Probability Theory

Solutions #3

3.1 $\mathbf{P}(\text{at least 1 pair}) = 1 - \mathbf{P}(\text{no pairs}) = 1 - \frac{\binom{12}{5}2^5}{\binom{24}{5}} = \frac{65}{161}$ (as in 2.7)

Or:

$$\mathbf{P}(\text{at least 1 pair}) = \mathbf{P}(\cup_{i=1}^{12} A_i),$$

where A_i is the event that the i^{th} pair is among the 5 chosen.

$$\mathbf{P}(A_i) = \frac{\binom{22}{3}}{\binom{24}{5}}, \mathbf{P}(A_i \cap A_j) = \frac{\binom{20}{1}}{\binom{24}{5}}, \mathbf{P}(A_i \cap A_j \cap A_k) = 0$$

and applying the sieve formula we get:

$$\mathbf{P}(\cup_{i=1}^{12} A_i) = 12\frac{\binom{22}{3}}{\binom{24}{5}} - \binom{12}{2}\frac{\binom{20}{1}}{\binom{24}{5}} = \frac{65}{161}$$

3.4 Let A be the event that the first box contains a ball marked 1, B the event that the second box contains a ball marked 2 and C that the third box contains a ball marked 3. Then the probability in question is:

$$\begin{aligned}\mathbf{P}(A \cup B \cup C) &= \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(A \cap B) - \mathbf{P}(A \cap C) - \mathbf{P}(B \cap C) \\ &\quad + \mathbf{P}(A \cap B \cap C) \\ &= \frac{1}{6} + \frac{1}{3} + \frac{1}{2} - \frac{1}{6}\cdot\frac{2}{5} - \frac{1}{6}\cdot\frac{3}{5} - \frac{1}{3}\cdot\frac{3}{5} + \frac{1}{6}\cdot\frac{2}{5}\cdot\frac{3}{4} = \frac{41}{60}\end{aligned}$$

3.6 Let A_1 be the event that the hand contains no $\clubsuit$, and the events A_2, A_3, A_4 that the hand does not contain $\diamondsuit, \heartsuit, \spadesuit$ resp. Then we need $1 - \mathbf{P}(A_1 \cup A_2 \cup A_3 \cup A_4)$ which may be easily computed using the sieve formula. (The numerical answer is 0.949 for the bridge and 0.264 for the poker hand.)

3.8 Let A_i be the event that the i^{th} person went home in his own hat and coat. Then we need $\mathbf{P}(\cup_{i=1}^{n} A_i)$ which may be computed by the sieve formula.

$$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_i) = \frac{(n-i)!^2}{n!^2}$$

and

$$\mathbf{P}(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\frac{(n-i)!^2}{n!^2} = \sum_{i=1}^{n}(-1)^{i+1}\frac{(n-i)!}{i!n!}$$

3.9 (a) Let A_i be the event that the i^{th} box remains empty. Then we need $1 - \mathbf{P}(\cup_{i=1}^{n} A_i)$. In general $\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_i) = \frac{(n-i)^k}{n^k}$ and we get

$$1 - \mathbf{P}(\cup_{i=1}^{n} A_i) = 1 - \sum_{i=1}^{n}\binom{n}{i}\frac{(n-i)^k}{n^k}.$$

(b) If $k < n$ then the previous probability is 0 (since at least one box will be empty) and if $k = n$ then the resp. probability is $\frac{n!}{n^n}$. Comparing the sum in question with the formula we got for (a), we get that for $k < n$ it is 0, while for $k = n$ it is $n!(-1)^{n+1}$.