Probability Theory

Problem set #10

Continuous random variables: Expectation, normal random variables

Homework problems to be handed in: 10.4, 10.7, 10.12, 10.13, 10.14

Bonus problems for extra credit: 10.8, 10.11

Due date: May 9

10.1 Let X be uniformly distributed on $[0,1]$. Find the distribution and density functions of the random variables $Y := X^{-1}$ and $Z := X(1+X)^{-1}$.

10.2 Let X be a standard normal random variable. Find the distribution and density functions of the random variable $Y := 2 + |X|$.

10.3 Let X be a $N(m,\sigma)$ normal random variable. Determine the density function of the random variable $Y := e^X$. The distribution of Y is called *log-normal* with parameters (m,σ), it is denoted by $LN(m,\sigma)$.

10.4 Let X be a log-normal random variable with parameters (m,σ) (see the previous problem). Prove that if $C > 0, \alpha$ are fixed real numbers then CX^α is also log-normal and find its parameters.

10.5 The momentum-generating function of a random variable X is defined as $H(t) := \mathbf{E}e^{tX}$ for those values of t where this is finite. Calculate the momentum-generating functions of the $U(0,1)$, $N(m,\sigma)$ and $EXP(\lambda)$ distributions and find the intervals where they are defined.

10.6 Find the expectation and variance of a log-normal random variable with parameters (m,σ).

10.7 In the following exercises the random variable ξ is given (by its distribution or density function). You need to determine the densities of the random variables X, Y which are defined as functions of ξ.
(a) ξ is uniform on $[-1,1]$; $X := \xi^2$, $Y := \tan(\frac{\pi}{2}\xi)$.
(b) ξ is exponential with parameter λ; $X := 3\xi + 2$, $Y := \sqrt{\xi}$.
(c) ξ is standard normal; $X := \xi^{-2}$.

10.8 Let X be a standard Cauchy random variable with density function $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.
Clearly, $\mathbf{E}(|X|) = \infty$, but for every $1 > \varepsilon > 0$ the expectation $\mathbf{E}(|X|^{1-\varepsilon})$ is finite. Prove, that the limit $\lim_{\varepsilon\to 0}\varepsilon\mathbf{E}(|X|^{1-\varepsilon})$ exists and find its value.

10.9 Using the fact, that $\int_{-\infty}^{\infty}\exp(-x^2/2)dx = \sqrt{2\pi}$, calculate the value of the following integral:

$$\int_{-\infty}^{\infty}\exp(-ax^2 + bx + c)dx$$

where a is a positive real and b, c are arbitrary real (or complex) constants.

10.10 Let X be a standard normal random variable. Calculate its absolute moments:

$$A_k := \mathbf{E}\,|X|^k = \int_{-\infty}^{\infty} \varphi(y)|y|^k dy, \qquad k = 1, 2, 3, \dots$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Hint: For even $k = 2l$ evaluate and use the following expression:

$$\frac{d^l}{d\lambda^l} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda y^2/2} dy \bigg|_{\lambda=1}.$$

For odd $k = 2l + 1$ use change of variables with $z = y^2$ in the defining integral of A_k.

10.11 Prove, that the following series expansions hold for every real x.

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{k=1}^{\infty} \frac{(k-1)!2^{k-1}}{(2k-1)!} x^{2k-1};$$

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \left(x - \frac{1}{2 \cdot 1!} \frac{x^3}{3} + \frac{1}{4 \cdot 2!} \frac{x^5}{5} - \frac{1}{8 \cdot 3!} \frac{x^7}{7} + \dots \right).$$

10.12 Let X be an $N(0, \sigma)$ random variable. Prove that for every $x > 0$ the following inequalities hold:

$$\frac{1}{\sqrt{2\pi}} e^{-(x^2/2\sigma^2)} \left(\frac{\sigma}{x} - \frac{\sigma^3}{x^3} \right) \quad < \quad \mathbf{P}(X > x) \quad < \quad \frac{1}{\sqrt{2\pi}} e^{-(x^2/2\sigma^2)} \frac{\sigma}{x}.$$

Remark: Although we do not have an explicit formula for $\mathbf{P}(X > x) = 1 - \Phi(x)$, for $x \gg 1$ this inequality gives a good approximation of its value.

Hint: Differentiate each side of the inequality and compare the result.

10.13 A certain group of people has average weight 60 kg, the standard deviation of the weights (the square root of the variance) is 3 kg. Find the probability that the difference of the weight of a randomly chosen individual and the average weight is less than 5 kg, in the following cases:
(a) the distribution of the weights is normal;
(b) the distribution of the weights is log-normal.
Hint: Try to deduce the probability in question to a probability concerning a standard normal random variable, then use the function Φ to express the result.

10.14 Let X be a standard normal random variable. Determine the following expectations and variances:
(a) $\mathbf{E}(X\cos(X))$, $\mathbf{E}(X/(1 + X^2))$, $\mathbf{E}(\sin(X))$;
(b) $\mathbf{E}(\cos(X))$, $\mathbf{Var}(\cos(X))$, $\mathbf{Var}(\sin(X))$.