Probability Theory

Solutions #4

4.1 Let $K > 0$ and $A = \{\text{exactly } K \text{ in the first}\}$, $B = \{\text{at least one in the first}\}$. Then $\mathbf{P}(B) = 1 - (\frac{n-1}{n})^N$, $\mathbf{P}(A) = \binom{N}{K}(\frac{n-1}{n})^{N-K}$ and

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{B} = \frac{\mathbf{P}(A)}{\mathbf{P}(B)} = \frac{\binom{N}{K}(\frac{n-1}{n})^{N-K}}{1 - (\frac{n-1}{n})^N}.$$

4.4 Let D be the event that a randomly chosen bolt is defective and A, B, C the events that it was manufactured by the respective machines. Then, e.g.

$$\begin{aligned} \mathbf{P}(A|D) &= \frac{\mathbf{P}(A \cap D)}{\mathbf{P}(D)} = \frac{\mathbf{P}(D|A)\mathbf{P}(A)}{\mathbf{P}(D|A)\mathbf{P}(A) + \mathbf{P}(D|B)\mathbf{P}(B) + \mathbf{P}(D|C)\mathbf{P}(C)} \\ &= \frac{0.06 \times 0.25}{0.04 \times 0.3 + 0.0 \times 0.25 + 0.03 \times 0.45}. \end{aligned}$$

4.5

$$\begin{aligned} \mathbf{P}(\text{exactly } k \text{ boys}) &= \sum_{n=k}^{\infty} \mathbf{P}(k \text{ boys}|n \text{ children})\mathbf{P}(n \text{ children}) \\ &= \sum_{n=k}^{\infty} \binom{n}{k} \frac{1}{2^n} \alpha \rho^n = \frac{2\alpha\rho^k}{(2-\rho)^{k+1}} \end{aligned}$$

The last equation may be proved with induction on k ($k = 0$ is just the sum of an infinite geometric series). Or: we can start from the identity $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ (which holds if $|x| < 1$) and differentiating it k times we get:

$$\sum_{n=0}^{\infty} k! \binom{n}{k} x^n = \frac{k!}{(1-x)^{k+1}}.$$

4.7 The probability that the passport is in one of the first k drawers is $\frac{k}{n}$ (see **2.4**). Thus the answer for (a) is $\frac{n-k}{n} + w\frac{k}{n}$. For part (b):

$$\mathbf{P}(\text{not in first } k|\text{hasn't found it in first } k) = \frac{\frac{n-k}{n}}{\frac{n-k}{n} + w\frac{k}{n}}.$$

4.8 Let A_1, A_2, A_3 be the events that they start on the same corner, two neighboring corners, two opposing corners. Then $\mathbf{P}(A_1) = 1/4, \mathbf{P}(A_2) = 1/2, \mathbf{P}(A_3) = 1/4$. It is easy to check, that $\mathbf{P}(\text{they don't meet in the first } 3n \text{ minutes}|A_2) = (\frac{3}{4})^n$ and $\mathbf{P}(\text{they don't meet in the first } 3n \text{ minutes}|A_3) = (\frac{1}{2})^n$. From this we get:

$$\mathbf{P}(\text{they don't meet in the first } 3n \text{ minutes}) = \frac{1}{2}\left(\frac{3}{4}\right)^n + \frac{1}{4}\left(\frac{1}{2}\right)^n$$

and $p_n = 1 - \frac{1}{2}\left(\frac{3}{4}\right)^n + \frac{1}{4}\left(\frac{1}{2}\right)^n$. For r_n we need the probability of

$$\{\text{they meet in the first } 3n \text{ minutes}\} \setminus \{\text{they meet in the first } 3n\text{-}3 \text{ minutes}\}$$

which is exactly $p_n - p_{n-1}$. To prove (e) we need to show $\sum_{n=0}^{\infty} r_n = 1$ which is an easy exercise.