Probability Theory

Problem set #3

Sieve Formula

Homework problems to be handed in: 3.1, 3.4, 3.6, 3.8, 3.9

Bonus problems for extra credit: 3.10, 3.11

Due date: March 2

3.1 12 pair of shoes are in a closet. 5 shoes are selected at random. Find the probability that there will be at least one pair among the shoes selected.

3.2 We roll a die 10 times. What is the probability that all the results $1, 2, \ldots 6$ will appear at least once?

3.3 We roll two dice 100 times. Find the probability that all the results $(1,1), (2,2), \ldots (6,6)$ will appear at least once.

3.4 We have 6 balls, one is marked with the number 1, two are marked with the number 2 and three are marked with the number 3. We also have three boxes numbered 1, 2 and 3. We choose 3 balls randomly and put one in each box. Find the probability that there is a ball with the same number as its box.

3.5 Find the probability that a bridge hand does not contain 4 cards of the same face value.

3.6 Find the probability that a bridge hand contains a card from each of the four suits. What is this probability in case of a poker hand?

3.7 We toss a coin 6 times. Find the probability that it will fall heads at least 4 times in succession.

3.8 A company of n men went out for dinner. At the restaurant they left their hats and coats at the cloakroom. After dining (and drinking some wine) they collected their hats and coats *completely at random.* What is the probability that somebody from the company went home with his own hat *and* own coat?

3.9 (a) We put k balls into n boxes at random. What is the probability that no box remains empty?
(b) Using the preceding result calculate

$$\sum_{j=1}^{n} (-1)^{j-1} j^k \binom{n}{j}$$

for $k \leq n$.

3.10 Prove the following generalization of the inclusion exclusion formula (due to Károly Jordán). Let $A_1, A_2, \ldots, A_n$ be arbitrary events, we use the notation

$$S_l = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_l \leq n} \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \ldots A_{i_l}).$$

Then for $0 \le r \le n$:

$$\mathbf{P}(\text{exactly } r \text{ of the } n \text{ events occur}) = \sum_{k=0}^{n-r} (-1)^k \binom{r+k}{k} S_{r+k}.$$

(Taking $r = 0$ and considering the complementary event we get back the original inclusion-exclusion formula.)

3.11 Consider again problem **3.8**.
(a) What is the probability that exactly k persons from the company went home with their own hats ($k \le n$)? What is the limit of this probability if k is kept fixed and $n \to \infty$?
(b) What is the probability that exactly k persons from the company went home with their own hats *and* own coats ($k \le n$)? What is the limit of this probability if k is kept fixed and $n \to \infty$?
(It is possible to solve this without the result of **3.10**.)

Further recommended exercises: Feller, IV. 6. pages 111-113 (some of them are already covered by the previous problems).