RESEARCH STATEMENT

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Earlier results

During my undergraduate and masters studies, I had some achievements in number theory. Working with a fellow student of mine we solved an open problem of Paul Erdős [DV]. Under the guidance of Prof. András Sárközy I did research work in analytic number theory. I have two papers from these years [V1, V2], both in the field of discrepancy theory.

Later however, my interest shifted to *probability theory.* I have worked here during the last six years and I am planning to continue my research in this area.

My first result in probability theory is a joint paper with S. Csörgő and W. B. Wu [CsVW]. My contribution was the analysis of the size of certain random multisets with the use of classical Poisson approximation.

Currently, my field of interest lies in the *theory of hydrodynamic limits* which is also the subject of my PhD research. I have four papers in this field: [TV1, TV2, TV3, V3], three of which are co-authored by my supervisor, Prof. Bálint Tóth.

In the following sections I give a very brief introduction to the theory of hydrodynamic limits, a short overview of the results of my PhD research and an account of the open problems I am planning to investigate in the near future.

Hydrodynamic limits

The central problem of non-equilibrium statistical physics is the study of the dynamics of interacting particle systems. Generally, the size of these systems is enormous, thus tracking every single particle is hopeless, even if we know everything about the microscopic dynamics. The proper way to look at this problem is from the macroscopic point of view. We characterize the state of the system with the *local densities* of certain physically relevant conserved quantities (e.g. particle number, momentum, energy). These quantities change slowly on a microscopic time-scale, and after appropriate rescaling, their local densities become deterministic. The time-evolution of these functions, which is usually driven by coupled partial differential equations (pde's), gives us the needed description.

Hydrodynamic limit (hdl) is the device to obtain these systems of pde's from the microscopic dynamics. In the physics literature there are a number of well-known phenomenological derivations of hdl's for several systems (see e.g. [LL]). It is a challenging and important program of mathematical physics to give *mathematically rigorous* versions of these derivations. For completely deterministic systems, this is still an unsolved problem. However, in the last decades there have been considerable advances in the case of stochastic interacting particle systems. In particular, much effort has been made in the analysis of lattice gas models with conserved quantities, which can be viewed as discrete approximations of the original physical problems (see the monographs [Sp, KL, F1]).

In general, these models live on a discrete lattice ($\mathbb{Z}$, the size n discrete torus $\mathbb{Z}/n\mathbb{Z}$, or d-dimensional versions of these), have *Markovian dynamics* and possess a certain number of *conserved quantities.* The simplest such model is the so-called 'simple exclusion'. This simulates the simultaneous random walks of particles on a lattice with an exclusion rule: one cannot have more than one particle on a site. If we consider it on a lattice Λ then the phase space is $\{0,1\}^\Lambda$ (since we have either 1 or 0 particle at a given site). We have one conserved quantity, the particle number.

We want to describe the local densities of the conserved quantities as the lattice constant goes to zero. This means that in the limit we get functions on the continuous versions of the relevant lattice ($\mathbb{R}$, $\mathbb{R}/\mathbb{Z}\dots$). In order to see something non-trivial, we also need to rescale time. The degree of the rescaling depends on the type of model we consider. To derive a hydrodynamic limit one has to prove that after proper rescaling of space and time the local densities of the conserved quantities will converge to deterministic functions which are solutions of a certain pde. This is essentially a law of large numbers for the macroscopic densities of the conserved quantities.

My field of interest is the *investigation of hyperbolic systems*. In that case the natural scaling one has to apply is Eulerian (i.e., space and time are scaled the same way) and in the limit we get *hyperbolic conservation laws*:

$$\partial_t u + \partial_x J(u) = 0.$$

If we have k conserved quantities in our microscopic model then this is a k-component pde: $u \in \mathbb{R}^k$, and the smooth function $J : \mathbb{R}^k \to \mathbb{R}^k$ depends on the microscopic features of the model. Hyperbolicity means that the Jacobian matrix of J has k distinct real eigenvalues.

The theory of these equations is a much investigated, important field of pde theory (see e.g. [E]). One of their most interesting features is the following: arbitrary smooth initial conditions yield *shocks* in finite time (apart from some very specially prepared cases), which means that we do not have global classical solutions. The study of weak solutions is the next natural step, but then we lose uniqueness: we have an infinite number of weak solutions with the same initial condition. It is possible to single out the relevant (so-called entropy-) solution, and for one-component hyperbolic equations the problem is essentially solved: the existence and uniqueness of entropy-solutions is established. For *multi-component systems* (i.e., when there are more than one conservation laws) the pde theory is still not completely resolved, which means that in these cases the hydrodynamic description is also quite difficult.

PhD results

The motivation for my work originates from a conjecture of B. Tóth and W. Werner. In [TW] the authors present the following hyperbolic pde system for modelling a general domain growth/deposition phenomena in dimension one:

$$\left\{ \begin{array}{rcl} \partial_t \rho + \partial_x(\rho u) & = & 0 \\ \partial_t u + \partial_x(\rho + \gamma u^2) & = & 0, \end{array} \right. \tag{1}$$

where $\rho = \rho(t,x) \in \mathbb{R}_+$, $u = u(t,x) \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ is a fixed parameter. The pde describes the growth of a deposition ('wall') which is built by a population ('builders'). ρ stands for the density of population performing the deposition, u is the negative gradient of the height of the wall. In the equations it is encoded that the deposition is done by the builders (more builders means faster deposition) and also that the movement of the builders is influenced by the gradient of the wall (they tend to go 'downhill', rather then 'uphill'). The parameter γ gives the rate of a self-generated deposition caused by the wall which is a natural feature in view of the Kardar-Parisi-Zhang theory of domain-growth (see [KPZ]).

In [TW] the pde (1) was derived using a formal, non-rigorous hydrodynamic limit and low density perturbation analysis from a specific two-component microscopic model. It was conjectured there that the arguments can be made rigorous and should hold for a large class of models with two conserved quantities. My thesis *proves this conjecture* and other related results about *multi-component hyperbolic systems*, as follows.

I consider a general family of interacting particle systems with two conserved quantities. These are one dimensional lattice models on the discrete tori $\mathbb{Z}/n\mathbb{Z}$ with non-reversible Markovian dynamics. They possess stationary measures which have a product structure (this is an important technical ingredient).

H. T. Yau's relative entropy method [Y] is essentially the only robust, model-independent method in the literature for proving hydrodynamic limits which works for hyperbolic interacting particle systems with two conserved quantities. It does not depend much on the microscopic properties of the model, but has one drawback: the technique only works for smooth solutions of the limiting pde. (Actually some finite differentiability conditions suffice.) However, hyperbolic conservation laws with generic initial conditions cannot have globally smooth solutions. Thus, the relative entropy method can only apply in a finite time-interval (whose length depends on the initial condition): up to the first appearance of shocks. We note that Olla, Varadhan and Yau proved the first major result about the Eulerian hydrodynamic limit for multi-component hyperbolic systems with this method in [OVY]: namely, for Hamiltonian systems perturbed by a weak noise.

Eulerian hydrodynamic limit

There are several heuristical derivations which (formally) yield that under Eulerian scaling the macroscopic density-profiles of the two conserved quantities evolve according to a pde of the form

$$\begin{cases} \partial_t \rho + \partial_x \Psi(\rho, u) &= 0, \\ \partial_t u + \partial_x \Phi(\rho, u) &= 0, \end{cases} \tag{2}$$

which is usually a hyperbolic conservation law. Here $(\rho, u) \in \mathcal{D}$, which is a convex domain of $\mathbb{R}^2$ (the 'physical domain'), Ψ and Φ are the so-called macroscopic flux functions, which can be explicitly calculated from the microscopic model.

In [TV2] we give a rigorous version of this result with the relative entropy method, i. e. we verify the hydrodynamic limit. The main novel ingredient was the proof of a certain symmetry between the macroscopic fluxes Ψ, Φ, reminiscent of the *Onsager reciprocity relations*. This was vital for proving the hdl, but also enabled us to derive certain interesting (although not surprising) facts about the pde (2). We were able to prove the (weak) hiperbolicity of the pde inside the physical domain $\mathcal{D}$ and that there exists a non-trivial globally convex Lax entropy for the system (which is essentially an extra conservation law). Each result of [TV2] holds also for multi-component systems with more than two conserved quantities.

Perturbation of a singular equilibrium — deriving the universal pde (1)

Although inside $\mathcal{D}$ the pde (2) is hyperbolic, there could be isolated non-hyperbolic points on the boundary. We consider such a *singular* point (ρ_0, u_0). By formal computations it is possible to show that *small perturbations* of the constant (ρ_0, u_0) solution of pde (2) will evolve according to the pde (1). This practically follows from the asymptotics of Ψ, Φ near (ρ_0, u_0).

This behavior of the perturbations can also be seen as a hydrodynamic limit. We consider the steady state of the interacting particle system corresponding to the densities (ρ_0, u_0) and investigate the evolution of *small nonequilibrium perturbations.* In [TV3] we prove for a very rich class of systems that under proper hydrodynamic limit the propagation of these small perturbations is *universally* driven by the system (1). In the limiting pde (1), the parameter γ is the only trace of the microscopic structure. The scaling is not Eulerian: if the lattice constant is n^{-1}, then we have to speed up time by $n^{1+\beta}$ and have to consider perturbations of order $n^{-2\beta}$ and $n^{-\beta}$ of the two conserved quantities (β is a small fixed positive constant.)

The proof relies on the relative entropy method, but there are essential new elements: there is an interesting interplay between probabilistic and pde techniques. In order to control the fluctuations of certain terms with Poissonian (rather than Gaussian) decay coming from the low density approximations we have to apply refined pde estimates. In particular *Lax entropies* of these pde systems play a *not merely technical* role in the main part of the proof. The Onsager-type relation derived in [TV2] is also an integral element.

In the considered models hydrodynamic behavior is determined only by the asymmetric part of the Markovian generator; one only sees the effect of the symmetric part if time is speeded up diffusively, i.e., by a factor of n^2. This fact is used in the paper [TV3] to define a more treatable model by speeding up the symmetric part subdiffusively. With this modification we get more control on the convergence to local equilibrium measures, which is an essential technical element of the proof.

Perturbation of a hyperbolic equilibrium

There are several results dealing with the perturbation analysis of hydrodynamic limits for interacting particle systems. In the landmark paper of Esposito, Marra and Yau [EMY] the authors prove that for the asymmetric simple exclusion, in dimensions higher than two, perturbations of order n^{-1} of a constant profile evolve according to a certain parabolic equation under diffusive scaling. Motivated by this result T. Seppäläinen investigated a similar problem in one dimension in [Se2] for the so-called totally asymmetric stick process (a one-component system). He proves that a perturbation of order $n^{-\beta}$ of the constant profile is governed by the Burgers equation (even after the appearance of shocks) if time is rescaled by $n^{1+\beta}$ and space by n^{-1}, where $\beta \in (0, \frac{1}{2})$ is a fixed constant. Independently, in [TV1] we derived a theorem which partially extends this result. We proved that one *universally* gets the Burgers equation in the hydrodynamic limit for similar perturbations of equilibrium for a wide class of one-dimensional interacting particle systems with one conservation law. As the proof uses the relative entropy method, it only applies in the regime of smooth solutions with $\beta \in (0, \frac{1}{5})$. It is conjectured that the result should hold in general for all $\beta \in (0, \frac{1}{2})$ and without the smoothness assumption, just as in [Se2].

In [V3] I extend the results of [Se2] and [TV1] for two-component systems. I consider small perturbations around a *hyperbolic equilibrium point* and prove that they essentially evolve according to two *decoupled* Burgers equations. This also complements [TV3]. This result agrees with the formal perturbation of the pde (2), e.g. by the method of weakly nonlinear geometric optics (see [DPM]). The scaling is similar to the one used in [Se2] and [TV1]: if the lattice constant is n^{-1}, then the perturbations are of order $n^{-\beta}$ and time is speeded up by $n^{1+\beta}$. (β is a small positive constant.) Besides the relative entropy method, the Onsager-type relation derived in [TV2] also plays an integral role in the proof.

Future work

Since each of my results described above rely on the use of the relative entropy method, the proofs are valid only in the regime of smooth solutions. It would be of great interest if these results (particularly the Eulerian scaling limit) could be *extended beyond the shocks*.

The investigation of this area is the main objective of my coming research. Below I describe some specific problems I am planning to explore.

For one-component microscopic systems which exhibit the so-called attractivity property there are deep and strong results about the hydrodynamic behavior, even after the loss of smoothness. The key idea, first used by F. Rezakhanlou [R], is the implementation of certain pde techniques (by Kruzkov and Oleinik) on a stochastic level. In [Se1] T. Seppäläinen uses a stochastic version of the Lax-Hopf variational formula to prove the existence of hydrodynamic limit for general K-exclusion models. In the recent paper [BGRS] Bahadoran et al. extend these results by an argument inspired by another pde method, the so-called Glimm's scheme.

In the case of multi-component systems attractivity is not a natural feature and thus the tools developed for attractive systems cannot be used. Recently, in [F2] J. Fritz introduced a new method, which consists of transposing some pde tools developed for the study of multi-component hyperbolic systems (by Tartar, Murat and DiPerna) to stochastic interacting particle systems. This technique (the method of *compensated compactness*) was successfully applied by J. Fritz and B. Tóth in [FT] to a particular two-component system. This was the first (and is still the only) result where hyperbolic hydrodynamic Euler equations are derived for a multi-component system which are valid beyond the appearance of shock waves. It is hoped that this method may be *extended to more general systems* (including non-attractive one-component models).

For this general case, *stochastic versions of other relevant pde tools* would be very useful. In pde theory the control of the size of the possible solutions of a given equation is always important. For hyperbolic conservation laws the Lax-Chuey-Conley-Smoller maximum principle provides this control. It would be essential to give a version of this maximum principle which could be used for stochastic particle systems.

In [FT] (just as in [TV3]) the symmetric part of the generator is speeded up to gain more control on the local equilibrium measure. This speeded up part does not show its effect in the *limit* as it gives a vanishing viscous (parabolic) term to the equation. The addition of a vanishing viscous term is also a common technique to stabilize the solutions in pde theory. It might be useful to exploit this idea in proving hdl's. It might be possible to prove hydrodynamic limits by comparing the macroscopic density profiles to the solutions of the modified equation, which behave more nicely than the solutions of the original hyperbolic pde.

Besides hydrodynamics I am also interested in other branches of modern probability theory. During my PhD years I have studied SLE theory, random combinatorial structures, percolation, large deviation theory (among others). Although I haven't achieved research results relating to these, I am open to collaboration in these areas.

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