Probability Theory

Solutions #10

10.4 If X is log-normal with parameters (m, σ) then it has the same distribution as $e^{\sigma Z+m}$ where Z is standard normal. Then CX^α has the same distribution as $e^{\alpha\sigma+\alpha m+\log C}$ thus it is log-normal with parameters $(\alpha\sigma, \alpha m + \log C)$.

10.7 You can find the density functions by several methods. You can use the formula we had in class: $g(y) = \sum_{x\in\Psi^{-1}(y)} \frac{x}{|\Psi'(x)|}$, but you have to remember that $\Psi^{-1}(y)$ is a set and if Ψ is not one-to-one then it will have more than one elements. (E.g. if $\Psi(x) = x^2$.) You can also find the density by first determining the distribution function and then differentiating it.

(a) We will calculate the distribution function of X for $x \in [0, 1]$:

$$F(x) = \mathbf{P}(\xi^2 < x) = \mathbf{P}(-\sqrt{x} < \xi < \sqrt{x}) = \sqrt{x}.$$

Thus $f(x) = \frac{1}{2\sqrt{x}}$ for $x \in [0, 1]$ and 0 otherwise.

For the density of Y it is easier to use the formula, since $\tan(\pi/2x)$ is invertible if $x \in (-1, 1)$. The density will be $g(y) = \frac{1}{2}\frac{2}{\pi}(\arctan y)' = \frac{1}{\pi}\frac{1}{1+y^2}$.

(b) Set $h(x) = \lambda e^{-\lambda x}$ for $x > 0$ and 0 otherwise (this is the density of ξ). To get the density of X we only need a linear transformation: the density will be $\frac{1}{3}h(\frac{x-2}{3})$.

Since the function $\sqrt{y}$ is invertible for $y > 0$ we get $g(y) = h(y^2)2y$ for the density of Y.

(c) If $x > 0$ then the distribution function of X is

$$\mathbf{P}(\xi^{-2} < x) = \mathbf{P}(\xi^2 > 1/x) = \mathbf{P}(\xi > \sqrt{1/x} \text{ or } \mathbf{P}(\xi < -\sqrt{1/x})) = 2 - 2\Phi(\sqrt{1/x}).$$

By differentiation $f(x) = \phi(\sqrt{1/x})\sqrt{1/x^3} = \frac{1}{2\pi x^3}e^{-1/2x}$ if $x > 0$ and 0 otherwise.

10.12 It is enough to prove the inequality for $\sigma = 1$, the general case follows from scaling. By differentiating all three terms, and also checking that there limits are 0 when $x \to \infty$ we get that

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}e^{-(x^2/2)}\left(\frac{1}{x} - \frac{1}{x^3}\right) &= \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-(y^2/2)}\left(1 - \frac{3}{y^4}\right)dy \\
1 - \Phi(x) &= \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-(y^2/2)}dy \\
\frac{1}{\sqrt{2\pi}}e^{-(x^2/2)}\frac{1}{x} &= \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-(y^2/2)}\left(1 + \frac{1}{y^2}\right)dy
\end{aligned}$$

Comparing the functions under the integrals we get the needed inequalities.

10.13 We need to find $\mathbf{P}(55 < X < 65)$ in the cases where X is normal/log-normal with $\mathbf{E}X = 60$ and $\mathbf{Var}X = 3^2$.

(a) If X is normal then it has to be $N(60, 3)$ which means that $\frac{X-60}{3}$ is standard normal. Thus

$$\mathbf{P}(55 < X < 65) = \mathbf{P}(-5/3 < \frac{X-60}{3} < 5/3) = 2\Phi(5/3) - 1.$$

(b) This needs a bit more calculation. If X is log-normal with parameters (m, σ), then $\mathbf{E}X = e^{\sigma^2/2+m}$ and $\mathbf{Var}X = e^{2\sigma^2+2m} - e^{\sigma^2+2m}$. From these the values of m and σ may be easily computed. (They will not be equal to 60 and 3!) If X is log-normal with parameters (m, σ) then $\frac{\log X - m}{\sigma}$ is standard normal. Then, by similar calculations as in the previous part, $\mathbf{P}(55 < X < 65) = \Phi(\frac{\log 65 - m}{\sigma}) - \Phi(\frac{\log 55 - m}{\sigma})$.

10.14 (a) Observe that in all three cases we have to find the expectation of a $f(X)$ where $f(\cdot)$ is anti-symmetric: $f(-x) = -f(x)$. Thus by the symmetry of the density function of the standard normal distribution all three expectations will be 0.

(b) You need to use the identities $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2}$ (where $i = \sqrt{-1}$), and also that the formula for the momentum-generating function of the standard normal also works for complex values: $\mathbf{E}e^{itX} = e^{-t^2/2}$ (you need some complex analysis to verify this properly). From these you can compute all these expectations and variances, e.g. $\mathbf{E}\cos X = \mathbf{E}\frac{e^{iX}+e^{-iX}}{2} = e^{-1/2}$ (the others may be computed similarly).

Since you needed complex analysis for this part, I gave full credit if you did part (a) of the problem.