Probability Theory

Problem set #2

Combinatorial Problems

Homework problems to be handed in: 2.2, 2.4, 2.7, 2.12, 2.15

Bonus problems for extra credit: 2.18, 2.19
(You can also hand in 2.20, but do it without using the Feller book!)

Due date: February 23

2.1 Throwing a dice 20 times, what is the probability that there were no aces?

2.2 A throws six dice and wins if he scores at least one ace. B throws twelve dice and wins if he scores at least two aces. Who has the greater probability to win?

2.3 If n balls are placed at random into n cells, find the probability that *exactly* one cell remains empty.

2.4 A man is given n keys of which only one fits his door. He tries them successively (sampling without replacement). This procedure may require $1, 2, \ldots, n$ trials. Show that each one of these n outcomes has probability n^{-1}.

2.5 Show that it is more probable to get at least one ace with one throw of four dice than at least one double ace in 24 throws of two dice.

2.6 From a population of n elements a sample of size r is taken. Find the probability that none of N prescribed elements will be included in the sample, assuming the sampling to be (a) without, (b) with replacement. Compare the numerical values of the two methods when (i) $n = 100$, $r = N = 3$, and (ii) $n = 100$, $r = N = 10$.

2.7 A closet contains n pairs of shoes. If $2r$ shoes are chosen at random (with $2r < n$), what is the probability that there will be (a) no complete pair, (b) exactly one complete pair, (c) exactly two complete pairs among them?

2.8 Throwing three fair dice, what is the probability that the sum be greater than 10?
Remark: This was the condition of winning in the popular game called "passe dix", in the XVII-th century.

2.9 Compute the probabilities of having 0, 1, 2, 3, 4 and 5 hits with one lottery ticket (Hungarian lottery system: '5 out of 90').

2.10 Find the probability of a poker hand (5 random cards out of 52: with suits $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$ and face values $2, 3, 4, \ldots, 10$, J, Q, K, A) to be (a) royal flush (10, J, Q, K, A in a single suit) (a) straight flush (5 cards in sequence in a single suit) (c) four of a kind (four cards of equal face value) (d) full house (one pair and one triple of cards with equal face values) (e) flush (5 cards in the same suit) (f) straight (five cards in sequence regardless of suit) (g) three of a kind (3 equal face values plus two extra cards) (h) two pairs (two pairs plus an extra card) (g) one pair (one pair plus 3 extra cards of different face values)

2.11 Find the probability that the bridge hands of North and South together contain exactly k aces, where $k = 0, 1, 2, 3, 4$. (Bridge: the 52 cards are randomly divided into four hands of 13 to North, East, South and West.) Calculate the exact probabilities and compare them to the approximate values calculated with Stirling's formula.

2.12 Let a, b, c, d be non-negative integers with $a + b + c + d = 13$. Find the probability that in a bridge game the players North, East, South and West have a, b, c, d spades, respectively.

2.13 A group of $2N$ boys and $2N$ girls is divided into two equal groups. Find the probability that each group will be equally divided into boys and girls. Give an asymptotic estimate to this probability using Stirling's formula.

2.14 We put 20 *indistinguishable* balls into 6 *distinguishable* boxes randomly in way that every possible configuration has the same probability.
(a) What is the probability that none of the boxes is empty?
(b) What is the probability that exactly two of the boxes will be empty?
(c) What is the probability that each box will contain at least two balls?

2.15 Prove the following identities:

$$\text{for } n \geq 1: \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0,$$

$$\text{for } n \geq 0: \quad \sum_{k=0}^{n} k \binom{n}{k} = n 2^{n-1},$$

$$\text{for } n \geq 2: \quad \sum_{k=0}^{n} (-1)^k k \binom{n}{k} = 0,$$

$$\text{for } n \geq 0: \quad \sum_{k=0}^{n} k(k-1) \binom{n}{k} = n(n-1)2^{n-2}.$$

2.16 Prove that for positive integers n and k

$$\sum_{j=0}^{k} (-1)^j \binom{n}{j} \binom{n-j}{k-j} = 0.$$

More generally

$$\sum_{j=0}^{k} \binom{n}{j} \binom{n-j}{k-j} t^j = \binom{n}{k} (1+t)^k.$$

2.17 Prove that for any natural number n

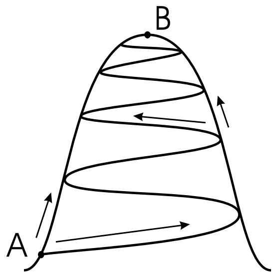
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Hint: Use the hypergeometric distribution.

2.18 We want to climb to the mountain-top (point B) starting from point A. We only want go uphill, but at every junction (including A) we may choose between the steep and the not-so-steep road. What is the probability that we arrive at the mountain-top coming from the right side, if

(a) we choose from each possible path randomly, giving each of them the same probability?

(b) we choose our path using by tossing a fair coin at every junction: if the result is head we go the steep way otherwise the not-so-steep way.



2.19 At a movie theater, the manager announces that they will give a free ticket to the first person in line whose birthday is the same as someone who has already bought a ticket. You have the option of getting in line at any time. What position in line gives you the greatest chance of being the first duplicate birthday? (Assume that you don't know anyone else's birthday, that birthdays are distributed randomly throughout the year and disregard leap years.)

2.20 Prove the following form of Stirling's formula:

The limit $\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}}$ exists and it is a finite positive number. (Of course, the *real* Stirling's formula also identifies the limit.)

Outline of a possible proof:

1. Consider the sequence $d_n = \log\left(\frac{n!}{n^{n+1/2}e^{-n}}\right) = \log n! - (n+\frac{1}{2})\log n + n$, it is enough to prove that $\lim_{n\to\infty} d_n$ is finite.

2. Calculate $d_{n+1} - d_n$ and show that

$$0 < d_{n+1} - d_n < \frac{1}{12n} - \frac{1}{12(n+1)}.$$

(You may need to use the series expansion of the function $\log \frac{1+t}{1-t}$.)

3. Show that the limit of d_n exists and it is finite.

(*Hint:* consider also the sequence $d_n - \frac{1}{12n}$.)

Remark: It is possible to identify $\lim_{n\to\infty} d_n$ as $\log\sqrt{2\pi}$ and from that we get

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Actually, one can also prove the sharper result

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \mathcal{O}(1/n^2)\right)$$

Further recommended exercises: Feller, II. 10-12. pages 54-66 (some of them are already covered by the previous problems).