Probability Theory

Problem set #7

Discrete random variables: expectation, variance, covariance

Homework problems to be handed in: 7.2, 7.7, 7.9, 7.11, 7.13

Bonus problem for extra credit: 7.12, 7.15, 7.16

Due date: April 11

7.1 Let X_1 and X_2 be independent random variables with Poisson distributions $p(k;\lambda_1)$, respectively, $p(k;\lambda_2)$. We have already proved that X_1+X_2 has the Poisson distribution $p(k;\lambda_1+\lambda_2)$. (Check problems 6.8 and 6.12.) Show that the *conditional distribution of* X_1 *given* X_1+X_2 is binomial, namely:

$$\mathbf{P}(X_1=k|X_1+X_2=n)=\binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}.$$

7.2 Let X, Y and Z be independent random variables with the same geometric distribution $g(k;p)=q^kp$ with $k=0,1,2,\ldots$.
(a) Find the following probabilities:

$$\mathbf{P}(X=Y),\quad \mathbf{P}(X\geq 2Y),\quad \mathbf{P}(X+Y\leq Z)$$

(b) Let $U:=\min\{X,Y\}$ and $V:=X-Y$. Show that U and V are independent.

7.3 Let X and Y be independent and have the common geometric distribution $g(k;p)=q^kp$ with $k=0,1,2,\ldots$. Show (preferably without calculation) that the conditional distribution of X given $X+Y$ is uniform, that is,

$$\mathbf{P}(X=k|X+Y=n)=\frac{1}{n+1},\qquad k=0,1,\ldots,n.$$

7.4 (a) We toss a fair coin. If the result is H then we toss the coin two more times, otherwise we toss it only one more time. What is the expectation of the total number of heads during these coin-tosses?
(b) We toss a fair coin until we get the same result twice in succession. What is the expectation of the number of tosses?

7.5 A man is given n keys of which only one fits his door. He tries them randomly until he finds the right one. Find the expectation of the number of his trials, if
(a) he doesn't discard the keys that didn't work: he always chooses out of n keys (sampling with replacement)
(b) he always tries a new key (sampling without replacement)

7.6 I throw two dice. What are the expectations of the maximum and minimum of the two numbers?

7.7 (a) Find the expectation of the number of hits on a lottery ticket (5 out of 90).
(b) Find the expectations of the largest and smallest number drawn out at the lottery.

7.8 Let X denote a random variable which can only take values from the natural numbers. Assume $\mathbf{E}(X)<\infty$. Prove the following identity:

$$\mathbf{E}(X)=\sum_{i=1}^{\infty}\mathbf{P}(X\geq i).$$

7.9 A and B are shooting at a target, A hits it with probability p_1, B hits it with probability p_2 ($p_1<p_2$). A shoots first, they take turns until somebody hits the target, this person wins the game. Find the probability that A wins. Find the expectation of the number of shots taken in the game .

7.10 Determine the expectation of the random variable $(1+X)^{-1}$ in the following two cases:
(a) if X is distributed according to binomial distribution $b(k;p,n)$;
(b) if X is distributed according to Poisson distribution $p(k;\lambda)$.

7.11 (a) Let X and Y be two independent random variables with range in $\mathbb{N}$ (the natural numbers). Assume $\mathbf{E}(X) < \infty$, $\mathbf{E}(Y) < \infty$. Prove that

$$\mathbf{E}(\min\{X,Y\}) = \sum_{i=1}^{\infty} \mathbf{P}(X \geq i)\mathbf{P}(Y \geq i),$$

$$\mathbf{E}(\max\{X,Y\}) = \sum_{i=0}^{\infty} \left[1 - \mathbf{P}(X \leq i)\mathbf{P}(Y \leq i)\right].$$

(b) Prove the following generalization of the first formula: consider k independent random variables with range in $\mathbb{N}$: X_1, X_2, $\ldots X_k$, all with finite expectation, then

$$\mathbf{E}(\min\{X_1, X_2, \ldots, X_k\}) = \sum_{i=1}^{\infty} \prod_{j=1}^{k} \mathbf{P}(X_j \geq i).$$

7.12 We toss several times a biased coin with $\mathbf{P}(\mathrm{H}) = p$, $\mathbf{P}(\mathrm{T}) = q$. Let X and Y denote the length of the first and second run of pure heads or pure tails. (E.g. in the case $HHHTTH\ldots$ $X=3$, $Y=2$; in the case $THHT\ldots$ $X=1$, $Y=2$.) Determine $\mathbf{E}(X)$, $\mathbf{E}(Y)$, $\mathbf{E}(X^2)$, $\mathbf{E}(Y^2)$, $\mathbf{Var}(X)$, $\mathbf{Var}(Y)$, $\mathbf{E}(XY)$, $\mathbf{Cov}(XY)$.

7.13 Let X be a non-negative integer valued random variable with finite second moment ($\mathbf{E}(X^2) < \infty$). Express $S := \sum_{k=1}^{\infty} k\mathbf{P}(X \geq k)$ in terms of $\mathbf{E}(X)$ és $\mathbf{Var}(X)$.

7.14 Let X be a non-negative random variable. Assume that both $\mathbf{E}(X)$ and $\mathbf{E}(X^{-1})$ are finite. Prove that $\mathbf{E}(X^{-1}) \geq (\mathbf{E}(X))^{-1}$.

7.15 Consider the problem 6.13. Find the expectation of the number of games Peter plays until he wins 1000$ or loses all his money. (He starts with $10n$ dollars and the probability that he wins a single game is p.)

7.16 (Weak Law of Large Numbers)

Let $p \in (0,1), 0 < \varepsilon$ be fixed real numbers and S_n be a binomially distributed random variable with parameters n, p. Using the outline below, prove

$$\lim_{n\to\infty} \mathbf{P}\left(\left|\frac{S_n}{n} - p\right| > \varepsilon\right) = 0.$$

(a) Let $r \geq np + p$ and $l \geq 0$ be arbitrary integers. Prove that

$$b(r+l) \leq b(r)y^l,$$

where $b(r) = b(r,n,p), y = \frac{(n-r)p}{rq}$, and $q = 1-p$.

(b) Show that $\mathbf{P}(S_n \geq r) \leq b(r)\frac{rq}{r-np}$.

(c) Prove that $b(r) \leq \frac{1}{r-np}$.

(d) Using the previous estimates show that

$$\mathbf{P}(S_n \geq n(p-\varepsilon)) < \frac{C}{n},$$

where C is a positive constant depending only on ε and p.

Further recommended exercises: Feller IX.9., pages 237-242