Probability Theory

Problem set #9

Continuous random variables: Distribution function, probability density function

Homework problems to be handed in: 9.2, 9.5, 9.6, 9.8, 9.9

Bonus problem for extra credit: 9.10

Due date: April 27

REMINDER:

Distribution function:
If X is a real-valued random variable, then its distribution function is defined as

$$F(x) := \mathbf{P}(X < x).$$

Properties:

0. $F : \mathbb{R} \to [0, 1]$,
1. monotone non-decreasing;
2. left continuous;
3. $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$. Probability density function:

If the distribution function $F(\cdot)$ is equal to $F(x) = \int_{-\infty}^{x} f(y)dy$ for any $x \in \mathbb{R}$ with a 'nice' function $f(\cdot)$, then $F(\cdot)$ *absolutely continuous* and $f : \mathbb{R} \to [0, \infty)$ is the *probability density function* of $F(\cdot)$.
In that case F (almost everywhere) differentiable and (almost everywhere) $F'(x) = f(x)$.
Properties:

1. measurable (i.e. 'nice');
2. non-negative;
3. $\int_{-\infty}^{\infty} f(y)dy = 1$.

9.1 Determine which of the following are probability distribution functions on $\mathbb{R}$:

$$(a) \quad F(x) := \frac{3}{4} + \frac{1}{2\pi}\operatorname{arctg}(x);$$

$$(b) \quad F(x) := \begin{cases} 0, & \text{if} \quad -\infty < x \le 0, \\ [x]/2, & \text{if} \quad 0 < x \le 2, \\ 1, & \text{if} \quad 2 < x < \infty; \end{cases}$$

$$(c) \quad F(x) := \begin{cases} 0, & \text{if} \quad -\infty < x \le 0, \\ x/(1+x), & \text{if} \quad 0 < x < \infty; \end{cases}$$

$$(d) \quad F(x) := \exp(-e^{-x})$$

$$(e) \quad F(x) := \begin{cases} 0, & \text{if} \quad -\infty < x \le 0, \\ 1 - (1 - \exp\{-x\})/x, & \text{if} \quad 0 < x < \infty. \end{cases}$$

9.2 For which values of α and c will the function F be a distribution function?

$$F(x) = \exp(-ce^{-\alpha x}).$$

9.3 Let $F(x)$ be a *continuous* distribution function with $F(0) = 0$. Prove that

$$G(x) := \begin{cases} 0, & \text{if} \quad -\infty < x \le 1, \\ F(x) - F(x^{-1}), & \text{if} \quad 1 < x < \infty. \end{cases}$$

is also a distribution function. How can you interpret this result?

9.4 We break a stick of length 1 at a randomly chosen point. Find the distribution function of the length of the shorter piece obtained.

9.5 Choose three points at random on the interval $[0, 1]$ (independently, with uniform distribution). Determine the distribution function of the coordinate of the central point.

9.6 Determine which of the following are probability density functions on $\mathbb{R}$:

(a) $f(x) := \begin{cases} 1/3 & \text{ha } 0 \le x \le 1, \\ 0 & \text{otherwise;} \end{cases}$ (b) $f(x) := \begin{cases} (\sin x)/2 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise;} \end{cases}$

(c) $f(x) := \begin{cases} x^{-2} & \text{if } 1 \le x, \\ 0 & \text{otherwise;} \end{cases}$ (d) $f(x) := \begin{cases} x/(1+x) & \text{if } 0 \le x, \\ 0 & \text{otherwise;} \end{cases}$

(e) $f(x) := \frac{1}{\pi}\frac{1}{1+x^2}, \quad -\infty < x < \infty$ (f) $f(x) := \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty$

(g) $f(x) := \begin{cases} 4x^3 e^{-x^4} & \text{if } 0 \le x, \\ 0 & \text{otherwise;} \end{cases}$ (h) $f(x) := \begin{cases} -\log x & \text{if } 0 < x \le 1, \\ 0 & \text{otherwise;} \end{cases}$

9.7 Choose a point at random on the interval $[0, 1]$ of the x-axis. Denote by X the distance between this random point and the point in the plane with coordinates $(0; 1)$. Determine the density of the distribution of the random variable X.

9.8 Choose a point at random in the unit square. Denote by X its distance from the nearest side. Determine the density function of the distribution of X.

9.9 Choose a point at random in the unit square. Denote by X its distance from the nearest corner of the square. Determine the density function of the distribution of X.

9.10 Choose n points at random on the interval $[0, 1]$ (independently, with uniform distribution). Determine the distribution function of the coordinate of the k^{th} largest point.