Probability Theory

Problem set #6

Discrete random variables

Homework problems to be handed in: 6.4, 6.7, 6.8, 6.10, 6.12

Bonus problems for extra credit:6.13

Due date: March 30

6.1 We observe a system each day for ten minutes. It contains n components which may work or not work independently of each other, we may also assume that what happens today is independent from the events of all the other days. The system breaks down, if at least k of its components malfunction. What is the probability that we first observe a breakdown on the m^{th} day, if for each component the probability that it works in the respective time interval is p (independently of the others).

6.2 The approximate ratio of left-handers in a population is 1%. Estimate the probability that out of 200 randomly chosen individuals there are at least 4 left-handers.

6.3 In a 400 page long book there are 200 misprints (randomly distributed). What is the probability that on the 13. page there are more than one of them?

6.4 How many raisins should a cookie factory put in its cookies *on average*, if they want to achieve that if we randomly choose a cookie, then there is at least one raisin in it with 99% probability.

6.5 (a) Find the probability that in 1000 consecutive poker games we have at least 4 full houses.
(b) Compute this probability numerically, using Poisson approximation.

6.6 We distribute 1000 (distinguishable) balls into 10 000 boxes. Estimate the probability that the total number of balls in the first 25 boxes is at least 4.

6.7 In a certain forest, the average number of trees per 100 m^2 is 6. Assume for simplicity that the trees have a circular cross-section with a diameter of 20 cm. If at some location in the forest we fire a shot in a direction where the end of the forest is only 120 m away, what is the probability that we will hit a tree?

6.8 Suppose that there exist two kinds of stars. The probability that a given volume of space contains j stars of the first kind is $p(j; a)$, and the probability that it contains k stars of the second kind is $p(k; b)$; the two events are assumed to be *independent*. Prove that the probability that the volume contains a total of n stars is $p(n; a + b)$. (Interpret the assertion and the assumptions abstractly.)

6.9 Suppose that the probability of an insect laying r eggs is $p(r; \lambda)$ and that the probability of an egg developing is p. Assuming mutual independence of the eggs, show that the probability of a total of k survivors is given by the Poisson distribution with parameter λp.

6.10 We buy a lotto ticket every week (Hungarian system: '5 out of 90'). For how many consecutive weeks should we do so in order that the probability of having three or more hits at least once, be greater than 0.5? (Use Poisson approximation...)

6.11 Prove the following identities:

$$\sum_{k=0}^{n} kb(k;p,n) = np, \qquad \sum_{k=0}^{n} k^2 b(k;p,n) = n^2p^2 + np(1-p);$$
$$\sum_{k=0}^{\infty} kp(k;\lambda) = \lambda, \qquad \sum_{k=0}^{\infty} k^2 p(k;\lambda) = \lambda^2 + \lambda.$$

6.12 Prove the following identities:

$$\sum_{l=0}^{k} b(l;p,n_1)b(k-l;p,n_2) = b(k;p,n_1+n_2),$$
$$\sum_{l=0}^{k} p(l;\lambda_1)p(k-l;\lambda_2) = p(k;p,\lambda_1+\lambda_2).$$

How would you interpret these results?

6.13 Peter plays a series of gambling games, in each game he wins 10 dollars with probability 1/2 or loses 10 dollars with probability 1/2 (independently of the previous outcomes). If he collects 1000$ then he stops gambling and he also has to stop if he is out of money. At the start of the games he had $10n$ dollars. What is the probability that he finishes with 1000$?

Solve the same problem if Peter has a probability p of winning in a single game (instead of 1/2).

Further recommended exercises: Feller VI.10., pages 169-173.