Probability Theory

Problem set #5

Independence, Geometric Probability

Homework problems to be handed in: 5.4, 5.7, 5.9, 5.10, 5.12

Bonus problems for extra credit: 5.11, 5.13

Due date: March 19 (next Saturday!)

5.1 Events A, B and C are independent and $\mathbf{P}(A) = p, \mathbf{P}(B) = q, \mathbf{P}(C) = r$. Calculate the value of $\mathbf{P}\left((A \cup B) \cap C\right)$.

5.2 We toss a fair coin three times. Let A be the event that there is at least one head and at least one tail among the three tosses, and let B be the event that there is at most one tail. Decide if A and B are independent.

5.3 Two fair dice are thrown. Show that the event that their sum is 7 is independent of the number shown by the first die.

5.4 We roll a die 10 times. Let $A_{i,j}$ be the event that the i^{th} and j^{th} rolls produce the same number. Show that the events $\{A_{i,j}, 1 \le i < j \le 10\}$ are pairwise independent, but not (completely) independent.

5.5 We choose a number from the set $\{1, 2, 3, \ldots, n\}$ with uniform distribution. We denote by A_p the event that the chosen number is divisible by the prime p.
(a) Suppose that the (different) primes $p_1, p_2, \ldots, p_k$ all divide n. Prove, that the events $A_{p_1}, A_{p_2}, \ldots, A_{p_k}$ are (completely) independent.
(b) We denote by C_n the event that the random chosen number is relative prime to n (i.e. the greatest common divisor of n and the chosen number is 1). Prove that

$$\mathbf{P}(C_n) = \prod_{p \text{ prime}, p|n} \left(1 - \frac{1}{p}\right).$$

5.6 (a) What is the probability that out of two independently filled out lottery tickets ('5 out of 90') at least one will have at least two hits?
(b) What is the probability that among 3 000 000 independently filled out lottery tickets there are exactly three with 5 hits.

5.7 On a test there are 20 true or false questions. Suppose that a student knows the correct answer with probability p, thinks he knows the correct answer, but actually he doesn't with probability q and with probability r he is aware of the fact that he doesn't know the answer (these happen independently for each question, $p+q+r = 1$). If the student is aware of the fact that he doesn't know the answer then makes a guess: he writes true or false with probability $1/2$. What is the probability that the student gives the correct answer for at least 19 questions?

5.8 Calculate $\mathbf{P}(A_1 \circ A_2 \circ \cdots \circ A_n)$ if the events $A_1, A_2, \ldots, A_n$ are (completely) independent and each has probability p.

5.9 The cities A, B, C and D are connected with the following routes: $A \leftrightarrow B$, $A \leftrightarrow C$, $C \leftrightarrow B$, $D \leftrightarrow B$, $D \leftrightarrow C$. After a winter night the snow may block each of the routes independently with probability p. What is the probability that next morning we are able to go from A to D (maybe going via some of the other cities)? What is this probability if $p = 1/2$?

5.10 Three points A, B and C are chosen at random independently, with uniform distribution on a circle. What is the probability that the triangle so obtained is acute-angled?

5.11 A stick of length l is broken at two points chosen at random independently, each with uniform distribution. What is the probability that:
(a) a triangle can be formed from the three pieces so obtained?
(b) (difficult!) an acute-angled triangle can be formed from the three pieces so obtained?
(c) all the three pieces so obtained are shorter then $a \in [l/3, l]$?

5.12 We throw 3 darts to a dartboard which has a radius of 30 cm. Each hits the board at a random position with uniform distribution, independently of the others. I measure the distances of these random points from the center of the board. What is the probability that none of the three distances is between 10 and 20 cm?

5.13 We break at random three pieces from three identical sticks. The three breaking points are chosen independently with uniform distribution. What is the probability that:
(a) a triangle can be formed from the three pieces so obtained?
(b) an acute-angled triangle can be formed from the three pieces so obtained?

5.14 Planet X is a ball with center O. Three spaceships A, B and C land at random on its surface, their positions being independent and each uniformly distributed on the surface. Spaceships A and B can communicate directly by radio if the angle $\widehat{\mathrm{AOB}}$ is less than 90°. Show that the probability that they can keep in touch (with, for example, A communicating with B via C if necessary) is $(\pi + 2)/(4\pi)$.

5.15 (a) We choose randomly n points, independently and with uniform distribution on the circumference of a circle. What is the probability that the convex n-gon determined by these points contains the center of the circle in its interior?
(b) We choose randomly n points, independently and with uniform distribution in the interior of a circle. What is the probability that the convex hull of these points contains the center of the circle in its interior?