Probability Theory

Problem set #4

Conditional Probability

Homework problems to be handed in: 4.1, 4.4, 4.5, 4.7, 4.8

Bonus problems for extra credit: 4.11, 4.12

Due date: March 9

4.1 We put N balls into n boxes randomly in a way that all n^N configurations are equally likely. Given the information that there is at least one ball in the first box, find the probability that there are actually K of them in there.

4.2 We throw a die until we get a 6. Given that we needed an even number of throws, find the probability that there were exactly 2 throws.

4.3 We throw three dice. What is the probability of having at least one 6 given that the dice show three different numbers?

4.4 In a bolt factory machines A, B and C manufacture, respectively, 25, 30 and 45 per cent of the total. Of their output 6, 4 and 3 per cent are defective bolts. A bolt is drawn at random from the produce and is found defective. What are the probabilities that it was manufactured by machines A, B, C?

4.5 Let the probability p_n that a family has exactly n children be $\alpha\rho^n$ when $n \geq 1$ and $p_0 = 1 - \alpha\rho(1 + \rho + \rho^2 + \rho^3 + \dots)$. ($\rho \in (0, 1)$ and $\alpha > 0$ are chosen so that $\alpha\rho \leq 1 - \rho$.) Suppose that all sex distribution of n children have the same probability. Show that for $k \geq 1$ the probability that a family has exactly k boys is $2\alpha\rho^k/(2-\rho)^{k+1}$.

4.6 Die A has four red and two white faces, whereas die B has two red and four white faces. A fair coin is flipped *once*. If it falls heads, the game continues by throwing die A alone; if it falls tails, die B is to be used.
(a) Show that the probability of red at any throw is $\frac{1}{2}$.
(b) If the first two throws resulted in red, what is the probability of red at the third throw?
(c) If red turns up at the first n throws, what is the probability that die A is being used?

4.7 A traveller put his passport in one of the n drawers of his desk, but he forgot which one. Before starting a journey he nervously tries to find it. He tries the drawers at random, without repetition. But, due to nervous haste, with probability $1 > w > 0$ he does not notice the passport even if he checks the drawer in which he had actually put it.
(a) What is the probability that he will not find his passport in the first k trials?
(b) If he hasn't found it in the first k drawers opened, what is the probability that the passport actually wasn't in any one of them?

4.8 Adam and Ben decides to meet at a given time at the crossing of two avenues. Unfortunately, they forget to set which of the four corners will be the meeting point. (There

is a really heavy traffic and they cannot see each other if they are at different corners.) Both of them arrive on time choosing a corner randomly (with 1/4-1/4 probability) and use the following strategy. If the other is not there then they wait 2.5 minutes and after that they cross one of the avenues to get to a neighboring corner (choosing the direction with 1/2-1/2 probability) which takes another 30 seconds. They continue this until they meet (it may happen that they meet while crossing the same avenue).
(a) What is the probability that they meet during the first 3 minutes?
(b) What is the probability that they meet during the first 6 minutes?
(c) Let p_n denote the probability that they meet during the first 3n minutes. Calculate p_n.
(d) Let r_n denote the probability that they meet during the $3n^{\text{th}}$ minute. Calculate r_n.
(e) Prove that they meet in a finite time with probability 1.

4.9 We have got two urns: one containing 5 white and 4 black balls, another with 7 white and 3 black balls.
(a) We draw a ball from the first urn and put it in the second one, after this we draw a ball from the second urn and put it in the first one. Finally we draw a ball from the first urn. What is the probability that at this final drawing we choose a white ball?
(b) What is the answer to the same question if first we draw simultaneously a ball from each urn and change their place and after this we draw a ball from the first urn?

4.10 We know about **A**dam, **B**en, **C**laire and **D**olly that each one of them tells the truth randomly, only once in three cases in the average.

A declares that **B** denies that **C** affirms that **D** has lied.
What is the probability that **D** however told the truth?
(We assume, of course, that **C** knew whether **D** lied or not, and **A**, respectively, **B** knew what **B**, respectively **C** told. They lie or not independently of one another.)

4.11 A hunter fires at a fox, which is at a distance of 30m. If not killed, the fox tries to escape running away at a speed of 10 m/s. The hitting probability decreases quadratically with the distance, being equal to $0.75 \cdot \frac{900}{x^2}$ (where x is the distance given in meters, $x \geq 30$), and the hunter continues to refill and fire every 3 seconds until the fox is either killed or it disappears at the horizon. Even being hit, the fox may be wounded but not killed, the probability of surviving a hit being $\frac{1}{4}$, independently of the number of previous wounds. What is the probability that the fox eventually escapes?

4.12 Sindbad, the famous hero of Arabian Nights, has the unique opportunity to choose one beauty out of N girls passing in front of him one by one. Sindbad makes his choice according to the following strategy (call it *k-strategy*): he lets k girls to walk away and after this he chooses the first one who is more beautiful then all the previous ones. What is the probability that applying the k-strategy Sindbad will indeed select the most beautiful of the girls? Determine the optimal k for N large.
(We assume that there exists a well defined order among the girls regarding their beauty and that each one of the $N!$ possible orders of appearance are equally likely.)

Further recommended exercises: Feller, V. 8. pages 140-143 (some of them are already covered by the previous problems).