Probability Theory

Solutions #2

2.2 $\mathbf{P}(A \text{ wins}) = 1 - \mathbf{P}(\text{no ace from six throws}) = 1 - \left(\frac{5}{6}\right)^6 \approx 0.665$

$$\begin{aligned}\mathbf{P}(B \text{ wins}) &= 1 - \mathbf{P}(\text{no ace from twelve throws}) - \mathbf{P}(\text{exactly one ace from twelve throws}) \\ &= 1 - \left(\frac{5}{6}\right)^1 2 - 12\,\frac{1}{6}\left(\frac{5}{6}\right)^1 1 \approx 0.619\end{aligned}$$

2.4

$$\begin{aligned}\mathbf{P}(k^{\text{th}} \text{ trial works}) &= \mathbf{P}(\text{none of the trials } 1, 2, \dots k-1 \text{ work and trial } k \text{ works}) \\ &= \frac{n-1}{n}\,\frac{n-2}{n-1} \cdots \frac{n-k+1}{n-k}\,\frac{1}{n-k} = \frac{1}{n}\end{aligned}$$

Or

$\mathbf{P}(k^{\text{th}} \text{ trial works})$

$$= \mathbf{P}(\text{in a random permutation of the first } n \text{ numbers 1 stands at the } k^{th}\text{position}) = \frac{1}{n}.$$

2.7 In general:

$$\mathbf{P}(\text{there will be exactly } k \text{ pairs}) = \frac{\binom{n}{k}\binom{n-k}{r-k}2^{r-k}}{\binom{2n}{2r}}$$

2.12 Number of ways to distribute the 52 cards into four hands of 13: $\frac{52!}{(13!)^4}$.

Number of ways to distribute the 13 spades into four sets of size a, b, c and d: $\frac{13!}{a!b!c!d!}$.

Number of ways to distribute the rest of the cards into four sets of size $13-a, 13-b, 13-c$ and $13-d$: $\frac{39!}{(13-a)!(13-b)!(13-c)!(13-d)!}$.

The requested probability: $\frac{\frac{13!39!}{a!b!c!d!(13-a)!(13-b)!(13-c)!(13-d)!}}{\frac{52!}{(13!)^4}}$

2.15

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = (1-1)^n = 0$$

$$\sum_{k=0}^{n}k\binom{n}{k} = \sum_{k=0}^{n}n\binom{n-1}{k-1} = n(1+1)^n = n2^n$$

The third and fourth identities may be proved with the same tricks.