Probability Theory

Solutions #9

9.2 If $c = 0$, then the function is constant 1, if $c < 0$ then it could take values greater than 1, thus $c > 0$. If we want the limit in $-\infty$ to be 0, then the exponent must converge to $-\infty$ which means that α has to be positive. We can check, that in case all properties are fulfilled, thus the answer is $c > 0, \alpha > 0$.

9.5

$$\begin{aligned} \mathbf{P}(\text{position of central point} < x) &= \mathbf{P}(\text{all 3 points are in } [0,x)) + \mathbf{P}(\text{exactly 2 points are in } [0,x)) \\ &= x^3 + 3x^2(1-x). \end{aligned}$$

9.6 The integrals of a), b) and d) are not 1, thus they are not densities. The others are nonnegative 'nice' functions and there integral on $\mathbb{R}$ is 1, thus they are densities.

9.8

$$F(x) = \begin{cases} 0 & x \le 0 \\ 4 - 8x & 0 < x \le 1/2 \\ 1 & 1/2 < x \end{cases}$$

Clearly the distance is between 0 and 1/2, this justifies the first and third lines. If $0 < x \le 1/2$ then the probability that the point is closer than x to a side is easily computable using geometric probabilities.

9.9

$$f(x) = \begin{cases} 0 & x \le 0 \\ 2\pi x & 0 < x \le 1/2 \\ 2\pi x - 8x \arccos(1/2x) & 1/2 < x \le 1/\sqrt{2} \\ 1 & 1/\sqrt{2} < x \end{cases}$$

One can solve this problem using the methods of the previous exercise: using geometric probabilities we can compute the distribution function and differentiating it we may get the density.

An other solution is to directly go after the density function, using the fact that $\mathbf{P}(X \in (x, x+\varepsilon)) \approx \varepsilon f(x)$. The probability $\mathbf{P}(X \in (x, x+\varepsilon))$ may be computed quite easily: e.g. in the case $0 < x \le 1/2$ this is just the area of four 'fattened' quarter arcs with radii x with width ε. This is approximately $\varepsilon 2\pi x$ which means that in this interval the density is $2\pi x$. The case $1/2 < x \le 1/\sqrt{2}$ may be handled similarly.