Geometric probability problems and general probability spaces

(Introduction to Probability Theory, BSM)

Geometric Problems

There are several probability problems when one may parameterize the the sample space Ω by a (sufficiently nice) subset of $\mathbb{R}^d$ (the d-dimensional Euclidean space). If by some symmetry considerations we can assume that the outcome of our experiment is uniform on this set then it is natural to consider the following probability measure on Ω:

$$\text{if } A \subset \Omega \text{ then } \mathbf{P}(A) = \frac{|A|}{|\Omega|},$$

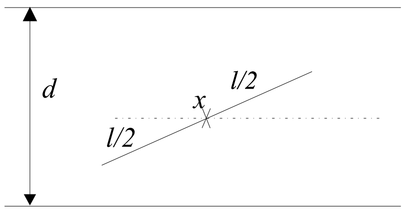
where $|A|$ denotes the d-dimensional measure (length, area, volume...) of A. The event-algebra $\mathcal{A}$ will consist of all the subsets of Ω which can be 'measured'. We will see later that *this does not contain all subsets of* Ω, i.e. there are subsets which are not measurable.

Let's look at some examples!

1. **Buffon's needle problem**

   We have a grid of parallel lines with spacing d. We drop a needle of length $l \leq d$ randomly. What is the probability that it will intersect a horizontal line?

   It is easy to see that the position of the needle is essentially determined by two following parameters: x, the distance of its midpoint from the line below and φ its angle.



   Thus our phase space is

$$\Omega = \{(x, \varphi) : x \in [0, d], \varphi \in [0, 2\pi]\}.$$

   By symmetry, we may assume that the outcome of the experiment is uniform, hence our probability measure is

$$\mathbf{P}(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{d \cdot \pi}.$$

   We want to compute the probability of the following event (check it on the picture):

$$A = \{(x, \varphi) \in \Omega : \frac{1}{2} l \sin\varphi > \min(x, d - x)\},$$

   for this we need its area. This is easily calculated by the integrating:

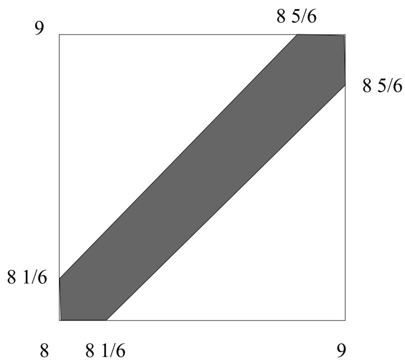
$$|A| = 2 \cdot \int_0^\pi \sin\varphi \, d\varphi = 2l.$$

   Therefore the probability of intersecting a line is $\frac{2l}{\pi d}$.

2. **'Meeting problem'**

   A and B are set to meet at a given place. Unfortunately, they forgot the exact time of their meeting, they only remember that is is between 8 and 9 PM. They both choose their arrival time uniformly in this interval and each waits 10 minutes (or until 9, whichever comes first) for the other to come. What is the probability that they meet?

   The sample space:
   $$\Omega = \{(t_A, t_B) : 8 \le t_A, t_b \le 9\},$$
   P is the uniform distribution.
   $$\{\text{A and B meet}\} = \{|t_A - t_B| \le \frac{1}{6}\}.$$



   The are of the shaded area (and thus the probability) is $1 - \frac{5^2}{6^2} = \frac{11}{36}$.

3. **Random chord problem ('Bertrand's Paradox')**

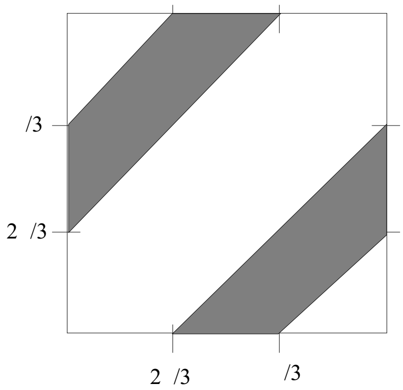
   We choose a chord randomly on a circle. What is the probability that its length is larger than the side of the regular triangle inscribed?

   There are several ways to parameterize this sample space, and since the *randomness* in the question was not defined, we can interpret it several different ways. We list three of them.

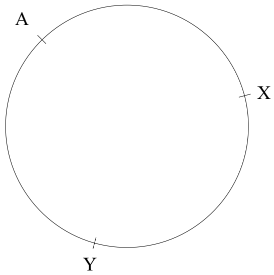
   (A) The midpoint identifies the chord. If it is distributed uniformly inside the disk:
   $$\begin{aligned} \Omega &= \{\mathbf{r} \in \mathbb{R}^2 : |\mathbf{r}| < 1\}, \\ A &= \{\mathbf{r} \in \mathbb{R}^2 : |\mathbf{r}| < 1/2\}. \end{aligned}$$
   The probability $= \frac{|A|}{|\Omega|} = \frac{1}{4}$.

   (B) The distance of the midpoint from the center identifies the length of the chord. If it is distributed uniformly inside [0,1]:
   $$\Omega = [0, 1], \quad A = [0, 1/2].$$
   The probability $= \frac{|A|}{|\Omega|} = \frac{1}{2}$.

   (C) If we choose the endpoints of the chord uniformly on the circle:
   $$\begin{aligned} \Omega &= [0, 2\pi] \times [0, 2\pi], \\ A &= \{(x, y) \in \Omega : \min(|x - y|, |x - y + 2\pi|) > 2\pi/3\}. \end{aligned}$$



The probability $= \frac{|A|}{|\Omega|} = \frac{1}{3}$. We could get this without computing the shaded area by using the following simple trick. Suppose that we have already chosen the first point (A).



Then it is obvious that the other point (B) has to be chosen from the arc XY in order to get the length of the chord large enough. The length of the arc XY is 1/3 of the whole circumference and thus the probability that point B will fall inside is also 1/3. This simple reasoning allowed us to reduce a 2-dimensional problem to a 1-dimensional one which was easier to deal with. (This trick is often used to simplify geometric probability problems.)

General probability spaces

Usually, we define a general probability space the following way. First we set Ω (this is usually given). We consider a collection of 'nice' subsets of Ω which we denote by $\mathcal{A}_0$. $\mathcal{A}_0$ usually is not a σ-algebra (it might not even be an algebra at all). We consider a σ-additive measure $\mathbf{P}_0$ on $\mathcal{A}_0$ (this is usually done in a natural way).

Now we are ready to define our probability space: first we extend $\mathcal{A}_0$ to be a σ-algebra (i.e. we include all those subsets of Ω which will make this a σ-algebra) and denote this new collection of subsets by $\mathcal{A}$. $\mathcal{A}$ won't be as nice as $\mathcal{A}_0$, it could contain really weird sets. Then we also extend $\mathbf{P}_0$ in a way that it is defined on $\mathcal{A}$ and it is a σ-additive measure and we denote this by $\mathbf{P}$. The triple $(\Omega, \mathcal{A}, \mathbf{P})$ will be our probability space. (It can be proved that if $\mathcal{A}_0$ is 'nice' enough then these extensions may be worked out.)

Examples:

1. $\Omega = [0, 1]$, $\mathcal{A}_0$ is the finite union of intervals, $\mathbf{P}_0$ is the length

   Then after the extension we get $\mathcal{A} = \{\text{Borel sets in } [0, 1]\}$ and $\mathbf{P}$ will be the so-called Lebesgue-measure on $[0, 1]$ (which is an extension of length).

2. $\Omega = [0,1] \times [0,1]$, $\mathcal{A}_0$ is the finite rectangles, $\mathbf{P}_0$ is the area

   After the extension we get $\mathcal{A} = \{\text{Borel sets in } [0,1]^2\}$ and $\mathbf{P}$ will be the Lebesgue-measure on $[0,1]^2$ (which is an extension of the area).

   Similar procedures may be done in d-dimension.

3. Infinite coin tossing with a biased coin

   (The probability of H is p, probability of T is $1-p$, we use 1 and 0 for H and T.)

   $\Omega = \{0,1\}^{\mathbb{Z}_+}$ infinite sequences of 0-1 $\mathcal{A}_0 = \{C_{F,\underline{\varepsilon}} : F \subset \mathbb{Z}_+, \underline{\varepsilon} \in \{0,1\}^F\}$ is the collection of finite based cylinders in Ω where

   $$C_{F,\underline{\varepsilon}} = \{(\omega_1, \omega_2, \omega_3, \dots) \in \{0,1\}^{\mathbb{Z}_+} : \omega_i = \varepsilon_i \text{ if } i \in F\}.$$

   I.e. we collect all those sequences where at the positions determined by the index set F we see the prescribed outcomes ε (all the other positions may be chosen freely). We define $\mathbf{P}_0$ on $\mathcal{A}_0$ as follows:

   $$\mathbf{P}_0(C_{F,\underline{\varepsilon}}) = p^{\sum_{i\in F}\varepsilon_i}(1-p)^{|F|-\sum_{i\in F}\varepsilon_i}.$$

   ($\sum_{i\in F}\varepsilon_i$ is the number of heads in
   $underlinevareps$, $|F| - \sum_{i\in F}\varepsilon_i$ is the number of tails.)

   The extension of $\mathcal{A}_0$ and $\mathbf{P}_0$ provides the event-algebra and probability measure of the infinite coin-tossing.

Vitali's construction of a non-measurable set in $[0,1]$

Usually $\mathbf{P}_0$ cannot be extended beyond some natural σ-algebra, it cannot contain all subsets of Ω. We will give an example of a subset of $[0,1]$ which cannot be measured by the Lebesgue measure (the extension of length).

We define the addition on $\Omega = [0,1]$ modulo 1, and say that $x \sim y$ if $x - y \in \mathbb{Q}$ (the rational numbers). It is clear that this is indeed an equivalence relation which divides Ω into equivalence classes. Choose *one element from each class* and denote this set V_0. For a rational number $q \in [0,1] \cap \mathbb{Q}$ define V_q as $\{v + q \mod 1 : v \in V_0\}$ which is just a translation of V_0. It is easy to see that if $q \neq q'$ then $V_q \cap V_{q'} = \emptyset$ and also that $\Omega = \cup_{q\in[0,1]\cap\mathbb{Q}} V_q$. (Try to verify this!) Suppose that V_0 is measurable. Since the Lebesgue measure is translation invariant, each set V_q must have the same measure. If this measure is 0 then $\Omega = \cup_{q\in[0,1]\cap\mathbb{Q}} V_q$ is also of measure 0 (by σ-additivity) which is a contradiction. If V_0 has positive measure then $\Omega = \cup_{q\in[0,1]\cap\mathbb{Q}} V_q$ has a measure ∞ which is again a contradiction (since the measure of Ω is 1). These contradictions show that V_0 is indeed not Lebesgue measurable.
(In 'everyday life' we will not meet such pathological sets. But it is worth noting that it may happen...)