Probability Theory

Problem set #1

Sample space, algebra of events, Boolean operations

Homework problems to be handed in: 1.2 (only d, e, h, l, u), 1.3, 1.5, 1.7, 1.8

Bonus problems for extra credit: 1.13, 1.14

Due date: February 16

1.1 Let A, B and C be three arbitrary events. Express the following events using the operations of the event-algebra:

(a) out of the events A, B and C exactly k occur $(k = 0, 1, 2)$;

(b) out of the events A, B and C at least i occur $(i = 1, 2)$;

(c) out of the events A, B and C at most j occur $(j = 1, 2)$.

1.2 Verify the following relations. Hand in only (d), (e), (h), (l) and (u):

(a) $A \circ A = \emptyset$; (b) $A \circ \emptyset = A$; (c) $A \circ \Omega = \overline{A}$; (d) $A \circ (B \circ C) = (A \circ B) \circ C$;

(e) $(A \circ B) \cap B = B \setminus A$; (f) $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$;

(g) $A \cap B \setminus C = (A \setminus C) \cap (B \setminus C)$; (h) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$;

(i) $A \setminus [A \setminus (B \setminus C)] = A \cap B \cap \overline{C}$; (j) $A \cup B = A \circ B \circ (A \cap B)$;

(k) $A \circ C \cup [B \setminus (A \cup C)] = [(A \circ B) \setminus (A \cap C)] \cup [(B \circ C) \setminus (A \cap C)]$;

(l) $A \circ C \subset (A \circ B) \cup (B \circ C)$; (m) $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (B \cap C) \cup (C \cap A)$;

(n) $(A \cup B) \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C) \cup (B \setminus C)$;

(o) $(A \cup B \cup C) \setminus (A \cap B \cap C) = [A \setminus (B \cap C)] \cup [B \setminus (C \cap A)] \cup [C \setminus (A \cap B)]$;

(p) $(A \cup B \cup C \cup D) \setminus (A \cup B \cup C) = D \setminus (A \cup B \cup C)$;

(r) $A \cap (\overline{A} \cup B) \cap (\overline{A} \cup \overline{B} \cup C) \cap (\overline{A} \cup \overline{B} \cup \overline{C} \cup D) = A \cap B \cap C \cap D$;

(s) $[(A \cap B) \setminus (A \cap C)] \setminus (B \cap C) = (A \cap B) \setminus C$;

(t) $(A \cap B) \cup C \setminus [(A \cap C) \cup B] = C \setminus (A \cup B)$;

(u) $[A \cap (B \cup C)] \cup [B \cap (C \cup A)] \cup [C \cap (A \cup B)] = (A \cap B) \cup (B \cap C) \cup (C \cap A)$.

1.3 We throw three dice at the same time. How many observable simple events do we have

(a) if the three dice are of the same color (indistinguishable),

(b) if two dice are black (indistinguishable) and the third one is red,

(c) if the three dice are of three different colors?

(d) What is the probability of the event that the sum of the three numbers is odd?

1.4 (a) We toss three different coins and throw two identical (indistinguishable) dice at the same time. How many different outcomes can occur? Represent the sample space.

(b) We throw three black and two white dice at the same time (dice of the same color are indistinguishable). How many different outcomes can occur? Represent the sample space.

1.5 Let $A_1, A_2, \ldots, A_n$ be arbitrary events. What is the meaning of the event $A_1 \circ A_2 \circ \ldots \circ A_n$?
Note: Due to (d) of problem 1.2, the operation $\circ$ is *associative*.

1.6 (a) Prove that for three arbitrary events A, B and C the following inequality holds

$$\mathbf{P}(A \circ C) \leq \mathbf{P}(A \circ B) + \mathbf{P}(B \circ C).$$

that is: if we consider $\Delta(A, B) = \mathbf{P}(A \circ B)$ as a distance between the events A and B, then for this distance the triangle inequality holds.

(b) Prove that if $\mathbf{P}(A \circ B) = 0$ then $\mathbf{P}(A) = \mathbf{P}(B)$.

1.7 A coin is tossed until for the first time the same result appears twice in succession. To every possible outcome requiring n tosses attribute the probability 2^{-n} (why?). Describe the sample space. Find the probability of the following events:

(a) the experiment ends before the sixth toss;
(b) an *even* number of tosses is required.

1.8 (a) Let A and B be two events of a sample space. Prove that if $\mathbf{P}(A) \geq 0.8$ and $\mathbf{P}(B) \geq 0.5$ then $\mathbf{P}(A \cap B) \geq 0.3$.

(b) Prove that for arbitrary events $A_1, A_2, \ldots, A_n$ of a sample space the following inequality holds

$$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n) \geq P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1).$$

1.9 Show that for any three events A, B, and C of a probability space the following inequality holds:

$$|\mathbf{P}(A \cap B) - \mathbf{P}(A \cap C)| \leq \mathbf{P}(B \circ C).$$

1.10 Show that for any two events A and B of a probability space the following inequalities hold:

$$-\frac{1}{4} \leq \mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B) \leq \frac{1}{4}.$$

1.11 A, B and C are equally good tennis players who play the following tournament. First A and B play a match. After that, the winner always stays on the court and the looser is replaced by the third player. This continues until a player wins twice in succession, thus becoming the winner of the tournament.
Find a representation for the sample space. To every possible outcome requiring n matches attribute the probability 2^{-n} (why?). Find the probability of winning for A, B and C.

1.12 Show that for any two events A and B of a probability space the following inequality holds:

$$\mathbf{P}^2(A \cap B) + \mathbf{P}^2(A \cap \overline{B}) + \mathbf{P}^2(\overline{A} \cap B) + \mathbf{P}^2(\overline{A} \cap \overline{B}) \geq \frac{1}{4}$$

Equality holds if and only if $\mathbf{P}(A) = \mathbf{P}(B) = 1/2$ and $\mathbf{P}(A \cap B) = 1/4$.
Hint: Apply Schwarz's inequality.

1.13 Suppose that from the events $A_1, A_2, \ldots, A_n$ we somehow constructed the events $B_1, B_2, \ldots, B_m$ with the usual operations of the event-algebra $(\cap, \cup, \setminus, \circ \ldots)$. (E.g. $B_1 = (A_1 \cap A_3)$, $B_2 = (A_4 \circ \overline{A_2}) \setminus A_6$, etc.) Let $\alpha_1, \alpha_2 \ldots, \alpha_m$ be real numbers.
Prove that

$$\sum_{i=1}^{m} \alpha_i \, \mathbf{P}(B_i) \geq 0$$

holds *on every probability space for any choice of* $A_1, A_2, \ldots, A_n$ **if and only if** it holds on the trivial probability space. (A probability space is trivial if we only have $\emptyset$ and Ω as events in it.)
This means, for example, that in order to prove 1.8 (b) it is enough to consider the case where each A_i is either $\emptyset$ or Ω.
Hint: Consider all the events which can be generated from $A_1, A_2, \ldots, A_n$ by using the operations and try to find ,,building blocks" among these.

1.14 Using the result of the previous problem, prove the sieve- (or inclusion-exclusion) formula:

$$\mathbf{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{k+1} \, \mathbf{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}).$$

(We will prove the sieve-formula in class with a different type of reasoning.)

Further recommended exercises: Feller, I. 8. pages 24-25 (some of them are essentially covered by the previous problems).