Probability Theory

Solutions #7

7.2 (a)

$$\mathbf{P}(X=Y)=\sum_{k=0}^{\infty}\mathbf{P}(X=k,\,Y=k)=\sum_{k=0}^{\infty}\mathbf{P}(X=k)\mathbf{P}(Y=k)=\sum_{k=0}^{\infty}p^2q^{2k}=\frac{p}{1+q}$$

$$\mathbf{P}(X\ge 2Y)=\sum_{k=0}^{\infty}\sum_{l=2k}^{\infty}\mathbf{P}(Y=k)\mathbf{P}(X=l)=\sum_{k=0}^{\infty}\sum_{l=2k}^{\infty}p^2q^{k+l}=\frac{1}{1+q+q^2}$$

$$\mathbf{P}(X+Y\le Z)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{j}\mathbf{P}(Z=k)\mathbf{P}(X=l)\mathbf{P}(Y=j-l)=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{j}p^3q^{k+j}=\frac{1}{(1+q)^2}$$

(b) We need to prove $\mathbf{P}(U=k,V=l)=\mathbf{P}(U=k)\mathbf{P}(V=l)$ for every pair (k,l). This may be done the same way as we calculated the probabilities in part (a).

7.7 (a) $X=\#$ of hits on a lottery ticket $=I_1+I_2+\cdots+I_5$, where I_i is the indicator of the event that our i^{th} number was chosen. Then by linearity

$$\mathbf{E}X=\mathbf{E}(I_1+I_2+\cdots+I_5)=5\,\mathbf{P}(\text{a fixed number is chosen})=\frac{5}{18}.$$

(b) If $Y=$ smallest number chosen, then $\mathbf{P}(Y\ge i)=\frac{\binom{91-i}{5}}{\binom{90}{5}}$. By problem 7.8 we have

$$\mathbf{E}Y=\sum_{i=1}^{86}=\frac{\binom{91-i}{5}}{\binom{90}{5}}=\frac{91}{6}$$

The expectation of the largest number is $91-\mathbf{E}Y$.

7.9

$$\mathbf{P}(\text{A wins})=\sum_{l=1}^{\infty}p_1q_1^{l-1}q_2^{l-1}=\frac{p_1}{1-q_1q_2}.$$

$$\mathbf{E}(\text{length of game})=\sum_{l=1}^{\infty}(2l-1)p_1q_1^{l-1}q_2^{l-1}+2lp_1q_1^{l}q_2^{l-1}=\frac{2-p_1}{p_1+p_2-p_1p_2}$$

One can get these result in a simpler way by conditioning on the outcomes of the first two shots.

7.11 Use $\mathbf{P}(\min(X,Y)\ge i)=\mathbf{P}(X\ge i)\mathbf{P}(Y\ge i)$ and problem 7.8, part (b) follows similarly. For the second part of (a): $\mathbf{P}(\max(X,Y)\ge i)=1-\mathbf{P}(\max(X,Y)<i)=1-\mathbf{P}(X<i)\mathbf{P}(Y<i)$.

7.13

$$S:=\sum_{k=1}^{\infty}k\mathbf{P}(X\ge k)=\sum_{k=1}^{\infty}k\sum_{i=k}^{\infty}\mathbf{P}(X=i)=\sum_{i=1}^{\infty}\mathbf{P}(X=i)\sum_{k=1}^{i}k=\sum_{i=1}^{\infty}\mathbf{P}(X=i)\frac{i^2+i}{2}$$

Thus $S=\frac{1}{2}(\mathbf{E}X^2+X)=\frac{1}{2}(\mathbf{Var}X+(\mathbf{E}X)^2+\mathbf{E}X)$.