Probability Theory

Problem set #8

Discrete random variables: expectation, variance, covariance

Homework problems to be handed in: 8.1, 8.3, 8.7, 8.9, 8.12

Bonus problem for extra credit: 8.13, 8.14, 8.15

Due date: April 18

8.1 For 60 randomly chosen people, find the expected number of days of the year which are the birthdays of 0, 1, 2, 3, 4 people.

8.2 Suppose we toss a needle of length l (less than 1) on a grid with both vertical and horizontal rulings spaced one unit apart. What is the expected number of lines the needle crosses? (Hint: Use the result in Buffon's needle problem.)

8.3 Find the expected duration of the game of craps. (Craps: we throw two dice. If the sum is 7 or 11 we win, if it is 2, 3, or 12 we lose. If the sum is something else, then we throw until we get the same sum as in the first throw (and we win) or 7 (then we lose).)

8.4 A, B, C, D throw with two dice in succession, the highest throw winning a prize of 100$. If two or more throw equal sums (higher than their competitors) they divide the prize. A having thrown 9, what is the expected value of money he receives?

8.5 A knight is placed at random on an empty chessboard. Find the expected number of its possible moves.

8.6 From a box containing N balls labelled 1 to N, we draw with replacement until all balls have turned up. If X denotes the number of drawings, compute $\mathbf{E}(X)$ and $\mathbf{Var}(X)$.

8.7 We roll a die n times; let X and Y denote the number of ones and sixes, respectively. Compute $\mathbf{Cov}(X, Y)$.

8.8 Let X and Y be random variables assuming only two values each, i.e. $\mathrm{Ran}(X) = \{x_1, x_2\}$, $\mathrm{Ran}(Y) = \{y_1, y_2\}$. Prove that if $E(XY) = E(X)E(Y)$ then X and Y are independent.

8.9 We toss a fair coin three times, getting X heads and Y tails. What is the expected value and the variance of XY?

8.10 A fair die is rolled ten times. Let the random variable X denote the number of times we see an even number followed immediately by an odd number. What is the expected value and variance of X?

8.11 Twelve passengers enter an elevator at the basement and independently choose to exit randomly at one of the ten above-ground floors. What is the expected number of stops that the elevator will have to make?

8.12 (a) If we roll a die six times, what is the expectation of the number of different outcomes? (I.e. how many different numbers we see in average?)
(b) If we roll a die until we have seen four distinct numbers, how many times do we expect to have to roll it?

8.13 Cards are drawn one by one (without replacement) from a pack of 52 cards until at least one card of every suit has been drawn. Find the expected number of cards drawn. Solve the problem also in the case when the drawings are with replacement.

8.14 Cards are drawn one by one (without replacement) from a pack of 52 cards until an ace is drawn. Find the expectation of the total value of the cards drawn. (The value of a number card is the number on it, while the value of face cards is 10.)

8.15 (a) In a Chinese restaurant there are infinitely many round tables. In the beginning the restaurant is empty, as customers arrive, they can either sit at a new table or join someone else's table. If there are k customers in the restaurant then the next person may choose to sit either to the left of a customer already present or at a new table, each with probability $\frac{1}{k+1}$. The seating in the restaurant at a given time corresponds to a permutation of the customers present (the people sitting at a given table correspond to a a cycle of the appropriate permutation). Prove that if there are k persons present then the corresponding random permutation is a uniformly chosen permutation.
(b) We chose a uniform random permutation of the first n numbers. Denote by X the number of cycles of the permutation, find $\mathbf{E}X$ and $\mathbf{Var}X$.

Further recommended exercises: Feller IX.9., pages 237-242