mathbf{X}}$ and $\mathcal{Q}_{\mathbf{Y}}$. It is well known that if $\mathbf{X}$ and $\mathbf{Y}$ are discrete random vectors with the same set of possible values $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ in $\mathbb{R}^d$, then

$$(3) \qquad d_{\mathrm{TV}}(\mathcal{Q}_{\mathbf{X}}, \mathcal{Q}_{\mathbf{Y}}) = \frac{1}{2} \sum_{k=0}^{\infty} \big| \mathbb{P}\{\mathbf{X} = \mathbf{x}_k\} - \mathbb{P}\{\mathbf{Y} = \mathbf{x}_k\} \big| \leq \mathbb{P}\{\mathbf{X} \neq \mathbf{Y}\}$$

for any joint distribution of $\mathbf{X}$ and $\mathbf{Y}$. Finally, let specifically $\mathcal{N}_n^{\alpha} = \mathcal{Q}_{N_n(\alpha)}$ denote the distribution of our random count $N_n(\alpha)$ in (1), let $\mathcal{B}_p^n$ be the Binomial(n, p) distribution with parameters $n \in \mathbb{N}$ and $p \in (0, 1)$, and let $\mathcal{P}_\lambda$ stand for the Poisson(λ) distribution with mean $\lambda > 0$, that is, $\mathcal{N}_n^{\alpha}(B) = \mathcal{Q}_{N_n(\alpha)}(B) = \mathbb{P}\{N_n(\alpha) \in B\}$,

$$\mathcal{B}_p^n(B) = \sum_{k \in B \cap \{0,1,\ldots,n\}} \binom{n}{k} p^k (1-p)^{n-k} \quad \text{and} \quad \mathcal{P}_\lambda(B) = e^{-\lambda} \sum_{k \in B \cap \{0,1,2,\ldots\}} \frac{\lambda^k}{k!}$$

for all $B \in \mathcal{B} = \mathcal{B}^1$, empty sums being understood as zero. One of our main results for $N_n(\alpha)$ is as follows.

Theorem 2. *If $\alpha > 2$ and $\lambda_n(\alpha) = n\sum_{m=n+1}^{\infty} \frac{m}{\lfloor m^\alpha \rfloor}$, then*

$$d_{\rm TV}\Big(\mathcal{N}_n^\alpha, \mathcal{B}_{p_n(\alpha)}^n\Big) \le \frac{\frac{c_\alpha^2}{\alpha-2} + \frac{c_\alpha^3(\alpha-1)}{2(\alpha-2)^2}}{n^{2\alpha-4}} =: \frac{C_\alpha}{n^{2\alpha-4}}, \quad \textit{where} \quad p_n(\alpha) := 1 - e^{-\frac{\lambda_n(\alpha)}{n}}$$

and $c_\alpha := (\lfloor 2^\alpha \rfloor + 1)/\lfloor 2^\alpha \rfloor$, and so

$$d_{\rm TV}\Big(\mathcal{N}_n^\alpha, \mathcal{P}_{np_n(\alpha)}\Big) \le \begin{cases} \frac{C_\alpha + A_\alpha}{n^{\alpha-2}}, & \textit{if} \quad \alpha < 3, \\ \frac{C_\alpha + B_\alpha}{n^{2\alpha-5}}, & \textit{if} \quad \alpha \ge 3, \end{cases}$$

for all $n \in \mathbb{N}$, where $A_\alpha := 13\, c_\alpha/[2(\alpha-2)]$ and $B_\alpha := c_\alpha^2/(\alpha-2)^2$. Furthermore,

$$d_{\rm TV}\Big(\mathcal{N}_n^\alpha, \mathcal{P}_{\lambda_n^*(\alpha)}\Big) = \begin{cases} O\Big(\frac{1}{n^{\frac{3\alpha-7}{2}}}\Big), & \textit{if} \quad \frac{7}{3} < \alpha < 3, \\ O\Big(\frac{1}{n^{\alpha-2}}\Big), & \textit{if} \quad \alpha \ge 3, \end{cases} \qquad \textit{where} \quad \lambda_n^*(\alpha) = \frac{n^{3-\alpha}}{\alpha-2}.$$

The second and third statements in Theorem 2 together establish precise versions of half of the second conjecture in Section 5 in [6]. The constants A_α, B_α and C_α in these statements can all be reduced if the inequalities are required not for all $n \in \mathbb{N}$, but only for all $n \ge n_0$ for a given threshold $n_0 \in \mathbb{N}$, since in that case an inspection of the proofs shows that the ingredient c_α can be reduced to $(\lfloor (n_0+1)^\alpha \rfloor + 1)/\lfloor (n_0+1)^\alpha \rfloor$ in all three of them.

With $\Phi(\cdot)$ still standing for the standard normal distribution function, the binomial approximation in the first statement and classical results on the rate of convergence in the central limit theorem imply that

$$\Delta_n^\alpha := \sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{N_n(\alpha) - np_n(\alpha)}{\sqrt{np_n(\alpha)[1-p_n(\alpha)]}} \le x \right\} - \Phi(x) \right| = \begin{cases} O\Big(\frac{1}{n^{2\alpha-4}}\Big), & \text{if} \quad 2 < \alpha \le \frac{11}{5}, \\ O\Big(\frac{1}{n^{\frac{3-\alpha}{2}}}\Big), & \text{if} \quad \frac{11}{5} \le \alpha < 3. \end{cases}$$

For $\alpha = 3$, the limiting distribution of $N_n(3)$ is standard Poisson, by the last statement we in fact have $d_{\rm TV}\big(\mathcal{N}_n^3, \mathcal{P}_1\big) = O(1/n)$.

Concerning the relationship between the two Poisson approximations, where of course $np_n(\alpha) \sim \lambda_n^*(\alpha)$, the replacement of $np_n(\alpha)$ by its asymptotic form is not feasible for $\alpha \in (2, 7/3]$. This is exactly the range where in the central limit theorem $\Delta_n^\alpha \to 0$, ignoring the rate, one cannot replace the approximate mean $np_n(\alpha)$ by $\lambda_n^*(\alpha)$ because $[\lambda_n^*(\alpha) - np_n(\alpha)]/\sqrt{np_n(\alpha)[1-p_n(\alpha)]} \sim [\lambda_n^*(\alpha) - np_n(\alpha)]/\sqrt{\lambda_n^*(\alpha)}$ does not go to zero for $\alpha \in (2, 7/3]$.

The motivating application to bootstrap means and randomly rarefied averages in the next section require the type of the statement in Theorem 3 below. Deriving it directly from the first version of the Poisson approximation in Theorem 2 would restrict α to $(3,\infty)$ at the outset, and hence could not give anything more than what is already known in [6], while the primary binomial approximation combined with standard results in large deviation theory would still give the statement of Theorem 3 for $\alpha > 5/2$ only. The direct proof of Theorem 3 circumvents this problem altogether, and produces half of the first conjecture in Section 5 in [6] in its full form. The trick in the proof to achieve this is an instance of what may be called 'Poisson stochastic majorization' of conditional random counts.

Theorem 3. *If $\alpha > 2$, then $\sum_{n=1}^{\infty} \mathbb{P}\big\{N_n(\alpha) \geq K_\alpha n^{3-\alpha}\big\} < \infty$ for some constant $K_\alpha > 0$.*

The results in this paper all belong to the triangular array model in [6]. The corresponding problems and conjectures in [6] for the sequence model, in which the multisets $\mathcal{I}_1(\alpha), \mathcal{I}_2(\alpha), \ldots$ are all based on a single sequence $U_1, U_2, \ldots$ of independent Uniform$(0,1)$ variables and hence are not independent, all remain open.

2. Proof of Theorems 1, 2 and 3

Proof of Theorem 1. Fix any possible value $k \in \{1, 2, \ldots, \min(n, \lfloor n^\alpha \rfloor)\}$ and consider any partition $\{0, 1, 2, \ldots, \lfloor n^\alpha \rfloor\} = \{m_1, \ldots, m_k\} \cup \{j_1, \ldots, j_{\lfloor n^\alpha \rfloor - k}\}$ into two disjoint sets. Then, anticipating that, once k is given, the probabilities

$$p_{n,k}(\alpha) := \mathbb{P}\{m_1, \ldots, m_k \in \mathcal{I}_n(\alpha),\ \ j_1, \ldots, j_{\lfloor n^\alpha \rfloor - k} \notin \mathcal{I}_n(\alpha)\}$$

will not depend on particular subsets $\{m_1, \ldots, m_k\} \subset \{0, 1, 2, \ldots, \lfloor n^\alpha \rfloor\}$ chosen, we have $\mathbb{P}\big\{\big|S[\mathcal{I}_n(\alpha)]\big| = k\big\} = \binom{\lfloor n^\alpha \rfloor}{k} p_{n,k}(\alpha)$.

Next, denoting by $\mathbb{P}_B\{A\} = \mathbb{P}\{A \,|\, B\} = \mathbb{P}\{A \cap B\}/\mathbb{P}\{B\}$ the conditional probability of A given the event B of positive probability, introducing the event $B^\alpha_{n,k} := \{j_1, \ldots, j_{\lfloor n^\alpha \rfloor - k} \notin \mathcal{I}_n(\alpha)\}$ and applying the sieve formula to the conditional probability $\mathbb{P}_{B^\alpha_{n,k}}\{\cdot\}$, we obtain

$$\begin{aligned}
p_{n,k}(\alpha) &= \mathbb{P}_{B^\alpha_{n,k}}\big\{ \cap_{l=1}^k \{m_l \in \mathcal{I}_n(\alpha)\}\big\} \mathbb{P}\big\{B^\alpha_{n,k}\big\} \\
&= \Big[1 - \mathbb{P}_{B^\alpha_{n,k}}\big\{ \cup_{l=1}^k \{m_l \notin \mathcal{I}_n(\alpha)\}\big\}\Big] \mathbb{P}\big\{B^\alpha_{n,k}\big\} \\
&= \mathbb{P}\big\{B^\alpha_{n,k}\big\} - \mathbb{P}\big\{B^\alpha_{n,k}\big\} \sum_{l=1}^k (-1)^{l-1} \sum_{1 \leq r_1 < \cdots < r_l \leq k} \mathbb{P}_{B^\alpha_{n,k}}\{m_{r_1}, \ldots, m_{r_l} \notin \mathcal{I}_n(\alpha)\} \\
&= \mathbb{P}\big\{B^\alpha_{n,k}\big\} + \sum_{l=1}^k (-1)^l \sum_{1 \leq r_1 < \cdots < r_l \leq k} \mathbb{P}\big\{\{m_{r_1}, \ldots, m_{r_l} \notin \mathcal{I}_n(\alpha)\} \cap B^\alpha_{n,k}\big\} \\
&= \mathbb{P}\big\{j_1, \ldots, j_{\lfloor n^\alpha \rfloor - k} \notin \mathcal{I}_n(\alpha)\big\} \\
&\quad + \sum_{l=1}^k (-1)^l \binom{k}{l} \mathbb{P}\big\{m_{r_1}, \ldots, m_{r_l}, j_1, \ldots, j_{\lfloor n^\alpha \rfloor - k} \notin \mathcal{I}_n(\alpha)\big\}
\end{aligned}$$

$$= \left(1 - \frac{\lfloor n^\alpha \rfloor - k}{\lfloor n^\alpha \rfloor}\right)^n + \sum_{l=1}^{k} (-1)^l \binom{k}{l} \left(1 - \frac{l + \lfloor n^\alpha \rfloor - k}{\lfloor n^\alpha \rfloor}\right)^n$$

$$= \frac{1}{\lfloor n^\alpha \rfloor^n} \sum_{l=0}^{k-1} (-1)^l \binom{k}{l} (k-l)^n = \frac{1}{\lfloor n^\alpha \rfloor^n} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n,$$

proving the first formula in the first statement of the theorem, where, in the sixth equation of the chain, we use the fact that if $i_1, i_2, \ldots, i_r \in \{0, 1, 2, \ldots, \lfloor n^\alpha \rfloor - 1\}$ are different integers, $r = 1, 2, \ldots, \lfloor n^\alpha \rfloor$, then for every $n \in \mathbb{N}$,

$$\begin{aligned} q_{n,\alpha}(r) := \mathbb{P}\{i_1, i_2, \ldots, i_r \notin \mathcal{I}_n(\alpha)\} &= \mathbb{P}\big\{ \cap_{j=1}^n \{\lfloor \lfloor n^\alpha \rfloor U_{n,j} \rfloor \neq i_1, \ldots, i_r\}\big\} \\ &= \prod_{j=1}^n \mathbb{P}\left\{ U_{n,j} \notin \bigcup_{l=1}^r \left[\frac{i_l}{\lfloor n^\alpha \rfloor}, \frac{i_l + 1}{\lfloor n^\alpha \rfloor}\right) \right\} \\ &= \left(1 - \frac{r}{\lfloor n^\alpha \rfloor}\right)^n. \end{aligned} \tag{4}$$

Next, the well-known identity (see [12], p. 169, for example)

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n = k! \left\{ {n \atop k} \right\} \text{ yields } \mathbb{P}\Big\{ \big|S[\mathcal{I}_n(\alpha)]\big| = k \Big\} = \frac{\lfloor n^\alpha \rfloor !}{(\lfloor n^\alpha \rfloor - k)!} \frac{1}{\lfloor n^\alpha \rfloor^n} \left\{ {n \atop k} \right\},$$

whence the second formula of the first statement also follows by simple calculation.

Since $\mathbb{E}(|S[\mathcal{I}_n(\alpha)]|) = \lfloor n^\alpha \rfloor \mathbb{P}\{m \in \mathcal{I}_n(\alpha)\} =: \lfloor n^\alpha \rfloor p_{n,\alpha}(1) = \lfloor n^\alpha \rfloor [1 - q_{n,\alpha}(1)]$ independently of the choice of $m \in \{0, 1, 2, \ldots, \lfloor n^\alpha \rfloor - 1\}$, the exact formula for the mean follows from the special case $r = 1$ of (4), while the three asymptotic equalities follow by standard arguments after exponentiating and expanding the resulting logarithm in the exponent. Similarly for the variance,

$$\begin{aligned} \mathrm{Var}\Big(\big|S[\mathcal{I}_n(\alpha)]\big|\Big) &= \sum_{l,m=0}^{\lfloor n^\alpha \rfloor - 1} \mathbb{P}\big\{l, m \in \mathcal{I}_n(\alpha)\big\} - \lfloor n^\alpha \rfloor^2 p_{n,\alpha}^2(1) \\ &= \lfloor n^\alpha \rfloor p_{n,\alpha}(1) + \big(\lfloor n^\alpha \rfloor^2 - \lfloor n^\alpha \rfloor\big) p_{n,\alpha}(3) - \lfloor n^\alpha \rfloor^2 p_{n,\alpha}^2(1) \\ &= \lfloor n^\alpha \rfloor \big\{p_{n,\alpha}(1) - p_{n,\alpha}(3)\big\} + \lfloor n^\alpha \rfloor^2 \big\{p_{n,\alpha}(3) - p_{n,\alpha}^2(1)\big\}, \end{aligned}$$

where $p_{n,\alpha}(3) = \mathbb{P}\{l, m \in \mathcal{I}_n(\alpha)\}$ for any $l, m \in \{0, 1, 2, \ldots, \lfloor n^\alpha \rfloor - 1\}$ such that $l \neq m$, and so $p_{n,\alpha}(3) = 1 - [\mathbb{P}\{l \notin \mathcal{I}_n(\alpha)\} + \mathbb{P}\{m \notin \mathcal{I}_n(\alpha)\} - \mathbb{P}\{l, m \notin \mathcal{I}_n(\alpha)\}] = 1 + q_{n,\alpha}(3) - 2q_{n,\alpha}(1)$. Substituting this and $p_{n,\alpha}(1) = 1 - q_{n,\alpha}(1)$ in and using (4) again for $r = 1, 2$, we obtain

$$\begin{aligned} \mathrm{Var}\Big(\big|S[\mathcal{I}_n(\alpha)]\big|\Big) = \lfloor n^\alpha \rfloor &\left[\left(1 - \frac{1}{\lfloor n^\alpha \rfloor}\right)^n - \left(1 - \frac{2}{\lfloor n^\alpha \rfloor}\right)^n\right] \\ &+ \lfloor n^\alpha \rfloor^2 \left[\left(1 - \frac{2}{\lfloor n^\alpha \rfloor}\right)^n - \left(1 - \frac{1}{\lfloor n^\alpha \rfloor}\right)^{2n}\right], \end{aligned}$$

which is the exact formula claimed, and from which the stated asymptotic expressions follow by a somewhat lengthy elementary analysis, the nature of which is indicated above for the mean.

We discussed the references for the asymptotic normality for $\alpha \in [1,2)$. Concerning the case $\alpha = 2$, when

$$\mathbb{P}\Big\{n - \big|S[\mathcal{I}_n(2)]\big| = k\Big\} = \frac{\left\{ {n \atop n-k} \right\}}{n^{2k}} \prod_{j=1}^{n-k-1}\left(1 - \frac{j}{n^2}\right), \quad k = 0,1,2,\ldots,n-1,$$

notice first by exponentiating that the last products converge to $e^{-1/2}$ for each fixed $k = 0,1,2,\ldots$. Since also

$$\frac{\left\{ {n \atop n-k} \right\}}{n^{2k}} \to \frac{1}{2^k\, k!} \quad \text{for each fixed } k = 0,1,2,\ldots,$$

which is a classical result of Jordan ([12], p. 174), the claim follows.

To prove the last statement, fix any $\alpha > 2$. Set $Z_n = Z_n(\alpha) := n - \big|S[\mathcal{I}_n(\alpha)]\big|$, so that the probability in question is $\mathbb{P}\{Z_n \geq m\}$. By the first statement, bounding all the products by their limiting value of 1 and noticing that $n^{\alpha r} \leq 4\lfloor n^\alpha \rfloor^r/3$, $r = 0,1,\ldots,n-1$, for all $n \geq 2$, for the moment generating function we have

$$M_n(t) := \mathbb{E}\big(e^{tZ_n}\big) \leq \frac{4}{3}\sum_{r=0}^{n-1} \frac{e^{tr}}{n^{\alpha r}} \left\{ {n \atop n-r} \right\} = \frac{4}{3}\sum_{r=0}^{n-1} e^{r(t-\alpha\log n)} \left\{ {n \atop n-r} \right\}$$

for all $t > 0$. Hence by the Markov inequality,

$$\mathbb{P}\{Z_n \geq m\} \leq \frac{M_n\big((\alpha-2)\log n\big)}{e^{m(\alpha-2)\log n}} \leq \frac{4}{3}\,\frac{\sum_{k=1}^n \frac{1}{n^{2(n-k)}} \left\{ {n \atop k} \right\}}{n^{(\alpha-2)m}}, \quad m = 1,2,\ldots,n-1,$$

for all $n \geq 2$. Now we use the elementary upper bound from Theorem 3 of Rennie and Dobson [14], which states that

$$\left\{ {n \atop k} \right\} \leq \frac{1}{2}\binom{n}{k} k^{n-k}, \quad k = 1,2,\ldots,n-1.$$

This gives

$$\begin{aligned}\sum_{k=1}^n \frac{1}{n^{2(n-k)}} \left\{ {n \atop k} \right\} &\leq 1 + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{k}{n}\right)^{n-k} \frac{1}{n^{n-k}} < \frac{1}{2} + \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^{n-k} \\ &= \frac{1}{2} + \frac{1}{2}\left(1 + \frac{1}{n}\right)^n \uparrow \frac{1+e}{2},\end{aligned}$$

completing the proof of the theorem. ■

For the proof of Theorem 2 four lemmas are required. In the first of these, a multivariate extension of independent interest of a well-known result of Lucien Le Cam, $\mathbf{0} = (0, \ldots, 0)$ denotes the zero vector in $\mathbb{R}^d$ for a fixed $d \in \mathbb{N}$ and, using the Kronecker notation $\delta_{ji} = I(j = i)$, we set $\mathbf{e}_i = (\delta_{1i}, \ldots, \delta_{di})$, $i = 1, \ldots, d$, for the unit vectors in $\mathbb{R}^d$. By the d-dimensional Poisson distribution $\mathcal{P}_{\lambda_1,\ldots,\lambda_d}$ with mean vector $(\lambda_1, \ldots, \lambda_d)$ we mean the distribution on $(\mathbb{R}^d, \mathcal{B}^d)$ of a d-dimensional random vector with independent components such that the i-th component has the Poisson distribution with mean $\lambda_i > 0$, $i = 1, \ldots, d$.

Lemma 1. *Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent d-dimensional random vectors with distributions defined as $\mathbb{P}\{\mathbf{X}_j = \mathbf{e}_i\} = p_{ij}$, $i = 1, \ldots, d$, and $\mathbb{P}\{\mathbf{X}_j = \mathbf{0}\} = 1 - \sum_{i=1}^d p_{ij}$ for $j = 1, 2, \ldots$, where the constants $p_{ij} \in (0, 1)$ are such that $\sum_{i=1}^d p_{ij} \leq 1$, $j \in \mathbb{N}$, and $\lambda_i := \sum_{j=1}^\infty p_{ij} < \infty$, $i = 1, \ldots, d$. If $\mathbf{S} := \sum_{j=1}^\infty \mathbf{X}_j$, then*

$$d_{\mathrm{TV}}\big(\mathcal{Q}_{\mathbf{S}}, \mathcal{P}_{\lambda_1,\ldots,\lambda_d}\big) \leq \sum_{j=1}^{\infty} \left[\sum_{i=1}^{d} p_{ij}\right]^2.$$

Proof. Setting $q_{lj} = p_{lj} / \sum_{i=1}^d p_{ij}$, $l = 1, \ldots, d$, so that $\sum_{l=1}^d q_{lj} = 1$ for all $j \in \mathbb{N}$, we construct independent $(d+1)$-dimensional random vectors $\mathbf{Y}_j$, $j \in \mathbb{N}$, with nonnegative integer-valued components by specifying their distributions as follows: the possible values of any of the first d components are all the integers $0, 1, 2, \ldots$, while the possible values of the $(d+1)$-st component are $0, 1, \ldots, d$, such that

$$\begin{aligned}
&\mathbb{P}\{\mathbf{Y}_j = (\mathbf{0}, 0)\} = 1 - \sum_{r=1}^{d} p_{rj}, \\
&\mathbb{P}\{\mathbf{Y}_j = (\mathbf{0}, l)\} = q_{lj}\left[e^{-\sum_{r=1}^d p_{rj}} - \left\{1 - \sum_{r=1}^{d} p_{rj}\right\}\right], \quad l = 1, \ldots, d, \\
&\mathbb{P}\{\mathbf{Y}_j = (\mathbf{e}_i, 0)\} = p_{ij} e^{-\sum_{r=1}^d p_{rj}}, \quad \mathbb{P}\{\mathbf{Y}_j = (\mathbf{e}_i, l)\} = 0, \quad i, l = 1, \ldots, d,
\end{aligned}$$

and if $k_1, \ldots, k_d \geq 0$ such that $\sum_{i=1}^d k_i \geq 2$, then $\mathbb{P}\{\mathbf{Y}_j = (k_1, \ldots, k_d, 0)\} = 0$ and

$$\mathbb{P}\{\mathbf{Y}_j = (k_1, \ldots, k_d, l)\} = q_{ij} e^{-\sum_{r=1}^d p_{rj}} \prod_{r=1}^{d} \frac{p_{rj}^{k_r}}{k_r!}, \quad l = 1, \ldots, d.$$

Since $e^{-x} \geq 1 - x$ for all $x \geq 0$ and

$$\sum_{k_1,\ldots,k_d \geq 0} \prod_{r=1}^{d} \frac{p_{rj}^{k_r}}{k_r!} = e^{\sum_{r=1}^d p_{rj}}$$

it is easy to check that this is a proper distribution, and so $\mathbf{Y}_j$ is a well-defined random variable, for each $j \in \mathbb{N}$.

In terms of the sequence $\{\mathbf{Y}_j\}$, we now define two sequences $\{\mathbf{V}_j\}$ and $\{\mathbf{W}_j\}$ of d-dimensional random vectors. As for $\mathbf{V}_j$, for every $j \in \mathbb{N}$ we put $\mathbf{V}_j = \mathbf{0}$ if $\mathbf{Y}_j = (\mathbf{0}, 0)$ and for each $i = 1, \ldots, d$ we put $\mathbf{V}_j = \mathbf{e}_i$ if either $\mathbf{Y}_j = (\mathbf{e}_i, 0)$ or $\mathbf{Y}_j = (k_1, \ldots, k_d, i)$ for any nonnegative integers $k_1, \ldots, k_d$. For $\mathbf{W}_j$, for every $j \in \mathbb{N}$ and any nonnegative integers $k_1, \ldots, k_d$ we put $\mathbf{W}_j = (k_1, \ldots, k_d)$ whenever $\mathbf{Y}_j = (k_1, \ldots, k_d, l)$ for any $l \in \{0, 1, \ldots, d\}$. Clearly, $\mathbb{P}\{\mathbf{V}_j = \mathbf{0}\} = 1 - \sum_{r=1}^{d} p_{rj}$ and we see that

$$\begin{aligned}
\mathbb{P}\{\mathbf{V}_j = \mathbf{e}_i\} &= q_{ij} e^{-\sum_{r=1}^{d} p_{rj}} \left[e^{\sum_{r=1}^{d} p_{rj}} - \left\{ 1 + \sum_{r=1}^{d} p_{rj} \right\} \right] \\
&\qquad + q_{ij} \left[e^{-\sum_{r=1}^{d} p_{rj}} - \left\{ 1 - \sum_{r=1}^{d} p_{rj} \right\} \right] + p_{ij} e^{-\sum_{r=1}^{d} p_{rj}} \\
&= p_{ij}
\end{aligned}$$

for every $i = 1, \ldots, d$, and also that

$$\mathbb{P}\{\mathbf{W}_j = (k_1, \ldots, k_d)\} = \prod_{r=1}^{d} \frac{p_{rj}^{k_r}}{k_r!} e^{-p_{rj}}, \quad k_1, \ldots, k_d \geq 0,$$

for all $j \in \mathbb{N}$. In other words, for each $j \in \mathbb{N}$ the distributional equality $\mathbf{V}_j \stackrel{\mathcal{D}}{=} \mathbf{X}_j$ holds and $\mathbf{W}_j$ is a d-dimensional Poisson random vector with mean $(p_{1j}, \ldots, p_{dj})$. Furthermore, since $\mathbf{Y}_1, \mathbf{Y}_2, \ldots$ are independent, the vectors $\mathbf{V}_1, \mathbf{V}_2, \ldots$ are also independent, and so are the vectors $\mathbf{W}_1, \mathbf{W}_2, \ldots$. Thus $\mathbf{V} := \sum_{j=1}^{\infty} \mathbf{V}_j \stackrel{\mathcal{D}}{=} \mathbf{S}$ and the distribution of $\mathbf{W} := \sum_{j=1}^{\infty} \mathbf{W}_j$ is $\mathcal{P}_{\lambda_1, \ldots, \lambda_d}$. Since

$$\begin{aligned}
\mathbb{P}\{\mathbf{V} \neq \mathbf{W}\} &\leq \sum_{j=1}^{\infty} \mathbb{P}\{\mathbf{V}_j \neq \mathbf{W}_j\} \\
&= \sum_{j=1}^{\infty} \left[1 - \mathbb{P}\{\mathbf{Y}_j = (\mathbf{0}, 0)\} - \sum_{i=1}^{d} \mathbb{P}\{\mathbf{Y}_j = (\mathbf{e}_i, 0)\} \right] \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{d} p_{ij} \left[1 - e^{-\sum_{r=1}^{d} p_{rj}} \right] \\
&\leq \sum_{j=1}^{\infty} \left[\sum_{i=1}^{d} p_{ij} \right]^2,
\end{aligned}$$

this proves the lemma by an application of the inequality in (3). ■

Lemma 2. *For $\alpha > 2$, let $\mathcal{N}_{k,n}^{\alpha}(B) = \mathbb{P}\{N_n(\alpha) \in B \,|\, |S[\mathcal{I}_n(\alpha)]| = k\}$, $B \in \mathcal{B}$, be the conditional distribution of $N_n(\alpha)$ given $|S[\mathcal{I}_n(\alpha)]| = k$, $k = 1, 2, \ldots, n$. Then*

$$d_{\mathrm{TV}}\Big(\mathcal{N}_{k,n}^{\alpha}, \mathcal{B}_{p_n(\alpha)}^{k} \Big) \leq d_\alpha \frac{1}{n^{2\alpha-4}} + d_\alpha \frac{\alpha-1}{\alpha-2} \frac{\min(k, n-k)}{n^{2\alpha-4}}$$

for all $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$, where $d_\alpha := c_\alpha^2/(2\alpha-4)$ and $c_\alpha = (\lfloor 2^\alpha \rfloor + 1)/\lfloor 2^\alpha \rfloor$.

Proof. For each $n \in \mathbb{N}$ and each $k \in \{1, 2, \dots, n\}$, and a fixed vector $\mathbf{l} = (l_1, l_2, \dots, l_k)$ for some $l_1, l_2, \dots, l_k \in \{0, 1, 2, \dots, \lfloor n^\alpha \rfloor - 1\} =: H_n^\alpha$, where we assume that $l_1 < l_2 < \cdots < l_k$, consider for the integers $m \ge n+1$ and $j \in \{1, 2, \dots, m\}$ the independent k-dimensional random vectors

$$\begin{aligned}\mathbf{X}_{m,j}(\mathbf{l}) &= \Big(X_{m,j}(l_1), X_{m,j}(l_2), \dots, X_{m,j}(l_k)\Big) \\ &:= \big(I(\lfloor \lfloor m^\alpha \rfloor U_{m,j} \rfloor = l_1), I(\lfloor \lfloor m^\alpha \rfloor U_{m,j} \rfloor = l_2), \dots, I(\lfloor \lfloor m^\alpha \rfloor U_{m,j} \rfloor = l_k)\big),\end{aligned}$$

introduce the component sums $S_n(l_i) := \sum_{m=n+1}^{\infty} \sum_{j=1}^{m} X_{m,j}(l_i)$, $i = 1, 2, \dots, k$, and form the vectors $\mathbf{S}_n(\mathbf{l}) := (S_n(l_1), S_n(l_2), \dots, S_n(l_k)) = \sum_{m=n+1}^{\infty} \sum_{j=1}^{m} \mathbf{X}_{m,j}(\mathbf{l})$. Consider also a k-dimensional Poisson random vector $\mathbf{W}_n = (W_{n,1}, W_{n,2}, \dots, W_{n,k})$ with mean $\big(\sum_{m=n+1}^{\infty} m/\lfloor m^\alpha \rfloor, \dots, \sum_{m=n+1}^{\infty} m/\lfloor m^\alpha \rfloor\big) = \big(\lambda_n(\alpha)/n, \dots, \lambda_n(\alpha)/n\big)$ and notice that since the independent variables $\min(W_{n,1}\,, 1), \dots, \min(W_{n,1}\,, 1)$ all have the Bernoulli $\mathcal{B}^1_{p_n(\alpha)}$ distribution, the distribution of $V_n := \sum_{i=1}^{k} \min(W_{n,i}\,, 1)$ is $\mathcal{B}^k_{p_n(\alpha)}$. Since $\mathbb{P}\{\lfloor \lfloor m^\alpha \rfloor U_{m,j} \rfloor = l\} = 1/\lfloor m^\alpha \rfloor$ for any $l \in H_n^\alpha$, the distribution $\mathcal{Q}_{\mathbf{S}_n} = \mathcal{Q}_{\mathbf{S}_n(\mathbf{l})}$ does not depend on $\mathbf{l} \in H_{k,n}^\alpha$, where $H_{k,n}^\alpha$ is the set of all vectors $\mathbf{l} = (l_1, l_2, \dots, l_k)$ with components taken from H_n^α such that $l_1 < l_2 < \cdots < l_k$. Thus the distribution $\mathcal{Q}_{Y_{k,n}} = \mathcal{Q}_{Y_{k,n}(\mathbf{l})}$ of $Y_{k,n}(\mathbf{l}) := \sum_{i=1}^{k} \min(S_n(l_i)\,, 1)$ does not depend on $\mathbf{l} = (l_1, l_2, \dots, l_k) \in H_{k,n}^\alpha$ either, and we have

$$\begin{aligned} d_{\mathrm{TV}}\Big(\mathcal{Q}_{Y_{k,n}}\,, \mathcal{B}^k_{p_n(\alpha)}\Big) \le d_{\mathrm{TV}}\big(\mathcal{Q}_{\mathbf{S}_n}\,, \mathcal{Q}_{\mathbf{W}_n}\big) &\le \sum_{m=n+1}^{\infty} m \bigg(\frac{m}{\lfloor m^\alpha \rfloor}\bigg)^2 \le c_\alpha^2 \sum_{m=n+1}^{\infty} \frac{1}{m^{2\alpha-3}} \\ &\le c_\alpha^2 \int_n^\infty \frac{1}{x^{2\alpha-3}}\, dx = \frac{d_\alpha}{n^{2\alpha-4}} =: \delta_n(\alpha) \end{aligned}$$

for all $n \in \mathbb{N}$ and $k \in \{1, 2, \dots, n\}$, where the first inequality is obvious, the second is by Lemma 1, and where $c_\alpha = (\lfloor 2^\alpha \rfloor + 1)/\lfloor 2^\alpha \rfloor$ in the third and $d_\alpha = c_\alpha^2/(2\alpha - 4)$.

For any $k \in \{1, 2, \dots, n\}$, let $R_{k,n}$ be the set of all vectors $\mathbf{r} = (r_1, r_2, \dots, r_k)$ with components all in $\mathbb{N}$ such that $\sum_{i=1}^{k} r_i = n$. Also, for any particular case $S[\mathcal{I}_n(\alpha)] = \mathbf{l}$ of the event $\{|S[\mathcal{I}_n(\alpha)]| = k\}$, for some vector $\mathbf{l} = (l_1, l_2, \dots, l_k) \in H_{k,n}^\alpha$ after ordering, let $R[\mathcal{I}_n(\alpha)] = (\#(l_1, \mathcal{I}_n(\alpha)), \dots, \#(l_k, \mathcal{I}_n(\alpha)))$ denote the corresponding vector of multiplicities. Recalling (1), notice that for any Borel set $B \in \mathcal{B}^1$,

$$\mathcal{N}_{k,n}^\alpha(B) = \frac{\mathbb{P}\{N_n(\alpha) \in B\,, |S[\mathcal{I}_n(\alpha)]| = k\}}{\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = k\}} = \frac{\sum_{\mathbf{l} \in H_{k,n}^\alpha} \sum_{\mathbf{r} \in R_{k,n}} p_{k,n}^{\alpha,\mathbf{l},\mathbf{r}}(B)}{\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = k\}}\,,$$

where, setting $A_{k,n}^{\alpha,\mathbf{l},\mathbf{r}} := \{S[\mathcal{I}_n(\alpha)] = \mathbf{l} = (l_1, \dots, l_k)\} \cap \{R[\mathcal{I}_n(\alpha)] = \mathbf{r} = (r_1, \dots, r_k)\}$,

$$\begin{aligned} p_{k,n}^{\alpha,\mathbf{l},\mathbf{r}}(B) &= \mathbb{P}\Bigg\{\Bigg\{\sum_{i=1}^{k} \min(S_n(l_i)\,, r_i) \in B\Bigg\} \bigcap A_{k,n}^{\alpha,\mathbf{l},\mathbf{r}}\Bigg\} \\ &= \mathbb{P}\Bigg\{\sum_{i=1}^{k} \min(S_n(l_i)\,, r_i) \in B\Bigg\} \mathbb{P}\Big\{A_{k,n}^{\alpha,\mathbf{l},\mathbf{r}}\Big\} =: \mathcal{N}_{k,n}^{\alpha,\mathbf{l},\mathbf{r}}(B)\, \mathbb{P}\Big\{A_{k,n}^{\alpha,\mathbf{l},\mathbf{r}}\Big\} \end{aligned}$$

by the total independence of the Uniform$(0,1)$ variables in the underlying array. Whence

$$\mathcal{N}^{\alpha}_{k,n}(B) - \mathcal{Q}_{Y_{k,n}}(B) = \frac{\sum_{\mathbf{l}\in H^{\alpha}_{k,n}} \sum_{\mathbf{r}\in R_{k,n}} \Big[\mathcal{N}^{\alpha,\mathbf{l},\mathbf{r}}_{k,n}(B) - \mathcal{Q}_{Y_{k,n}(\mathbf{l})}(B)\Big] \mathbb{P}\Big\{A^{\alpha,\mathbf{l},\mathbf{r}}_{k,n}\Big\}}{\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = k\}}$$

for all $B \in \mathcal{B}^1$, and so, by (3),

$$d_{\mathrm{TV}}\big(\mathcal{N}^{\alpha}_{k,n}\,,\mathcal{Q}_{Y_{k,n}}\big) \le \frac{\sum_{\mathbf{l}\in H^{\alpha}_{k,n}} \sum_{\mathbf{r}\in R_{k,n}} q^{\alpha,\mathbf{l},\mathbf{r}}_{k,n}\, \mathbb{P}\Big\{A^{\alpha,\mathbf{l},\mathbf{r}}_{k,n}\Big\}}{\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = k\}}\,,$$

where, since $S_n(l)$ for any $l \in H^{\alpha}_n$ has the same distribution as $S_n(0)$, say, and since at most $n-k$ of the integers $r_1,\ldots,r_k$ is greater than 1 if $\mathbf{r} = (r_1,\ldots,r_k) \in R_{k,n}$,

$$\begin{aligned}
q^{\alpha,\mathbf{l},\mathbf{r}}_{k,n} &:= \mathbb{P}\Bigg\{\sum_{i=1}^{k} \min(S_n(l_i)\,,r_i) \ne \sum_{i=1}^{k} \min(S_n(l_i)\,,1)\Bigg\} \\
&\le \sum_{i=1}^{k} \mathbb{P}\big\{\min(S_n(l_i)\,,r_i) > \min(S_n(l_i)\,,1)\big\} \\
&\le \sum_{i=1}^{k} \mathbb{P}\big\{S_n(0) > 1\big\} I(r_i > 1) \\
&\le \min(k, n-k)\mathbb{P}\{S_n(0) > 1\}.
\end{aligned}$$

Combining the two parts of the proof above,

$$\begin{aligned}
d_{\mathrm{TV}}\Big(\mathcal{N}^{\alpha}_{k,n}\,,\mathcal{B}^{k}_{p_n(\alpha)}\Big) &\le d_{\mathrm{TV}}\big(\mathcal{N}^{\alpha}_{k,n}\,,\mathcal{Q}_{Y_{k,n}}\big) + d_{\mathrm{TV}}\Big(\mathcal{Q}_{Y_{k,n}}\,,\mathcal{B}^{k}_{p_n(\alpha)}\Big) \\
&\le \min(k,n-k)\mathbb{P}\{S_n(0) > 1\} + \delta_n(\alpha) \\
&\le \min(k,n-k)\left[\delta_n(\alpha) + \frac{\lambda_n^2(\alpha)}{2n^2}\right] + \delta_n(\alpha) \\
&\le \min(k,n-k)\,\delta_n(\alpha)\,\frac{\alpha-1}{\alpha-2} + \delta_n(\alpha),
\end{aligned}$$

where the penultimate inequality holds because, by the special case $k = 1$ of the main inequality in the first part of the proof, $|\mathbb{P}\{S_n(0) > 1\} - \mathcal{P}_{\lambda_n(\alpha)/n}(\{2,3,\ldots\})| \le \delta_n(\alpha)$, and because of the special case $k = 2$ of the following fact: for the Poisson distribution $\mathcal{P}_\lambda$ with mean $\lambda > 0$ we have

$$\begin{aligned}
\mathcal{P}_\lambda(\{k,k+1,k+2\ldots\}) &= e^{-\lambda}\sum_{j=k}^{\infty}\frac{\lambda^j}{j!} = \frac{\lambda^k}{k!}\,e^{-\lambda}\sum_{l=0}^{\infty}\frac{\lambda^l}{(k+1)(k+2)\cdots(k+l)} \\
&\le \frac{\lambda^k}{k!}
\end{aligned} \tag{5}$$

for every $k \in \mathbb{N}$. The last inequality in the bound for the total variation distance above is due to the fact that

$$\frac{\lambda_n(\alpha)}{n} = \sum_{m=n+1}^{\infty} \frac{m}{\lfloor m^\alpha \rfloor} \le c_\alpha \sum_{m=n+1}^{\infty} \frac{1}{m^{\alpha-1}} \le c_\alpha \int_n^\infty \frac{1}{x^{\alpha-1}}\, dx = \frac{c_\alpha}{\alpha-2}\, \frac{1}{n^{\alpha-2}}\,, \tag{6}$$

so that $\lambda_n^2(\alpha)/\big[2n^2\big] \le \delta_n(\alpha)/[\alpha-2]$, concluding the proof of the lemma. ■

The next lemma for the approximation of binomial distributions by Poisson distributions on $(\mathbb{R}, \mathcal{B})$ is a combination of Lemmas 4.2.2 and 4.2.3, as obtained in Corollary 4.3.2, in [1]; according to Lemma 8.4.1 in [1], the stated uniform bound is optimal modulo constants.

Lemma 3. $d_{\mathrm{TV}}(\mathcal{B}_p^n, \mathcal{P}_{np}) \le \min\big(\frac{13}{2}\, p\,, np^2\big)$ *for all* $n \in \mathbb{N}$ *and* $p \in (0,1)$.

The last lemma for Poisson distributions on $(\mathbb{R}, \mathcal{B})$ is just a simple elementary exercise, proved as Lemma 2.2 in [7].

Lemma 4. *If* $0 < \lambda < \mu$, *then* $d_{\mathrm{TV}}(\mathcal{P}_\lambda, \mathcal{P}_\mu) \le \min\Big(1, \frac{1}{\sqrt{\lambda}}\Big)(\mu - \lambda)$.

Proof of Theorem 2. Let $\alpha > 2$. Since, in the notation of Lemmas 2 and 3,

$$d_{\mathrm{TV}}\Big(\mathcal{N}_n^\alpha\,, \mathcal{B}_{p_n(\alpha)}^n\Big) \le \sum_{k=1}^{n} d_{\mathrm{TV}}\Big(\mathcal{N}_{k,n}^\alpha\,, \mathcal{B}_{p_n(\alpha)}^n\Big)\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = k\}$$

and $d_{\mathrm{TV}}(\mathcal{B}_p^n, \mathcal{B}_p^k) \le \mathcal{B}_p^{n-k}(\{1, 2, \ldots, n-k\}) \le (n-k)p$ for any positive integers $k \le n$ and any $p \in (0,1)$ by an easy application of (3) and by Markov's inequality, and, furthermore, for all $n \in \mathbb{N}$ on account of (6),

$$p_n(\alpha) = 1 - e^{-\frac{\lambda_n(\alpha)}{n}} \le \frac{\lambda_n(\alpha)}{n} \le \frac{c_\alpha}{\alpha-2}\, \frac{1}{n^{\alpha-2}} =: \frac{a_\alpha}{n^{\alpha-2}}\,, \tag{7}$$

setting $b_\alpha := d_\alpha[\alpha-1]/[\alpha-2]$ and applying Lemma 2 we obtain

$$d_{\mathrm{TV}}\Big(\mathcal{N}_{k,n}^\alpha\,, \mathcal{B}_{p_n(\alpha)}^n\Big) \le \begin{cases} \frac{d_\alpha}{n^{2\alpha-4}} + \frac{b_\alpha k}{n^{2\alpha-4}} + \frac{a_\alpha(n-k)}{n^{\alpha-2}}, & \text{if} \quad k \le n/2, \\ \frac{d_\alpha}{n^{2\alpha-4}} + \frac{b_\alpha(n-k)}{n^{2\alpha-4}} + \frac{a_\alpha(n-k)}{n^{\alpha-2}}, & \text{if} \quad k > n/2. \end{cases}$$

For all $n \in \mathbb{N}$ this yields

$$\begin{aligned} d_{\mathrm{TV}}\Big(\mathcal{N}_n^\alpha, \mathcal{B}^n_{p_n(\alpha)}\Big) &\le \frac{d_\alpha}{n^{2\alpha-4}} + \frac{b_\alpha}{n^{2\alpha-4}} \sum_{k=1}^{\lfloor n/2 \rfloor} k\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = k\} \\ &\quad + \frac{b_\alpha}{n^{2\alpha-4}} \sum_{k=\lfloor n/2\rfloor+1}^{n} (n-k)\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = k\} \\ &\quad + \frac{a_\alpha}{n^{\alpha-2}} \sum_{k=1}^{n} (n-k)\mathbb{P}\{|S[\mathcal{I}_n(\alpha)]| = k\} \\ &\le \frac{d_\alpha}{n^{2\alpha-4}} + \frac{b_\alpha}{2\,n^{2\alpha-5}}\, \mathbb{P}\Big\{|S[\mathcal{I}_n(\alpha)]| \le \frac{n}{2}\Big\} \\ &\quad + \frac{a_\alpha + b_\alpha}{n^{\alpha-2}} \big[n - \mathbb{E}\big(|S[\mathcal{I}_n(\alpha)]|\big)\big], \end{aligned}$$

upon enlarging all k to $n/2$ in the first sum and extending the second sum to all $k \in \{1, 2, \ldots, n\}$. Using the exact formula for the mean in Theorem 1, we see that $n - \mathbb{E}(|S[\mathcal{I}_n(\alpha)]|)$ is not greater than

$$(8) \qquad n - \lfloor n^\alpha \rfloor \Big[1 - e^{-\frac{n}{\lfloor n^\alpha \rfloor}}\Big] \le n - \lfloor n^\alpha \rfloor \left[\frac{n}{\lfloor n^\alpha \rfloor} - \frac{n^2}{2\lfloor n^\alpha \rfloor^2}\right] = \frac{n^2}{2\lfloor n^\alpha \rfloor} \le \frac{c_\alpha}{2}\,\frac{1}{n^{\alpha-2}}\,,$$

since $1 - e^{-x} \ge x - \frac{x^2}{2}$ for all $x \ge 0$, the final bound holding for all $n \in \mathbb{N}$, and hence also

$$\mathbb{P}\Big\{|S[\mathcal{I}_n(\alpha)]| \le \frac{n}{2}\Big\} = \mathbb{P}\Big\{n - |S[\mathcal{I}_n(\alpha)]| \ge \frac{n}{2}\Big\} \le \frac{c_\alpha}{2}\,\frac{1}{n^{\alpha-2}}\,\frac{2}{n} = \frac{c_\alpha}{n^{\alpha-1}}$$

by Markov's inequality. Therefore,

$$d_{\mathrm{TV}}\Big(\mathcal{N}_n^\alpha, \mathcal{B}^n_{p_n(\alpha)}\Big) \le \frac{D_\alpha}{n^{2\alpha-4}} + \frac{b_\alpha c_\alpha}{2\,n^{3\alpha-6}}\,, \quad \text{where} \quad D_\alpha := d_\alpha + \frac{[a_\alpha + b_\alpha]c_\alpha}{2}\,,$$

and hence finally

$$d_{\mathrm{TV}}\Big(\mathcal{N}_n^\alpha, \mathcal{B}^n_{p_n(\alpha)}\Big) \le \frac{C_\alpha}{n^{2\alpha-4}}\,, \quad \text{where} \quad C_\alpha = d_\alpha + \frac{a_\alpha c_\alpha}{2} + b_\alpha c_\alpha = 2d_\alpha + c_\alpha d_\alpha \frac{\alpha-1}{\alpha-2}\,,$$

for all $n \in \mathbb{N}$. Collecting the constants from Lemma 2, this proves the first statement of the theorem.

For the proof of the second statement we need only to note that

$$\begin{aligned} d_{\mathrm{TV}}\Big(\mathcal{B}^n_{p_n(\alpha)}, \mathcal{P}_{np_n(\alpha)}\Big) &\le \min\left(\frac{13}{2}\,p_n(\alpha)\,, np_n^2(\alpha)\right) \le \min\left(\frac{13\,a_\alpha}{2\,n^{\alpha-2}}\,, \frac{a_\alpha^2}{n^{2\alpha-5}}\right) \\ &\le \begin{cases} \frac{13\,a_\alpha}{2\,n^{\alpha-2}}, & \text{if } \alpha < 3, \\ \frac{a_\alpha^2}{n^{2\alpha-5}}, & \text{if } \alpha \ge 3, \end{cases} \end{aligned}$$

by Lemma 3 and (7). This and the first statement then imply the second.

Finally, to change the mean of the approximating Poisson distribution to $\lambda_n^*(\alpha)$, consider any $\alpha > 2$ and $n \in \mathbb{N}$, and note that by an obvious version of (6) and (7),

$$np_n(\alpha) \leq \lambda_n(\alpha) \leq \left[1 + \frac{1}{\lfloor (n+1)^\alpha \rfloor}\right]\lambda_n^*(\alpha) \leq \left[1 + \frac{1}{n^\alpha}\right]\lambda_n^*(\alpha)$$

and, as in (8),

$$np_n(\alpha) \geq n\left[\frac{\lambda_n(\alpha)}{n} - \frac{\lambda_n^2(\alpha)}{2n^2}\right] = \lambda_n(\alpha)\left[1 - \frac{\lambda_n(\alpha)}{2n}\right] \geq \lambda_n(\alpha)\left[1 - \frac{1}{2n}\left(1 + \frac{1}{n^\alpha}\right)\lambda_n^*(\alpha)\right],$$

where

$$\lambda_n(\alpha) \geq n \sum_{m=n+1}^{\infty} \frac{1}{m^{\alpha-1}} \geq n \int_{n+1}^{\infty} \frac{1}{x^{\alpha-1}}\, dx = \left(\frac{n}{n+1}\right)^{\alpha-2} \lambda_n^*(\alpha) \geq \left(1 - \frac{1}{n}\right)^{\alpha-2} \lambda_n^*(\alpha)$$

and, setting $\lceil x \rceil = \min\{k \in \mathbb{N} \colon k \geq x\}$ for $x > 0$, separating the cases $2 < \alpha \leq 3$ and $\alpha > 3$, and using the Bernoulli inequality in the latter case,

$$\left(1 - \frac{1}{n}\right)^{\alpha-2} \geq \left(1 - \frac{1}{n}\right)^{\lceil \alpha-2 \rceil} \geq 1 - \frac{\lceil \alpha - 2 \rceil}{n}\,.$$

Whence, simplifying the lower bound,

$$\left[1 - \frac{\lceil \alpha - 2 \rceil}{n}\right]\left[1 - \frac{1}{(\alpha-2)n^{\alpha-2}}\right]\lambda_n^*(\alpha) \leq np_n(\alpha) \leq \left[1 + \frac{1}{n^\alpha}\right]\lambda_n^*(\alpha)\,,$$

from which by further elementary considerations,

$$[1 - v_n(\alpha)]\lambda_n^*(\alpha) := \left[1 - \frac{m_\alpha}{n^{\min(\alpha-2,1)}}\right]\lambda_n^*(\alpha) \leq np_n(\alpha) \leq \left[1 + \frac{1}{n^\alpha}\right]\lambda_n^*(\alpha)\,,$$

where $m_\alpha := 2\max((\alpha-2)^{-1}, \lceil \alpha - 2 \rceil)$, which in particular implies the asymptotic equality $np_n(\alpha) \sim \lambda_n^*(\alpha)$. So, putting $c_\alpha^* := m_\alpha/(\alpha-2)$ and $d_\alpha^* := 1/(\alpha-2)$,

$$-\frac{c_\alpha^*}{n^{\alpha+\min(\alpha-2,1)-3}} \leq np_n(\alpha) - \lambda_n^*(\alpha) \leq \frac{d_\alpha^*}{n^{2\alpha-3}}\,.$$

Hence, using Lemma 4, for all n large enough,

$$\begin{aligned} d_{\mathrm{TV}}\Big(\mathcal{P}_{np_n(\alpha)}\,, \mathcal{P}_{\lambda_n^*(\alpha)}\Big) &\leq \min\left(1, \frac{1}{\sqrt{[1 - v_n(\alpha)]\lambda_n^*(\alpha)}}\right)\big|np_n(\alpha) - \lambda_n^*(\alpha)\big| \\ &\leq \min\left(1, \frac{\sqrt{\alpha-2}}{\sqrt{[1 - v_n(\alpha)]n^{3-\alpha}}}\right)\frac{c_\alpha^*}{n^{\alpha+\min(\alpha-2,1)-3}}\,, \end{aligned}$$

and so, for all n large enough,

$$d_{\mathrm{TV}}\Big(\mathcal{P}_{np_n(\alpha)}\,,\mathcal{P}_{\lambda_n^*(\alpha)}\Big) \le \begin{cases} \frac{\sqrt{2(\alpha-2)}\,c_\alpha^*}{n^{\frac{3\alpha-7}{2}}}\,, & \text{if } \alpha<3,\\ \frac{\sqrt{2}\,c_3^*}{n}\,, & \text{if } \alpha=3,\\ \frac{c_\alpha^*}{n^{2\alpha-5}}\,, & \text{if } \alpha>3. \end{cases}$$

Inspecting the cases separately when this inequality is combined with the second statement of the theorem, the last statement also follows. ■

Proof of Theorem 3. Fix any $\alpha > 2$. We introduce

$$N_n^*(\alpha) := \sum_{m=n+1}^{\infty}\sum_{j=1}^{m} I\big(\lfloor\lfloor m^\alpha\rfloor U_{m,j}\rfloor \in \mathcal{I}_n(\alpha)\big) \ge N_n(\alpha)$$

and prove the statement for $N_n^*(\alpha)$ instead of $N_n(\alpha)$. Since, clearly,

$$\mathbb{P}\big\{N_n^*(\alpha) \ge Kn^{3-\alpha}\,\big|\,|S[\mathcal{I}_n(\alpha)]| = n\big\} \ge \mathbb{P}\big\{N_n^*(\alpha) \ge Kn^{3-\alpha}\,\big|\,|S[\mathcal{I}_n(\alpha)]| = k\big\}$$

for every $k \in \{1, 2, \ldots, n\}$ and constant $K > 0$, we have

$$\begin{aligned}\mathbb{P}\big\{N_n^*(\alpha) \ge Kn^{3-\alpha}\big\} &= \sum_{k=1}^{n}\mathbb{P}\big\{N_n^*(\alpha) \ge Kn^{3-\alpha}\,\big|\,|S[\mathcal{I}_n(\alpha)]| = k\big\}\mathbb{P}\big\{|S[\mathcal{I}_n(\alpha)]| = k\big\}\\ &\le \mathbb{P}\big\{N_n^*(\alpha) \ge Kn^{3-\alpha}\,\big|\,|S[\mathcal{I}_n(\alpha)]| = n\big\}\end{aligned}$$

for all $n \in \mathbb{N}$, and hence it suffices to prove that

$$\sum_{n=1}^{\infty}\mathbb{P}\big\{N_n^*(\alpha) \ge K_\alpha n^{3-\alpha}\,\big|\,|S[\mathcal{I}_n(\alpha)]| = n\big\} < \infty$$

for some constant $K_\alpha > 0$.

To do this, consider for each $n \in \mathbb{N}$ the array $\{W_{m,1}^{n,\alpha}, W_{m,2}^{n,\alpha}, \ldots, W_{m,m}^{n,\alpha}\}_{m=n+1}^{\infty}$ of independent random variables such that $W_{m,1}^{n,\alpha}, W_{m,2}^{n,\alpha}, \ldots, W_{m,m}^{n,\alpha}$ have the same Poisson distribution with mean $-\log\big(1-\{n/\lfloor m^\alpha\rfloor\}\big)$, $m = n+1, n+2, \ldots$, and their sum $N_n^\diamond(\alpha) := \sum_{m=n+1}^{\infty}\sum_{j=1}^{m} W_{m,j}^{n,\alpha}$ that has the Poisson distribution with mean

$$\lambda_n^\diamond(\alpha) := -\sum_{m=n+1}^{\infty} m\log\bigg(1-\frac{n}{\lfloor m^\alpha\rfloor}\bigg).$$

The underlying idea in the Poisson majorization trick is to notice that

$$\mathbb{P}\big\{W_{m,j}^{n,\alpha} = 0\big\} = 1-\frac{n}{\lfloor m^\alpha\rfloor} = \mathbb{P}\big\{I\big(\lfloor\lfloor m^\alpha\rfloor U_{m,j}\rfloor \in \mathcal{I}_n(\alpha)\big) = 0\,\big|\,|S[\mathcal{I}_n(\alpha)]| = n\big\},$$

which implies that the distribution of the Bernoulli variable $\min\bigl(W_{m,j}^{n,\alpha},1\bigr)$ and the conditional distribution of the Bernoulli variable $I\bigl(\lfloor\lfloor m^\alpha\rfloor U_{m,j}\rfloor \in \mathcal{I}_n(\alpha)\bigr)$ given $|S[\mathcal{I}_n(\alpha)]| = n$ are the same for all $j \in \{1,2,\ldots,m\}$, $m \in \{n+1,n+2,\ldots\}$, $n \in \mathbb{N}$. Hence for all these parameters we have

$$\begin{aligned}\mathbb{P}\bigl\{W_{m,j}^{n,\alpha} \geq x\bigr\} &\geq \mathbb{P}\bigl\{\min\bigl(W_{m,j}^{n,\alpha},1\bigr) \geq x\bigr\} \\ &= \mathbb{P}\bigl\{I\bigl(\lfloor\lfloor m^\alpha\rfloor U_{m,j}\rfloor \in \mathcal{I}_n(\alpha)\bigr) \geq x \bigm| |S[\mathcal{I}_n(\alpha)]| = n\bigr\}\end{aligned}$$

for every $x \in \mathbb{R}$, which for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$ implies that

$$\mathbb{P}\bigl\{N_n^\diamond(\alpha) \geq x\bigr\} \geq \mathbb{P}\bigl\{N_n^*(\alpha) \geq x \bigm| |S[\mathcal{I}_n(\alpha)]| = n\bigr\}.$$

Thus it suffices to show that $s_\alpha := \sum_{n=1}^\infty \mathbb{P}\bigl\{N_n^\diamond(\alpha) \geq K_\alpha n^{3-\alpha}\bigr\} < \infty$ for some constant $K_\alpha > 0$.

Notice first that $n/\lfloor(n+1)^\alpha\rfloor \leq 1/2$ for all $n \in \mathbb{N}$ since $\alpha > 2$. Hence, using the inequality $-\log(1-x) \leq 2x$, valid for all $x \in [0,1/2]$, for all $n \in \mathbb{N}$ we obtain

$$\lambda_n^\diamond(\alpha) \leq 2n \sum_{m=n+1}^\infty \frac{m}{\lfloor m^\alpha\rfloor} \leq \frac{2c_\alpha n^{3-\alpha}}{\alpha-2} \leq \frac{4c_\alpha \lfloor n^{3-\alpha}\rfloor}{\alpha-2}$$

by the inequalities in (6). Using (5), it therefore follows that

$$s_\alpha \leq \sum_{n=1}^\infty \mathbb{P}\bigl\{N_n^\diamond(\alpha) \geq K_\alpha \lfloor n^{3-\alpha}\rfloor\bigr\} \leq \sum_{n=1}^\infty \frac{\Bigl(\frac{4c_\alpha}{\alpha-2}\lfloor n^{3-\alpha}\rfloor\Bigr)^{K_\alpha\lfloor n^{3-\alpha}\rfloor}}{\bigl(K_\alpha\lfloor n^{3-\alpha}\rfloor\bigr)!} < \infty,$$

the finiteness justified by Stirling's formula if the integer $K_\alpha > 4ec_\alpha/(\alpha-2)$. ■

3. Randomly rarefied averages and bootstrap means

Let $X = X_0, X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables, given on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where the triangular array $\{U_{n,1},\ldots,U_{n,m_n}\}_{n=1}^\infty$ of independent Uniform$(0,1)$ random variables is given, where $\{m_n\}_{n=1}^\infty$ is a sequence of positive integers, such that the sequence $\{X_j\}_{j=0}^\infty$ and the array $\{U_{n,1},\ldots,U_{n,m_n}\}_{n=1}^\infty$ are also independent; the array in itself is the same as in the previous sections when $m_n \equiv n$. Throughout this section, we use the notation $S_n := \sum_{j=0}^{n-1} X_j$, $n \in \mathbb{N}$, for the partial sums and assume that $\mathbb{E}(|X|) < \infty$. The notation associated with the array $\{U_{n,1},\ldots,U_{n,n}\}_{n=1}^\infty$ in Section 2 will also be used here.

We say that a sequence $\{\xi_n\}_{n=1}^\infty$ of random variables converges *completely* to the random variable ξ and, to honor Erdős, Hsu and Robbins as in [6], write $\xi_n \xrightarrow{\text{EHR}} \xi$, if and only if $\sum_{n=1}^\infty \mathbb{P}\{|\xi_n - \xi| > \varepsilon\} < \infty$ for every $\varepsilon > 0$. So, $\xi_n \xrightarrow{\text{EHR}} \xi$ implies $\xi_n \xrightarrow{\text{a.s.}} \xi$, almost sure convergence. The converse is generally not true: for example the Erdős–Hsu–Robbins theorem from [6, 7, 8] (cf. also [5], pp. 132, 375) itself says that $S_n/n \xrightarrow{\text{EHR}} \mathbb{E}(X)$ if and only if $\mathbb{E}(X^2) < \infty$, and one of the extensions by Baum and Katz [3] states that $\sum_{n=1}^{\infty} n^{p\gamma-2}\mathbb{P}\{|S_n - n\mathbb{E}(X)| > \varepsilon n^\gamma\} < \infty$ for every $\varepsilon > 0$ for some p and γ such that $p\gamma > 1$ and $\gamma > 1/2$ if and only if $\mathbb{E}(|X|^p) < \infty$; both will be used in the proofs below.

Let $m_n \equiv n$. The following strong law of large numbers for randomly rarefied averages is an extension of half of Theorem 3.1 in [6] from $\alpha > 3$ to all fixed $\alpha > 2$. While these rarefied averages and hence the strong law for them appear to be of self-explanatory interest from the pure probabilistic point of view, they also proved to be important both in [6] and here when tackling corresponding strong laws for bootstrap means.

Theorem 4. *If* $\alpha > 2$, *then*

$$\frac{1}{n}\sum_{j=1}^{n} X_{\lfloor \lfloor n^\alpha \rfloor U_{n,j} \rfloor} \xrightarrow{\text{a.s.}} \mathbb{E}(X)$$

if and only if $\mathbb{E}(X^2) < \infty$.

The sufficiency of the condition of a finite second moment is a special case of Theorem 3.2 in [6]. A little weaker form of the necessity part of the theorem, namely that almost sure convergence here implies $\mathbb{E}(|X|^\beta) < \infty$ for all $\beta \in (0,2)$, was cautiously conjectured in Section 5 of [6]. However, even for this weaker form it is somewhat hastily stated in [6] that once the conjecture that is now Theorem 3 is shown to be true, the necessity statement in question would follow by the proof of Theorem 3.1 in [6]. This is an overstatement: as the proof of Theorem 4 in the next section shows, a more sophisticated argument is needed.

Given the statistical sample $X_0, X_1, \dots, X_{n-1}$ of size n, for a general sequence $\{m_n\}$ the variables $X_{\lfloor nU_{n,1}\rfloor}, X_{\lfloor nU_{n,2}\rfloor}, \dots, X_{\lfloor nU_{n,m_n}\rfloor}$ may be viewed as a bootstrap sample of size m_n, which results from resampling m_n times the sequence $X_0, \dots, X_{n-1}$ with replacement such that at each stage any one element has probability $1/n$ to be selected. In fact, since the rows of the array $\{U_{n,1}, \dots, U_{n,m_n}\}$ are also independent, what we have here is that one of the two sequential bootstrap models considered in [6], in which each time n the statistician obtains an additional observation, he will bootstrap the upgraded sample of size $n+1$ by generating m_{n+1} random numbers anew, independently of all previous generations. This model, in comparison with the single-sequence model also considered in [6], reflects an attitude of convenience: no storage is needed and, after all, random number generation is cheap. From the universal convergence results for bootstrap means in [6] it follows that $m_n^{-1}\sum_{j=1}^{m_n} X_{\lfloor nU_{n,j}\rfloor}$ converges to $\mathbb{E}(X)$ in probability whenever $m_n \to \infty$, and, considering the bootstrap sample sizes $m_n = \lfloor n^{1/\alpha}\rfloor$, $n \in \mathbb{N}$, for some $\alpha \geq 1$, we have $m_n^{-1}\sum_{j=1}^{m_n} X_{\lfloor nU_{n,j}\rfloor} \xrightarrow{\text{EHR}} \mathbb{E}(X)$ if and only if $\mathbb{E}(|X|^{1+\alpha}) < \infty$, while one of the best sufficient conditions to date for almost sure convergence states that if $\mathbb{E}((|X| \log^+ |X|)^\alpha) < \infty$ then (9) below holds true; along with similar other results in Theorem 4.4 in [6], this was achieved through a slight improvement of the technique from [2]. Indeed, almost sure convergence has proved to be a deep question, the converse direction requiring a completely new approach: the first necessary condition, both for the present array model and for the single-sequence model, appears in Theorem 4.2 of [6]. It is only for $\alpha > 3$, and in that case in the present array model it states that (9) below implies $\mathbb{E}(|X|^\beta) < \infty$ for all $\beta \in \big[1, 1 + \frac{\alpha}{3}\big)$. This result is extended and improved here by the following

Theorem 5. *If* $\alpha > 2$ *and*

$$\frac{1}{\lfloor n^{1/\alpha} \rfloor} \sum_{j=1}^{\lfloor n^{1/\alpha} \rfloor} X_{\lfloor n U_{n,j} \rfloor} \xrightarrow{\text{a.s.}} \mathbb{E}(X), \tag{9}$$

then $\mathbb{E}(|X|^\beta) < \infty$ *for all* $\beta \in \{2\} \cup (2, 2\alpha/3)$.

Again we must point out a hastily made overstatement of Section 5 in [6] to the effect that once the conjectured Theorem 3 here is proved, then by the proof of Theorem 4.2 there for each $\alpha > 2$ it would follow from (9) that $\mathbb{E}(|X|^\beta) < \infty$ for all $\beta \in \big[1, 1 + \frac{\alpha}{2}\big)$. Here not only a more complicated proof is needed, but in fact we cannot prove this statement for the time being for $\alpha \in (2, 6)$: for this interval of α values Theorem 5 is weaker, while it is stronger for $\alpha > 6$. In any case Theorem 5 gives finite variance for all $\alpha > 2$ and is strictly better than Theorem 4.2 in [6] for $\alpha > 3$. Other than this, here we can only repeat the full conjecture made in [6]: (9) implies $\mathbb{E}(|X|^\alpha) < \infty$ for all $\alpha \geq 1$. While unclear at this writing, it still may be possible that some clever modification and elaboration of the ideas in the present paper could give this for $\alpha > 2$, but the case $\alpha \in [1, 2]$ seems real hard. And even between the full conjecture and the best sufficient condition available, cited above, there is a logarithmic gap, in which it is unclear what to believe.

We underline once more that the counterparts of the present Theorems 2–5 all remain open questions for the single-sequence model in [6].

4. Proof of Theorems 4 and 5

Besides (2) and Theorem 3, the proof of Theorem 4 requires two more lemmas.

Lemma 5. *Suppose that* $2 < \alpha \leq 3$ *and* $\mathbb{E}(X) = 0$, *and let* $\{\mathcal{H}_n\}_{n=1}^\infty$ *be any sequence of random sets of nonnegative integers that is independent of the basic sequence* $\{X_k\}_{k=0}^\infty$ *such that* $|\mathcal{H}_n| \leq \lceil K n^{3-\alpha} \rceil$ *almost surely for all* $n \in \mathbb{N}$, *where*

$K > 0$ is an arbitrary constant. If the number δ is such that $2(1+\delta) > 3-\alpha$ and $\mathbb{E}\big(|X|^{(4-\alpha)/(1+\delta)}\big) < \infty$, then

$$\frac{1}{n^{1+\delta}} \sum_{j\in\mathcal{H}_n} X_j \xrightarrow{\text{EHR}} 0.$$

Proof. Fix any $K \geq 1$. For $\alpha = 3$, when $|\mathcal{H}_n| \leq \lceil K \rceil$ almost surely and the conditions are $1+\delta > 0$ and $\mathbb{E}\big(|X|^{1/(1+\delta)}\big) < \infty$, the statement is trivial since it reduces to $\sum_{n=1}^{\infty} \mathbb{P}\big\{|X|^{1/(1+\delta)} \geq \varepsilon n\big\} < \infty$ for all $\varepsilon > 0$, a well-known characterization of the finiteness of $\mathbb{E}\big(|X|^{1/(1+\delta)}\big)$.

Assume $2 < \alpha < 3$ hereafter. Applying the theorem of Baum and Katz, cited above, with $p = (4-\alpha)/(1+\delta)$ and $\gamma = (1+\delta)/(3-\alpha)$, we obtain

$$\sum_{m=1}^{\infty} \mathbb{P}\Big\{\big|S_m\big| \geq c\, m^{\frac{1+\delta}{3-\alpha}}\Big\} m^{\frac{1}{3-\alpha}-1} < \infty \quad \text{for any } c > 0. \tag{10}$$

Keeping only the terms $m = \lceil K n^{3-\alpha} \rceil$, $n \in \mathbb{N}$, simple manipulation gives

$$\sum_{n=1}^{\infty} \mathbb{P}\Big\{\big|S_{\lceil Kn^{3-\alpha}\rceil}\big| \geq \varepsilon\, n^{1+\delta}\Big\} \leq \sum_{n=1}^{\infty} \mathbb{P}\Big\{\big|S_{\lceil Kn^{3-\alpha}\rceil}\big| \geq \varepsilon\, n^{1+\delta}\Big\} n^{\alpha-2} < \infty$$

for every $\varepsilon > 0$. Thus, assuming without loss of generality that the underlying probability space is rich enough to carry an independent copy $\{X_j'\}_{j=0}^{\infty}$ of the sequence $\{X_j\}_{j=0}^{\infty}$ and setting $\tilde{X}_j := X_j - X_j'$, $j = 0, 1, 2, \ldots$, with $S_n' := \sum_{j=0}^{n-1} X_j'$ and $\tilde{S}_n := \sum_{j=0}^{n-1} \tilde{X}_j = S_n - S_n'$, $n \in \mathbb{N}$, it follows that

$$\frac{S_{\lceil Kn^{3-\alpha}\rceil}}{n^{1+\delta}} \xrightarrow{\text{EHR}} 0 \quad \text{and} \quad \frac{S'_{\lceil Kn^{3-\alpha}\rceil}}{n^{1+\delta}} \xrightarrow{\text{EHR}} 0 \quad \text{and hence} \quad \frac{\tilde{S}_{\lceil Kn^{3-\alpha}\rceil}}{n^{1+\delta}} \xrightarrow{\text{EHR}} 0,$$

where the last implication comes from the inequality $\mathbb{P}\big\{\big|\tilde{S}_{\lceil Kn^{3-\alpha}\rceil}\big| > 2\varepsilon n^{1+\delta}\big\} \leq 2\mathbb{P}\big\{\big|S_{\lceil Kn^{3-\alpha}\rceil}\big| > \varepsilon n^{1+\delta}\big\}$, holding for all $\varepsilon > 0$ and $n \in \mathbb{N}$. The common distribution of $\tilde{X}_0, \tilde{X}_1, \ldots$ being symmetric about zero, by Lévy's inequality (Corollary 5 on p. 72 in [5]) we have

$$\mathbb{P}\Big\{\max_{1\leq k\leq\lceil Kn^{3-\alpha}\rceil} \big|\tilde{S}_k\big| > \varepsilon n^{1+\delta}\Big\} \leq 2\mathbb{P}\Big\{\big|\tilde{S}_{\lceil Kn^{3-\alpha}\rceil}\big| > \varepsilon n^{1+\delta}\Big\}.$$

Setting now $m_{k,n} := \inf\{x \in \mathbb{R} \colon \mathbb{P}\{S_k \leq x n^{1+\delta}\} \geq 1/2\}$ for the common median of $S_k/n^{1+\delta}$ and $S_k'/n^{1+\delta}$, as in the proof of Lemma 3.2 in [6] we also have

$$\mathbb{P}\bigg\{\bigg|\frac{S_k}{n^{1+\delta}} - m_{k,n}\bigg| > \varepsilon\bigg\} \leq 2\mathbb{P}\bigg\{\frac{|\tilde{S}_k|}{n^{1+\delta}} > \varepsilon\bigg\}, \quad k = 1, 2, \ldots, \lceil Kn^{3-\alpha}\rceil,$$

for any $\varepsilon > 0$ and $n \in \mathbb{N}$. Hence

$$M_n(\varepsilon) := \max_{k=1,2,\dots,\lceil Kn^{3-\alpha}\rceil} \mathbb{P}\Big\{|S_k| > \varepsilon n^{1+\delta}\Big\} \le 4\mathbb{P}\Big\{\big|\tilde{S}_{\lceil Kn^{3-\alpha}\rceil}\big| > \varepsilon n^{1+\delta}/2\Big\}$$

for all n for which $m_n := \max\{|m_{k,n}| : k = 1,2,\dots,\lceil Kn^{3-\alpha}\rceil\} < \varepsilon/2$. This will follow for all sufficiently large n if we show that $m_n \to 0$, which by the independence of $|\mathcal{H}_n|$ and $\{X_j\}_{j=0}^{\infty}$ will then imply that $\sum_{n=1}^{\infty} \mathbb{P}\{|\sum_{j\in\mathcal{H}_n} X_j| > \varepsilon n^{1+\delta}\} \le \sum_{n=1}^{\infty} M_n(\varepsilon) < \infty$, and hence the lemma.

Clearly, $m_n \to 0$ follows in turn if we show that $M_n(\varepsilon) \to 0$ for every $\varepsilon > 0$. Depending on ε and all the parameters, one can find a constant $c_\varepsilon > 0$ such that

$$\begin{aligned}
M_n(\varepsilon) &\le \max_{1\le m\le\lceil Kn^{3-\alpha}\rceil} \mathbb{P}\Big\{|S_m| \ge c_\varepsilon \lceil Kn^{3-\alpha}\rceil^{\frac{1+\delta}{3-\alpha}}\Big\} \\
&\le \max\bigg[\max_{1\le m\le m_0} \mathbb{P}\Big\{|S_m| \ge c_\varepsilon \lceil Kn^{3-\alpha}\rceil^{\frac{1+\delta}{3-\alpha}}\Big\}, \sup_{m>m_0} \mathbb{P}\Big\{|S_m| \ge c_\varepsilon m^{\frac{1+\delta}{3-\alpha}}\Big\}\bigg] \\
&\le \max\bigg[\max_{1\le m\le m_0} \mathbb{P}\Big\{|S_m| \ge c_\varepsilon \lceil Kn^{3-\alpha}\rceil^{\frac{1+\delta}{3-\alpha}}\Big\}, \eta\bigg] \\
&\le \eta
\end{aligned}$$

for all n large enough, where, given any $\eta > 0$, the threshold $m_0 = m_0(\varepsilon,\eta) \in \mathbb{N}$ is chosen on account of (10) for the third inequality to be satisfied. ∎

Lemma 6. *Suppose that $\alpha > 2$ and $\mathbb{E}(X) = 0$. If*

$$\text{(11)} \qquad \sum_{n=1}^{\infty} \mathbb{P}\Bigg\{\frac{1}{n}\Bigg|\sum_{j=1}^{n-\lceil Kn^{3-\alpha}\rceil} X_j\Bigg| \ge \varepsilon\big(1+n^{\delta}\big)\Bigg\} < \infty \quad \textit{for every } \varepsilon > 0$$

for some constants $K > 0$ and $\delta > -1/2$, i.e. for which $2(1+\delta) > 1$, then $\mathbb{E}\big(|X|^{\min[2,2/(1+\delta)]}\big) < \infty$.

Proof. For $\delta > 0$ a direct application of the theorem of Baum and Katz cited above yields $\mathbb{E}\big(|X|^{2/(1+\delta)}\big) < \infty$, while for $\delta \le 0$ the statement follows already from the original Erdős–Hsu–Robbins theorem. ∎

Proof of Theorem 4. For $\alpha > 3$ the equivalence statement is Theorem 3.1 in [6], so we only have to consider the case $2 < \alpha \le 3$. Assuming this from now on, it is still a special case of Theorem 3.2 in [6] that $\mathbb{E}(X^2) < \infty$ is sufficient for the almost sure convergence in question.

To prove necessity, as in [6] we assume without loss of generality that $\mathbb{E}(X) = 0$, and brake up the rarefied average as follows:

$$a_n^{(1)}(\alpha) + a_n^{(2)}(\alpha) = \frac{1}{n}\sum_{j\in\mathcal{I}_n(\alpha)} X_j = \frac{1}{n}\sum_{j=1}^{n} X_{\lfloor\lfloor n^\alpha\rfloor U_{n,j}\rfloor} \xrightarrow{\text{a.s.}} 0,$$

where

$$a_n^{(1)}(\alpha) = \sum_{j \in \mathcal{I}_n(\alpha) \setminus S[\mathcal{I}_n(\alpha)]} X_j \quad \text{and} \quad a_n^{(2)}(\alpha) = \sum_{j \in S[\mathcal{I}_n(\alpha)]} X_j.$$

Since $|\mathcal{I}_n(\alpha) \setminus S[\mathcal{I}_n(\alpha)]| = n - |S[\mathcal{I}_n(\alpha)]|$ for all $n \in \mathbb{N}$, for the first term we have

$$\begin{aligned}\mathbb{P}\big\{\big|a_n^{(1)}(\alpha)\big| > \varepsilon\big\} &\le \mathbb{P}\big\{\big|a_n^{(1)}(\alpha)\big| > \varepsilon,\, \big|\mathcal{I}_n(\alpha) \setminus S[\mathcal{I}_n(\alpha)]\big| \le m_\alpha\big\} \\ &\quad + \mathbb{P}\big\{\big|\mathcal{I}_n(\alpha) - S[\mathcal{I}_n(\alpha)]\big| > m_\alpha\big\} \\ &\le m_\alpha \mathbb{P}\big\{|X| > n\varepsilon/m_\alpha\big\} + \mathbb{P}\big\{|S[\mathcal{I}_n(\alpha)]| < n - m_\alpha\big\},\end{aligned}$$

where m_α is the integer figuring in (2), which, using also our underlying assumption that $\mathbb{E}(|X|) < \infty$, is summable in n for any $\varepsilon > 0$. This means that $a_n^{(1)}(\alpha) \xrightarrow{\text{EHR}} 0$. Thus $a_n^{(2)}(\alpha) \xrightarrow{\text{a.s.}} 0$.

Letting $R_n(\alpha) := S[\mathcal{I}_n(\alpha)] \setminus \cup_{j=n+1}^{\infty} S[\mathcal{I}_j(\alpha)]$, by (2) and Theorem 3 there exists a positive constant $K = K(\alpha)$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}\big\{|R_n(\alpha)| \le n - \lceil K n^{3-\alpha} \rceil\big\} < \infty. \tag{12}$$

Hence we work with this basic set (modified, in comparison with that in the proof of Theorem 3.1 in [6]) and, as in that proof, assume without loss of generality that the underlying probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is rich enough to carry the sequence $\{X_z\}_{z \in \mathbb{Z}}$ of independent and identically distributed random variables, an extended version of the given sequence $X_0, X_1, \dots$, and here redefine the integer-valued random variables $T_n(j)$, $j = 1, \dots, n$, and the random multisets $\mathcal{H}_n(\alpha)$, $n \in \mathbb{N}$, as follows: if $|R_n(\alpha)| \le n - \lceil K n^{3-\alpha} \rceil$, then let $T_n(j) := -(n^2 + j)$, $j = 1, \dots, n$, and $\mathcal{H}_n(\alpha) := \{1, 2, \dots, \lceil K n^{3-\alpha} \rceil\}$, $n \in \mathbb{N}$; while if $|R_n^\alpha| > n - \lceil K n^{3-\alpha} \rceil$, then the variables $T_n(1), T_n(2), \dots, T_n(n - \lceil K n^{3-\alpha} \rceil)$ are chosen to be the $n - \lceil K n^{3-\alpha} \rceil$ smallest integers in $R_n(\alpha)$, in the given order, and the remaining random variables $T_n(n - \lceil K n^{3-\alpha} \rceil + 1), T_n(n - \lceil K n^{3-\alpha} \rceil + 2), \dots, T_n(n)$ are chosen arbitrarily from the multiset $\mathcal{H}_n(\alpha) := \mathcal{I}_n(\alpha) \setminus \{T_n(1), T_n(2), \dots, T_n(n - \lceil K n^{3-\alpha} \rceil)\}$. Clearly, any two elements of the triangular array $\{T_n(1), T_n(2), \dots, T_n(n - \lceil K n^{3-\alpha} \rceil)\}_{n=\lceil K n^{3-\alpha} \rceil + 1}^{\infty}$ are different and the whole array is independent of the sequence $\{X_z\}_{z \in \mathbb{Z}}$, and hence the elements of the triangular array $\{X_{T_n(1)}, X_{T_n(2)}, \dots, X_{T_n(n - \lceil K n^{3-\alpha} \rceil)}\}_{n=\lceil K n^{3-\alpha} \rceil + 1}^{\infty}$ are independent and all are distributed as X. For all $n \in \mathbb{N}$ introduce also the events $A_n(\alpha) := \{|S[\mathcal{I}_n(\alpha)]| > n - m_\alpha\} \cap \{|R_n^\alpha| > n - \lceil K n^{3-\alpha} \rceil\}$ and the averages

$$b_n(\alpha) := \frac{1}{n} \sum_{j=1}^{n - \lceil K n^{3-\alpha} \rceil} X_{T_n(j)} \quad \text{and} \quad b_n^{(\delta)}(\alpha) := \frac{1}{n^{1+\delta}} \sum_{j \in \mathcal{H}_n(\alpha)} X_j.$$

Since $a_n^{(2)}(\alpha) \xrightarrow{\text{a.s.}} 0$, for every $\varepsilon > 0$ and $\delta \in \mathbb{R}$ we have

$$\begin{aligned}
0 &= \mathbb{P}\Big\{ \limsup_{n\to\infty} \big|a_n^{(2)}(\alpha)\big| > \varepsilon \Big\} = \mathbb{P}\Big\{ \big|a_n^{(2)}(\alpha)\big| > \varepsilon \quad \text{i.o.} \Big\} \\
&\geq \mathbb{P}\Big\{ \big\{ \big|a_n^{(2)}(\alpha)\big| > \varepsilon \big\} \cap A_n(\alpha) \quad \text{i.o.} \Big\} \qquad (13) \\
&\geq \mathbb{P}\Big\{ \big\{ \big|b_n(\alpha)\big| > \varepsilon\big(1 + n^{\delta}\big) \big\} \cap \big\{ \big|b_n^{(\delta)}(\alpha)\big| \leq \varepsilon \big\} \cap A_n(\alpha) \quad \text{i.o.} \Big\}
\end{aligned}$$

in Chung's abbreviation for infinitely often, that is, for the respective events holding for infinitely many $n \in \mathbb{N}$.

Now, take $\delta_0 = 3 - \alpha$. Then $1 + \delta_0 = 4 - \alpha$, and so $\mathbb{E}\big(|X|^{(4-\alpha)/(1+\delta_0)}\big) = \mathbb{E}(|X|) < \infty$, and $2(1 + \delta_0) > 3 - \alpha$ by the assumption that $2 < \alpha \leq 3$. Then, by (2), (12) and Lemma 5,

$$b_n^{(\delta_0)}(\alpha) \xrightarrow{\text{EHR}} 0, \quad \text{and hence} \quad \frac{b_n(\alpha)}{1 + n^{\delta_0}} \xrightarrow{\text{a.s.}} 0 \quad \text{by (13), and so} \quad \frac{b_n(\alpha)}{1 + n^{\delta_0}} \xrightarrow{\text{EHR}} 0.$$

The last implication, the crux of the argument, is by the independence of the averages $b_1(\alpha), b_2(\alpha), \ldots$, through the second Borel–Cantelli lemma. But, in view of the properties of the array $\{X_{T_n(1)}, X_{T_n(2)}, \ldots, X_{T_n(n-\lceil Kn^{3-\alpha}\rceil)}\}_{n=\lceil Kn^{3-\alpha}\rceil+1}^{\infty}$ listed above, this conclusion may be rewritten as (11), holding with $\delta = \delta_0$, and hence by Lemma 6 it follows that $\mathbb{E}\big(|X|^{\min[2,2/(1+\delta_0)]}\big) < \infty$. In particular, this completes the proof for $\alpha = 3$, for then $\delta_0 = 0$.

Suppose henceforth $2 < \alpha < 3$. Starting with the present $\delta_0 = 3 - \alpha$ for which $\mathbb{E}\big(|X|^{\min[2,2/(1+\delta_0)]}\big) < \infty$, define the sequence $\delta_1, \delta_2, \ldots$ inductively by the following equations:

$$\frac{1 + \delta_0}{1 + \delta_{k+1}} = \frac{4 - \alpha}{1 + \delta_{k+1}} = \min\left[2\,, \frac{2}{1 + \delta_k}\right], \qquad k = 0, 1, 2, \ldots\,.$$

Since $2(1+\delta_{k+1}) \geq 4 - \alpha > 1$ for all $k = 0, 1, 2, \ldots$, both Lemma 5 and Lemma 6 are applicable for δ_k if the moment condition $\mathbb{E}\big(|X|^{(4-\alpha)/(1+\delta_k)}\big) < \infty$ for the former and the convergence condition (11) with $\delta = \delta_k$ for the latter are satisfied, $k \in \mathbb{N}$.

Beginning with the moment condition $\mathbb{E}\big(|X|^{(4-\alpha)/(1+\delta_1)}\big) < \infty$, achieved in the preliminary round above, and repeating the basic argument following (13) for δ_1, we obtain $\mathbb{E}\big(|X|^{\min[2,2/(1+\delta_1)]}\big) = \mathbb{E}\big(|X|^{(4-\alpha)/(1+\delta_2)}\big) < \infty$. Using then this condition as input, the procedure can be continued and thus iterative applications of Lemmas 5 and 6 in the basic argument "bootstrap" us to see that $\mathbb{E}\big(|X|^{\min[2,2/(1+\delta_k)]}\big) < \infty$ for all $k \in \mathbb{N}$.

Finally, we claim that there exists an $r \in \mathbb{N}$ such that $2(1 + \delta_r) \geq 2$. Indeed, if $2(1 + \delta_k) < 2$ for all $k \in \mathbb{N}$ to the contrary, then, since $\alpha > 2$,

$$\frac{1 + \delta_0}{1 + \delta_k} = \left(\frac{2}{4 - \alpha}\right)^k \to \infty \quad \text{as } k \to \infty,$$

which is a contradiction since the left side is not greater than 2 for any $k \in \mathbb{N}$.

It follows, then, that $(4-\alpha)/(1+\delta_k) = 2$, that is, $\delta_k = 1-(\alpha/2) =: \delta_*(\alpha)$ for all $k > r$. Since $2/[1+\delta_*(\alpha)] = 4/(4-\alpha) > 2$, again because $\alpha > 2$, we conclude that $\mathbb{E}(X^2) < \infty$. ■

Besides Theorem 4, we need another two lemmas for the proof of Theorem 5.

Lemma 7. *Let $K > 0$ be any fixed integer and, for any $\alpha > 2$, let $\beta, \delta \in \mathbb{R}$ be such that $\delta + \frac{\beta}{\alpha} > 0$. If $Y = Y_1, Y_2, \ldots$ are identically distributed random variables with mean zero and $\mathbb{E}\big(|Y|^{1/(\delta+\frac{\beta}{\alpha})}\big) < \infty$, then*

$$\frac{1}{n^{\delta+\frac{\beta}{\alpha}}} \sum_{j=1}^{K} Y_j \xrightarrow{\text{EHR}} 0.$$

Proof. This is trivial since $|Y|/n^{\delta+\frac{\beta}{\alpha}} \xrightarrow{\text{EHR}} 0$. ■

Lemma 8. *Let $\alpha > 2$ and consider the array $\{X_{n,1}, X_{n,2}, \ldots, X_{n,\lfloor n^{1/\alpha}\rfloor}\}_{n=1}^{\infty}$ of independent, identically distributed random variables with mean zero. If*

$$\sum_{l=1}^{\infty} \mathbb{P}\left\{ \max_{\lfloor l^\beta \rfloor < n \le \lfloor l^\beta \rfloor + a(l)} \left| \sum_{j=1}^{\lfloor n^{1/\alpha}\rfloor} X_{n,j} \right| > (1+l^\delta) l^{\beta/\alpha} \right\} < \infty$$

for some $\beta \in (2,\alpha)$ and $\delta \in \mathbb{R}$, where $a(l) = \lfloor l^{1-\frac{\beta}{\alpha}}/2 \rfloor$, then $\mathbb{E}\big(|X|^{2/(\delta^+ + \frac{\beta}{\alpha})}\big) < \infty$ for the positive part $\delta^+ = \max(\delta, 0)$.

Proof. In our basic notation $S_n = X_0 + X_1 + \cdots + X_{n-1}$, $n \in \mathbb{N}$, the condition clearly implies

$$\sum_{l=1}^{\infty} \left[1 - \prod_{n=\lfloor l^\beta \rfloor+1}^{\lfloor l^\beta \rfloor + a(l)} \mathbb{P}\Big\{ \big| S_{\lfloor n^{1/\alpha}\rfloor} \big| \le 2 l^{\delta^+ + \frac{\beta}{\alpha}} \Big\} \right] < \infty.$$

Since for $\lfloor l^\beta \rfloor + 1 \le n \le \lfloor l^\beta \rfloor + a(l)$ we have

$$l^{\beta/\alpha} < n^{1/\alpha} \le l^{\beta/\alpha} \left[1 + \frac{l^{1-\frac{\beta}{\alpha}-\beta}}{2} \right]^{1/\alpha} < l^{\beta/\alpha} + \frac{l^{1-\beta}}{2}, \tag{14}$$

the latter form of the condition may be simplified as

$$\sum_{l=1}^{\infty} \Big[1 - (1-p_l)^{a(l)} \Big] = \sum_{l=1}^{\infty} \left[1 - \Big(\mathbb{P}\Big\{ \big| S_{\lfloor l^{\beta/\alpha} \rfloor} \big| \le 2 l^{\delta^+ + \frac{\beta}{\alpha}} \Big\} \Big)^{a(l)} \right] < \infty,$$

where $p_l = \mathbb{P}\{|S_{\lfloor l^{\beta/\alpha}\rfloor}| > 2l^{\delta^+ + \frac{\beta}{\alpha}}\}$. This forces $(1-p_l)^{a(l)} \to 1$ or $a(l)p_l \to 0$, and consequently $\{1-(1-p_l)^{a(l)}\}/\{a(l)p_l\} \to 1$ as $l \to \infty$, yielding $\sum_{l=1}^{\infty} a(l)p_l < \infty$, and hence

$$\sum_{l=1}^{\infty} l^{1-\frac{\beta}{\alpha}} \mathbb{P}\Big\{\big|S_{\lfloor l^{\beta/\alpha}\rfloor}\big| > 2l^{\delta^+ + \frac{\beta}{\alpha}}\Big\} < \infty.$$

Since $\beta/\alpha < 1$, probabilities containing S_n occur for all $n \in \mathbb{N}$ in this series, in fact the number of times this term appears is asymptotically $(n+1)^{\alpha/\beta} - n^{\alpha/\beta} \sim \alpha n^{\frac{\alpha}{\beta}-1}/\beta$. Combined with other easy considerations, this implies

$$\sum_{n=1}^{\infty} n^{2\frac{\alpha}{\beta}-2} \mathbb{P}\Big\{|S_n| > 2n^{1+\delta^+\frac{\alpha}{\beta}}\Big\} = \sum_{n=1}^{\infty} n^{\frac{2}{\delta^+ + \frac{\beta}{\alpha}}\left(1+\delta^+\frac{\alpha}{\beta}\right)-2} \mathbb{P}\Big\{|S_n| > 2n^{1+\delta^+\frac{\alpha}{\beta}}\Big\} < \infty.$$

Hence, using the theorem of Baum and Katz for the third time in this section, the lemma follows. ■

Proof of Theorem 5. If $2 < \alpha \le 3$, then we simply take the subsequence $n_k = \lfloor k^\alpha \rfloor$, $k \in \mathbb{N}$, in (9) and conclude by Theorem 4 that $\mathbb{E}(X^2) < \infty$.

Suppose from now on that $\alpha > 3$ and, without loss of generality, also that $\mathbb{E}(X) = 0$. Let $3 < \beta < \alpha$, introduce again $a(l) = \lfloor l^{1-\frac{\beta}{\alpha}}/2 \rfloor$ from Lemma 8, and consider the multisets

$$\mathcal{J}_l(\beta) = \big\{\lfloor nU_{n,j}\rfloor \colon\ 1 \le j \le \lfloor n^{1/\alpha}\rfloor, \lfloor l^\beta \rfloor < n \le \lfloor l^\beta\rfloor + a(l)\big\}, \quad l \in \mathbb{N}.$$

(Note that, intuitively speaking, the multiset $\mathcal{J}_l(\beta)$ is very similar to the multiset $\mathcal{M}_l(\beta) = \{\lfloor \lfloor l^\beta \rfloor U_{\lfloor l^\beta \rfloor, j}\rfloor \colon\ 1 \le j \le |M(l)|\}$, where $|M(l)|$ is the number of elements in the set $M(l) = \{(n,j) \colon\ 1 \le j \le \lfloor n^{1/\alpha}\rfloor, \lfloor l^\beta\rfloor < n \le \lfloor l^\beta \rfloor + a(l)\}$, so that $|M(l)| \sim l/2$ as $l \to \infty$.) In the rest of the proof, all asymptotic relations are meant as $l \to \infty$.

Given any integer $l > 1$, for any integers n, m, i, j for which $\lfloor l^\beta \rfloor < n, m \le \lfloor l^\beta \rfloor + a(l)$ and $1 \le i \le \lfloor n^{1/\alpha}\rfloor$, $1 \le j \le \lfloor m^{1/\alpha}\rfloor$, but $(n,i) \ne (m,j)$, we have

$$\mathbb{P}\big\{\lfloor nU_{n,i}\rfloor = \lfloor mU_{m,j}\rfloor\big\} = \sum_{k=0}^{m-1} \mathbb{P}\Big\{\frac{k}{n} \le U_{n,i} < \frac{k+1}{n}\Big\} \mathbb{P}\big\{\lfloor mU_{m,j}\rfloor = k\big\} = \frac{1}{n} \le \frac{1}{l^\beta}$$

if $m \le n$, so summing for the same n, m, i, j and using (14),

$$\mathbb{P}\big\{\mathcal{J}_l(\beta) \ne S[\mathcal{J}_l(\beta)]\big\} \le \sum_{n,m,i,j} \mathbb{P}\big\{\lfloor nU_{n,i}\rfloor = \lfloor mU_{m,j}\rfloor\big\} \le \frac{\big[a(l)2l^{\beta/\alpha}\big]^2}{l^\beta}.$$

Therefore, for all $l \in \mathbb{N}$,

$$\mathbb{P}\big\{\mathcal{J}_l(\beta) \ne S[\mathcal{J}_l(\beta)]\big\} \le l^{2-\beta}, \quad \text{and so} \quad \sum_{l=1}^{\infty} \mathbb{P}\big\{\mathcal{J}_l(\beta) \ne S[\mathcal{J}_l(\beta)]\big\} < \infty.$$

Using this, the proof of Theorem 2.1 in [6] gives a finite constant $K_\beta > 0$ for which

$$\sum_{l=1}^{\infty} \mathbb{P}\Big\{ \big| \mathcal{J}_l(\beta) \cap \big(\cup_{i=l+1}^{\infty} \mathcal{J}_i(\beta) \big) \big| \geq K_\beta \Big\} < \infty.$$

Hence, introducing the events

$$D_l(\beta) := \big\{ \big| \mathcal{J}_l(\beta) \cap \big(\cup_{i=l+1}^{\infty} \mathcal{J}_i(\beta) \big) \big| < K_\beta \big\} \cap \big\{ \mathcal{J}_l(\beta) = S[\mathcal{J}_l(\beta)] \big\}, \quad l \in \mathbb{N},$$

and combining the last two conclusions, we have

$$\sum_{l=1}^{\infty} \big[1 - \mathbb{P}\{D_l(\beta)\} \big] < \infty. \tag{15}$$

Furthermore, $a(l) \sim l^{1-\frac{\beta}{\alpha}}/2$ and $(l+1)^\beta - l^\beta \sim \beta l^{\beta-1}$, and so $a(l)/[(l+1)^\beta - l^\beta] \to 0$ whenever $\beta > 2$. Hence the choice $3 < \beta < \alpha$ and (9) imply

$$M_l(\alpha, \beta) := \frac{1}{l^{\beta/\alpha}} \max_{\lfloor l^\beta \rfloor < n \leq \lfloor l^\beta \rfloor + a(l)} \left| \sum_{j=1}^{\lfloor n^{1/\alpha} \rfloor} X_{\lfloor n U_{n,j} \rfloor} \right| \xrightarrow{\text{a.s.}} 0. \tag{16}$$

Now, if the event $D_l(\beta)$ occurs for some $l \in \mathbb{N}$, then we can find $a(l)$ disjoint sets $E_n \subset \{1, 2, \ldots, \lfloor n^{1/\alpha} \rfloor\}$, $\lfloor l^\beta \rfloor < n \leq \lfloor l^\beta \rfloor + a(l)$ such that for $F_n = \{1, 2, \ldots, \lfloor n^{1/\alpha} \rfloor\} - E_n$, we have

$$\sum_{n=\lfloor l^\beta \rfloor + 1}^{\lfloor l^\beta \rfloor + a(l)} |F_n| < K_\beta \tag{17}$$

and for any $\varepsilon > 0$ and $\delta \in \mathbb{R}$,

$$\begin{aligned} \{M_l(\alpha,\beta) > \varepsilon\} \supset & \left\{ \frac{1}{l^{\beta/\alpha}} \max_{\lfloor l^\beta \rfloor < n \leq \lfloor l^\beta \rfloor + a(l)} \left| \sum_{j \in E_n} X_{\lfloor n U_{n,j} \rfloor} \right| > \big(1 + l^\delta\big)\varepsilon \right\} \\ & \bigcap \left\{ \frac{1}{l^{\beta/\alpha}} \max_{\lfloor l^\beta \rfloor < n \leq \lfloor l^\beta \rfloor + a(l)} \left| \sum_{j \in F_n} X_{\lfloor n U_{n,j} \rfloor} \right| \leq l^\delta \varepsilon \right\}. \end{aligned} \tag{18}$$

Thus, if $D_l(\beta)$ occurs, $\cup_{n=\lfloor l^\beta \rfloor+1}^{\lfloor l^\beta \rfloor + a(l)} F_n$ consists of those of the $\sum_{n=\lfloor l^\beta \rfloor+1}^{\lfloor l^\beta \rfloor + a(l)} \lfloor n^{1/\alpha} \rfloor$ different elements of $\mathcal{J}_l(\beta)$ which are also in $\cup_{i=l+1}^{\infty} \mathcal{J}_i(\beta)$. But, almost surely, all but a finite number of $D_1(\beta), D_2(\beta), \ldots$ occur, as follows from (15) and the first Borel–Cantelli lemma.

Setting $U_{n,-j} := -(n^2 + j)/n$, $j = 1, 2, \ldots, \lfloor n^{1/\alpha} \rfloor$, $n \in \mathbb{N}$, in the irrelevant case when $D_l(\beta)$ does not occur for some $l \in \mathbb{N}$, we define $E_n := \{-1, -2, \ldots, -\lfloor n^{1/\alpha} \rfloor\}$

and $F_n := \emptyset$ for $\lfloor l^\beta \rfloor + 1 \le n \le \lfloor l^\beta \rfloor + a(l)$. Also, to cope with this case similarly as in the proof of Theorem 4, we assume without loss of generality that the basic underlying sequence $X_0, X_1, \ldots$ is extended to a two-sided sequence $\{X_z\}_{z\in\mathbb{Z}}$ of independent and identically distributed random variables with mean zero.

Let $\delta_0 = 1 - \frac{\beta}{\alpha}$ to begin with, so that $\delta_0 + \frac{\beta}{\alpha} = 1$. Then by (17) and Lemma 7,

$$\frac{1}{l^{\delta_0+\frac{\beta}{\alpha}}} \max_{\lfloor l^\beta \rfloor < n \le \lfloor l^\beta \rfloor + a(l)} \Bigl| \sum_{j\in F_n} X_{\lfloor nU_{n,j} \rfloor} \Bigr| \xrightarrow{\text{EHR}} 0,$$

which by (16) and (18) implies that

$$M_l(\delta_0,\alpha,\beta) := \frac{1}{1+l^{\delta_0}} \frac{1}{l^{\beta/\alpha}} \max_{\lfloor l^\beta \rfloor < n \le \lfloor l^\beta \rfloor + a(l)} \Bigl| \sum_{j\in E_n} X_{\lfloor nU_{n,j} \rfloor} \Bigr| \xrightarrow{\text{a.s.}} 0.$$

By the construction of the sets $\bigl\{E_n\colon \lfloor l^\beta \rfloor + 1 \le n \le \lfloor l^\beta \rfloor + a(l)\bigr\}_{l=1}^{\infty}$, all the sums here are independent, and hence the almost sure convergence here again can be upgraded to complete convergence by the second Borel–Cantelli lemma, as in the proof of Theorem 4, that is, $M_l(\delta_0,\alpha,\beta) \xrightarrow{\text{EHR}} 0$. But then, again similarly as in the proof of Theorem 4, the construction of the sets $\bigl\{E_n\colon \lfloor l^\beta \rfloor + 1 \le n \le \lfloor l^\beta \rfloor + a(l)\bigr\}_{l=1}^{\infty}$ also allows to conclude that the conditions of Lemma 8 are satisfied with $\alpha > 0$ and $\delta = \delta_0$, from which fact we infer that $\mathbb{E}\bigl(|X|^{2/(\delta_0^+ + \frac{\beta}{\alpha})}\bigr) < \infty$. Of course, literally this is nothing but $\mathbb{E}(X^2) < \infty$, again, but the notation is to suggest that this is just an initial step, similarly as in the proof of Theorem 4.

Indeed, define iteratively δ_k by the equation $\delta_k + \frac{\beta}{\alpha} = \bigl(\delta_{k-1}^+ + \frac{\beta}{\alpha}\bigr)/2$, $k \in \mathbb{N}$. Then, by the alternative applications of Lemmas 7 and 8, facilitated by the argument above, we see that $\mathbb{E}\bigl(|X|^{2/(\delta_k^+ + \frac{\beta}{\alpha})}\bigr) < \infty$ for all $k \in \mathbb{N}$. But, necessarily $\delta_k < 0$ for all k large enough, because otherwise $\frac{\beta}{\alpha} \le \delta_k + \frac{\beta}{\alpha} = \bigl(\delta_0 + \frac{\beta}{\alpha}\bigr)/2^k = 1/2^k \to 0$ as $k \to \infty$, an obvious contradiction. Hence $\mathbb{E}\bigl(|X|^{2\alpha/\beta}\bigr) = \mathbb{E}\bigl(|X|^{2/(\beta/\alpha)}\bigr) < \infty$, which implies the theorem for $\alpha > 3$ since $\beta \in (3,\alpha)$ may be chosen as close to 3 as desired. ■

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S. Csörgő, Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6726 Szeged, Hungary; *e-mail:* csorgo@math.u-szeged.hu

B. Valkó, Mathematical Institute, Technical University of Budapest, Műegyetem rakpart 9, 1111 Budapest, Hungary; *e-mail:* valko@math.bme.hu

W. B. Wu, Department of Statistics, University of Michigan, 4062 Frieze Building, Ann Arbor, MI 48109–1285, USA; *e-mail:* wbwu@umich.edu