Probability Theory

Solutions #8

8.1 In general, if X_k is the number of days with birthdays of k people, then it may be written as a sum of 365 indicator variables, where the n^{th} is the indicator of the event of having k birthdays on the n^{th} day of the year. By linearity and symmetry,

$$\mathbf{E}X_k = 365\mathbf{P}(\text{there are } k \text{ birthdays on the first day}) = 365b(60, 1/365, k) = 365\binom{60}{k}\frac{364^{60-k}}{365^{60}}.$$

8.3 The way to do this problem is by conditioning on the outcome of the first throw and using the theorem of complete expectation. If we get 2,3,7,11 or 12 for the first throw (this has a probability of 1/3) then the game is over after 1 throw. If we get something else (say n) then we have to wait until the first appearance of n or 7, and this will give us a geometric distribution with parameter $\mathbf{P}$(we get n or 7as a sum) (plus the first throw) and we already computed the expectation of that. Thus the answer is

$$1/3 + 2(\frac{3}{36}(1+\frac{1}{9/36}) + \cdot\frac{4}{36}(1+\frac{1}{10/36}) + \cdot\frac{5}{36}(1+\frac{1}{11/36})) = \frac{557}{165}$$

8.7 1. Solution: Observe that X, Y and $X+Y$ are binomials with parameters $(n, 1/6)$ in the first two cases and $(n, 1/3)$ in the third case. We may write $\mathbf{Cov}(X,Y)$ as $1/2(\mathbf{Var}(X+Y) - \mathbf{Var}X - \mathbf{Var}Y)$ and since we have a formula for the variance of the binomial distribution (npq) this may be readily computable. The answer is $-\frac{n}{36}$.
2. Solution: We write X and Y as sums of indicator functions: $X = I_1 + I_2 + \cdots + I_n$, $Y = J_1 + J_2 + \cdots + J_n$, where I_k, J_k shows if we had a 1 or a 6 on the k^{th} throw. For the covariance we need $\mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y)$. By linearity $\mathbf{E}(X) = \mathbf{E}(Y) = n/6$. Expanding the product in XY and using linearity: $\mathbf{E}XY = \sum_{i,j}\mathbf{E}I_iJ_j$. If $i = j$ then $I_iJ_j = 0$ (since we cannot have a 1 and a 6 for the same throw). If $i \neq j$ then $\mathbf{E}I_iJ_j = \frac{1}{36}$ (the probability of having a 1 at the i^{th} and a 6 at the j^{th} throw). From this we get again $-\frac{n}{36}$ as the answer.

8.9 Since XY is either 0 or 2, we can easily calculate the expectation and the variance using brute force. However, it may help to notice that $XY = 2I$, where I is the indicator of the event that we get 1 or 2 heads. $\mathbf{E}I = 3/4$, $\mathbf{Var}I = 3/41/4$ and from that $\mathbf{E}XY = 3/2$, $\mathbf{Var}XY = 3/4$.

8.12 (a) Using indicator variables we may write $X = I_1 + \cdots + I_6$ where I_k is the indicator that the number k came out among the 6 throws. $\mathbf{E}I_k = 1 - (\frac{5}{6})^6$ and by linearity the expectation in question is $6(1-(\frac{5}{6})^6)$.
(b) This is a variation of the 'coupon collector' problem we discussed in class. Using similar arguments it can be seen that the random variable may be written as $1+GEO(5/6)+GEO(4/6)+GEO(3/6)$ (we just decompose the total number of throws needed into the first throw, the number of throws needed to see the second different number, the number of throws needed to see the third different number and the number of throws needed to see the fourth different number). From this the expectation is $1 + 6/5 + 6/4 + 6/3 = 57/10$.