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# ON SEQUENCES OF "PURE HEADS"

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Statistics connected with the maximum of moving averages of series of i.i.d. random variables are investigated.

**1. Introduction.** If someone tries to produce (without much thinking) the results of a series of coin tossing, *not* using a coin or any other device generating random numbers, he will carefully check that the rate of "heads" be near to $\frac{1}{2}$, and perhaps in short blocks the "heads" and "tails" will follow each other quite randomly. However, the maximal length of blocks of consecutive heads will surely be far less than would be expected in a real coin tossing situation. Let us denote by $\nu_n$ the length of the longest block of consecutive heads (a block of "pure heads") in the first $n$ outcomes in an infinite series of coin tosses. In his book, Rényi (1970) proved that

$$P\left(\lim_{n\to\infty} \frac{\nu_n}{\log_2 n} = 1\right) = 1 .$$

Generally, we may ask for the maximal length of blocks in which the frequency of heads exceeds a fixed $x$. In case $x > \frac{1}{2}$ the maximal length $\nu_n(x)$ of such blocks is expected still to be around $c \log n$, where $c$ depends on $x$.

Let $\xi_1, \xi_2, \cdots, \xi_m, \cdots$ be i.i.d. nondegenerate random variables with distribution function $F$ (left-continuous) having expectation 0. For a fixed $m$ define $\eta_m(i)$ by

$$\eta_m(i) = \frac{1}{m} \sum_{j=i}^{i+m-1} \xi_j \qquad (i = 1, 2, \cdots) . \tag{1.1}$$

For $x > 0$, let $\nu_n(x)$ be the maximum of those $m$'s for which there exists an $i \leqq n$ such that $\eta_m(i) \geqq x$, and define $\zeta_n(m)$ by

$$\zeta_n(m) = \max_{1 \leqq i \leqq n} \eta_m(i) . \tag{1.2}$$

For a fixed $x > 0$, the elements of the process

$$\tau_m(x, 1), \tau_m(x, 2), \cdots, \tau_m(x, k), \cdots \tag{1.3}$$

are the indices $i$ for which $\eta_m(i) \geqq x$ (in increasing order). The first term of this process will be denoted by $\tau_m(x)$: $\tau_m(x) = \tau_m(x, 1)$.

Erdös and Rényi (1970) proved that

$$P(\lim_{n\to\infty} \zeta_n([c \log n]) = h(c)) = 1 .$$

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The function $h(c)$ is given in Section 4, which contains our main results. In Section 5 the asymptotic behaviour of the process $\tau_m(x, k)$ is investigated. Sections 2 and 3 deal with the necessary preliminaries.

**2. Theorems on large deviations.** Theorems on large deviations state that the probability of the events

$$A_i = \{\eta_m(i) < x\} \tag{2.1}$$

tends to 1 exponentially, where $\eta_m(i)$ is defined by (1.1). (The events $A_i$ depend on $x$ and on the length $m$ of the blocks, too, but for the sake of simplicity, we do not indicate this dependence in our notations.) The first version of this statement was formulated by Chernoff (1956) in the form

$$\lim_{m\to\infty} \frac{1}{m} \log P(\bar{A}_1) = \log \rho ,$$

where $\rho = \rho(x)$ is the so-called Chernoff function defined in (2.4). This theorem was extended by Bahadur and Ranga Rao (1960). They proved—roughly speaking—that $m^{\frac{1}{2}}\rho^{-m}P(\bar{A}_1)$ tends to some constant. Petrov (1965) proved that this convergences is uniform on closed intervals; we shall state his theorem later in detail.

The finiteness of the moment generating function,

$$R(t) = Ee^{t\xi_1} = \int_{-\infty}^{\infty} e^{ty}F(dy) \tag{2.2}$$

at least for some point $t > 0$, is needed for proving such a theorem. Write $a_1$ and $a_2$ for the endpoints (possibly infinite) of the domain of $R(t)$; that is,

$$a_1 = \inf\{t : R(t) < \infty\} ; \qquad a_2 = \sup\{t : R(t) < \infty\} . \tag{2.3}$$

Denote the open interval $(a_1, a_2)$ by $T$. It is easy to prove the following three statements:

i) $R$ is differentiable over $T$.
ii) The logarithmic derivative of $R$,

$$\psi(t) = \frac{R'(t)}{R(t)}$$

is continuous and monotone increasing. Thus it has an inverse function:

$$\alpha(x) = t \qquad \text{if} \quad \psi(t) = x , \quad t \in T ,$$

which is also continuous. Note that $\alpha(0) = \psi(0) = 0$, since the expectation of $\xi_1$ is assumed to be 0.

iii) Denote the interior of the range of $\psi$ by $X$: $X = \{\psi(t); a_1 < t < a_2\}$, and the set of positive elements of $X$ by $X^+$. For a given $x \in X^+$, the function

$$e^{-tx}R(t) \quad (a_1 < t < a_2)$$

takes its minimum at the point $\alpha = \alpha(x)$. (We shall mostly use the shorter form $\alpha$ for $\alpha(x)$, thus not indicating the parameter $x$.) Denote this minimal value by $\rho(x)$:

(2.4)
$$\rho(x) = e^{-\alpha x}R(\alpha) = \inf_{a_1<t<a_2} e^{-tx}R(t) .$$

The function $\rho$ is differentiable. In fact, $\rho'(x) = -\alpha\rho(x)$, i.e. the logarithmic derivative of $\rho$ is $-\alpha(x)$, and $\alpha(x)$ in turn is the inverse of the logarithmic derivative of $R(t)$.

The estimation of the probability

$$P(\sum_{i=1}^{m} \xi_i \geqq mx + y)$$

for $x \in X^+$ is usually based on the so-called conjugate distribution. For $\alpha = \alpha(x)$, define the distribution function $F_\alpha$ as

$$F_\alpha(\mu) = \frac{1}{R(\alpha)} \int_{-\infty}^{u} e^{\alpha z}F(dz) ,$$

and let $\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_n, \cdots$ be i.i.d. random variables with the common distribution $F_\alpha$. Denote the distribution function of the sum $\sum_{i=1}^{m} \tilde{\xi}_i$ by $G_m$.

With these definitions

$$\begin{aligned} P(\sum_{i=1}^{m} \xi_i \geqq mx + y) &= R^m(\alpha) \int_{mx+y}^{\infty} e^{-\alpha z}G_m(dz) \\ &= \rho^m(x)e^{-\alpha y} \int_{mx+y}^{\infty} e^{-\alpha(z-mx-y)}G_m(dz) . \end{aligned}$$

The distribution $G_m$ is centered at $mx$, hence

$$G_m(dz) \sim \frac{dz}{\sigma_\alpha(2\pi m)^{\frac{1}{2}}} ,$$

which yields the approximation

$$P(\sum_{i=1}^{m} \xi_i \geqq mx + y) \sim \rho^m(x) \cdot e^{-\alpha y} \cdot \frac{1}{\sigma_\alpha(2\pi m)^{\frac{1}{2}}} \cdot \frac{1}{\alpha} ,$$

where $\sigma_\alpha{}^2 = \operatorname{Var} \tilde{\xi}_1$.

The only exception is the case when $\xi_1$ is lattice-valued. Assume that $\Delta$ is the largest value such that

$$\sum_{k=-\infty}^{\infty} P(\xi_1 = k\Delta + \omega) = 1 .$$

$\Delta$ is called the width of the lattice. In this case the distribution $G_m$ is concentrated on the lattice $\mathscr{L}_m = \{k\Delta + m\omega\}_{k=-\infty}^{\infty}$.

THEOREM (V. V. Petrov). *Let $I \subset X^+$ be a closed interval. Then for any $\varepsilon > 0$ there exists a number $m_0$ such that*

$$\left| P(\sum_{i=1}^{m} \xi_i \geqq mx) - \frac{\rho^m(x)}{b_m(x)m^{\frac{1}{2}}} \right| < \varepsilon \frac{\rho^m(x)}{m^{\frac{1}{2}}}$$

*for all $x \in I$ whenever $m > m_0$, where*

$$\begin{aligned} b_m(x) &= \alpha\sigma_\alpha(2\pi)^{\frac{1}{2}} & &\textit{if } \xi_1 \textit{ is not lattice-valued} \\ &= \frac{1-e^{-\alpha\Delta}}{\Delta} e^{-\alpha\Delta x}\sigma_\alpha(2\pi)^{\frac{1}{2}} & &\textit{if } \xi_1 \textit{ is lattice-valued.} \end{aligned}$$

*Here $\Delta$ denotes the width of the lattice $\mathscr{L}$ of $\xi_1$, and*

$$\Delta_x = \inf_{v \in \mathscr{L}_m, v \geqq mx} (v - mx)$$

*is the distance between $mx$ and the first element on the right-hand side of $mx$ of the lattice $\mathscr{L}_m$ of $\sum_{i=1}^m \xi_i$.*

COROLLARY. *If $I \subset X^+$ is a closed interval, then there is a constant $K_1$ and an integer $m_0$ such that*

$$P(\textstyle\sum_{i=1}^m \xi_i \geqq mx + y \mid \sum_{i=1}^m \xi_i \geqq mx) \leqq K_1 e^{-\alpha y}$$

*for all $m > m_0$, $x \in I$ and arbitrary $y$.*

Note that strictly speaking only the case $x + y/m \in X^+$ is a corollary of the above theorem. The proof, however, is easily extended for arbitrary $y$. Actually, in case of non-lattice-valued distributions, $K_1$ could be chosen arbitrarily near to 1.

The fact that the rare event $\bar{A}_1 = \{\sum_{i=1}^m \xi_i \geqq mx\}$ occurred has a strong effect on the elements of the whole block $\{\xi_1, \xi_2, \cdots, \xi_m\}$. Roughly speaking the elements of a block with a large average are forced to be as near to $x$ as possible. The theoretical background of this phenomenon was investigated by Vincze (1972). It is expressed in the following theorems.

THEOREM (Bártfai). *If $I \subset X^+$ is a closed interval, then*

$$\lim_{m \to \infty} P(\xi_1 < z \mid \textstyle\sum_{i=1}^m \xi_i \geqq mx) = F_\alpha(z)\,,$$

*uniformly in $x \in I$, where $F_\alpha(z)$ is the conjugate distribution defined above.*

THEOREM 1. *If $I \subset X^+$ is a closed interval and $k$ is fixed, then*

$$\begin{aligned}\lim_{m \to \infty} P(\textstyle\sum_{i=1}^m \xi_i < L_m(mx) + y_0, \xi_1 < y_1, \cdots, \xi_k < y_k \mid \sum_{i=1}^m \xi_i \geqq mx)\\ = G(y_0) \textstyle\prod_{i=1}^k F_\alpha(y_i)\end{aligned}$$

*uniformly for all $x \in I$,, where $L_m(u) = u$ and*

$$G(u) = 1 - e^{-\alpha u}\,, \quad (u \geqq 0)$$

*if $\xi_i$ is non-lattice valued, otherwise*

$$L_m(u) = \min\{v \in \mathscr{L}_m;\, v \geqq u\}\,,$$

*and*

$$\begin{aligned} G(u) &= 1 - e^{-\alpha k \Delta} \quad &&\textit{for} \quad (k-1)\Delta < u \leqq k\Delta \qquad (k = 1, 2, \cdots)\\ &= 0 &&\textit{for} \quad u \leqq 0\,.\end{aligned}$$

PROOF. This theorem is an extension of the theorem of Bártfai. Since his methods apply in our case, we only sketch the proof.

$$\begin{aligned}&P(\textstyle\sum_{i=1}^m \xi_i \geqq L_m(mx) + y_0, \xi_1 < y_1, \cdots, \xi_k < y_k \mid \sum_{i=1}^m \xi_i \geqq mx)\\ &\quad = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_k} \frac{P(\sum_{i=k+1}^m \xi_i \geqq L_m(mx) + \tilde{y}_0 - \sum_{i=1}^k z_i)}{P(\sum_{i=1}^m \xi_i \geqq mx)} F(dz_1) \cdots F(dz_k)\,,\end{aligned}$$

where $\tilde{y}_0 = \inf\{j\Delta : j\Delta \geqq y_0\}$ for $y_0 > 0$, and $\tilde{y}_0 = 0$ for $y_0 \leqq 0$. The sequence of the integrands tends to $R^{-k}(\alpha)\exp\{\alpha \sum_{i=1}^k z_i - \alpha\tilde{y}_0\}$ and is dominated by $K(\tilde{y}_0, k, x)\exp\{\alpha \sum_{i=1}^k z_i\}$. An application of the Lebesgue theorem completes the proof.

**3. A conditional large deviation theorem.** Let the events $A_i$ be defined by (2.1) and let $\tilde{\xi}_0, \tilde{\xi}_1, \cdots, \tilde{\xi}_k, \cdots$ be random variables having the joint distribution given in Theorem 1:

$$P(\tilde{\xi}_0 < y_0, \tilde{\xi}_1 < y_1, \cdots, \tilde{\xi}_k < y_k) = G(y_0)\prod_{i=1}^k F_\alpha(y_i) .$$

Assume further that the systems $\{\xi_i\}$ and $\{\tilde{\xi}_i\}$ are independent. Denote the probability $P(\tilde{\xi}_1 + \cdots + \tilde{\xi}_j > \tilde{\xi}_0 + \xi_1 + \cdots + \xi_j; j = 1, 2, \cdots, i)$ by $p_i(x)$ and their limit by $p(x) = P(\tilde{\xi}_1 + \cdots + \tilde{\xi}_j > \tilde{\xi}_0 + \xi_1 + \cdots + \xi_j; j = 1, 2, \cdots)$.

Note that the distribution function of the variables $\tilde{\xi}_i$ continuously varies with the parameter $x$, and hence the function $p$ is continuous.

THEOREM 2. *For any $\varepsilon > 0$ and any closed interval $I \subset X^+$ there exist $n_0 = n_0(\varepsilon, I)$ and $m_0 = m_0(\varepsilon, I)$ such that*

$$\left|\frac{P(\bar{A}_n \mid A_1 \cdots A_{n-1})}{P(\bar{A}_n)} - p(x)\right| < \varepsilon$$

*if $m > m_0$, $n > n_0$ and $x \in I$.*

PROOF. Note that

$$\frac{P(\bar{A}_n \mid A_1 \cdots A_{n-1})}{P(\bar{A}_n)} = \frac{P(\bar{A}_1 A_2 \cdots A_n)}{P(\bar{A}_1)P(A_2 \cdots A_n)} = \frac{P(A_2 \cdots A_n \mid \bar{A}_1)}{P(A_2 \cdots A_n)} .$$

Introduce the following events:

$$\begin{aligned} A &= A_1 \\ B &= A_2 \cdots A_{k+1} \\ C &= A_{k+2} \cdots A_m \\ D_i &= A_{im+1} \cdots A_{(i+1)m} && (i = 0, 1, \cdots, l) \\ E_i &= A_{im+1} \cdots A_n && (i = 0, 1, \cdots, l) , \end{aligned}$$

where $l < n/m \leqq l + 1$, and $k < m_0 < n_0$ will be defined later.

In case $n \leqq 3m$ we use the inequalities

$$\begin{aligned} P(B \mid \bar{A}) \geqq P(A_2 \cdots A_n \mid \bar{A}_1) &\geqq P(BCD_1D_2 \mid \bar{A}) \\ &\geqq P(B \mid \bar{A}) - P(\bar{C} \mid \bar{A}) - P(\bar{D}_1) - P(\bar{D}_2) . \end{aligned}$$

In case $n > 3m$ $A_2 \cdots A_n = BCD_1E_2$, whence

$$\begin{aligned} P(A_2 \cdots A_n \mid \bar{A}_1) &\leqq P(BE_2 \mid \bar{A}) = P(B \mid \bar{A})P(E_2) \\ P(A_2 \cdots A_n \mid \bar{A}_1) &= P(BE_2 \mid \bar{A}) - P(BE_2\overline{CD_1} \mid \bar{A}) \\ &\geqq P(B \mid \bar{A})P(E_2) - P(\bar{C} \mid \bar{A})P(E_2) - P(\bar{D}_1)P(E_3) . \end{aligned}$$

We shall prove that

i) for fixed $k$ $P(B \mid \bar{A})$ tends to $p_k$ as $m \to \infty$;

ii) $P(\bar{C} \mid \bar{A})$ is arbitrarily small if $k$ and $m$ are large enough;

iii) $P(E_i \mid E_{i+1})$ is arbitrarily near to 1 for $0 \leqq i < l$ if $m$ and $n$ are large enough.

It is easy to see that our theorem is a consequence of these statements.

Part i) is a consequence of Theorem 1. Now we prove ii).

$$P(\bar{C} \mid \bar{A}) = P(\textstyle\bigcup_{j=k+2}^{m} \bar{A}_j \mid \bar{A}_1) \leqq P\left(\eta_m(1) \geqq x + \dfrac{k\delta}{m} \,\middle|\, \eta_m(1) \geqq x\right)$$
$$+ \textstyle\sum_{j=k+2}^{m} P(\eta_{j-1}(1) \leqq 2\delta \mid \eta_m(1) \geqq x)$$
$$+ \textstyle\sum_{j=k+2}^{m} P(\eta_{j-1}(m+1) \geqq \delta) = U + V + W .$$

The theorem of Bártfai and the corollary of Petrov's theorem imply

$$U \leqq K_1 e^{-k\delta\alpha}$$
$$W \leqq \textstyle\sum_{j=k+2}^{m} \rho_1^{\,j-1} \leqq \dfrac{\rho_1^{\,k+1}}{1-\rho_1} ,$$

where $\rho_1 = \rho(\delta)$. The corollary of Petrov's theorem yields

$$V \leqq \textstyle\sum_{j=k+2}^{m} P(\sum_{i=j}^{m} \xi_i \geqq mx - 2\delta(j-1) \mid \eta_m(1) \geqq x)$$
$$\leqq \textstyle\sum_{j=k+2}^{m} \dfrac{1}{P(\eta_m(1) \geqq x)} \cdot P(\sum_{i=j}^{m} \xi_i \geqq (m-j+1)x + (x-2\delta)(j-1))$$
$$\leqq K_2 \textstyle\sum_{j=k+2}^{m} \rho^{-m} \cdot e^{-(x-2\delta)(j-1)\alpha} \cdot \rho^{m-j+1} \cdot \left(\dfrac{m}{m-j+1}\right)^{\frac{1}{2}} .$$

Fortunately $e^{-x\alpha} < \rho(x)$ because

$$\textstyle\int_{-\infty}^{\infty} e^{\alpha z} F(dz) > \exp\{\int_{-\infty}^{\infty} \alpha z F(dz)\} = 1 .$$

This inequality assures that $V$ is small enough for a suitably chosen $\delta$ and for $m$ and $k$ large enough. This completes the proof of ii), and now we pass to the proof of iii).

From the inequality $P(E_{i+1}) \geqq P(E_i) \geqq P(E_{i+1}) - P(\bar{D}_i)P(E_{i+2})$ we get

$$1 \geqq P(E_i \mid E_{i+1}) \geqq 1 - \frac{P(\bar{D}_i)}{P(E_{i+1} \mid E_{i+2})} .$$

If $m$ is large enough $P(E_l)$, $P(E_{l-1})$, $P(D_{l-2})$ are close to 1, so we may assume $P(E_{l-1} \mid E_l) \geqq \frac{1}{2}$, $P(D_{l-2}) \geqq \frac{3}{4}$.

Step by step we get $P(E_i \mid E_{i+1}) \geqq \frac{1}{2}$ for all $i = 0, 1, \cdots, l-1$ (note that the probabilites $P(D_j)$ are equal). Hence $P(E_i \mid E_{i+1}) \geqq 1 - 2P(\bar{D}_i) \geqq 1 - 2mP(\bar{A})$. Thus $P(E_i \mid E_{i+1})$ is arbitrarily close to 1 if $m$ is large enough.

**4. The first large block.** The investigation of the random variables $\nu_n(x)$, $\tau_m(x)$ and $\zeta_n(m)$ defined in Section 1 is based on the following theorem.

THEOREM 3. *For an arbitrary closed interval* $I \subset X^+$

$$p_{mn}(x) = P(\tau_m(x) > n) = P(\zeta_n(m) < x)$$
$$= \left(1 - (1 + o_{m,n}(1)) \frac{p(x)\rho^m(x)}{b_m(x)m^{\frac{1}{2}}}\right)^n$$

*for all* $x \in I$, *where* $|o_{m,n}(1)|$ *is less than* $\varepsilon > 0$ *if* $m > m_1$, *and* $n > n_1$.

PROOF. The event indicated in the theorem means that there is no block of length $m$ among the first $n$ blocks with an average greater than or equal to $x$, hence

$$p_{mn}(x) = P(A_1 \cdots A_n) = P(A_1) \prod_{i=2}^n P(A_i \mid A_i \cdots A_{i-1})$$
$$= P(A_1) \prod_{i=2}^n (1 - P(\bar{A}_i \mid A_1 \cdots A_{i-1})) .$$

Theorem 2 implies that $P(\bar{A}_i \mid A_1 \cdots A_{i-1})$ is near to $p\rho^m b_m^{-1} m^{-\frac{1}{2}}$ if $i$ and $m$ are large enough: $i \geqq n_0$, $m \geqq m_0$. Since $p(x)$ is continuous and positive, it has a positive lower bound on I. Hence we could write

$$P(\bar{A}_i \mid A_1 \cdots A_{i-1}) = (1 + o_{m,n}(1)) \frac{p(x)\rho^m(x)}{b_m(x)m^{\frac{1}{2}}}$$

for $i \geqq n_0$, $m \geqq m_0$. For $i < n_0$ these probabilities tend to 0 as $m \to \infty$, and so does the term $p\rho^m b_m^{-1} m^{-\frac{1}{2}}$. Hence the given approximation is valid for the whole product.

THEOREM 4. *For arbitrary closed interval* $I \subset X^+$, *and for arbitrary* $Z > 0$,

$$P\left(\frac{p(x)\rho^m(x)}{b_m(x)m^{\frac{1}{2}}} \tau_m(x) < z\right) = 1 - e^{-z} + o_m(1) ,$$

*uniformly for all* $x \in I$, $0 \leqq z \leqq Z$, *and*

$$P(m(\zeta_n(m) - x) < z) = \exp\left\{-\frac{p(x)}{\tilde{\alpha}\sigma_\alpha(2\pi)^{\frac{1}{2}}} e^{-\alpha z}\right\} + o_{m,\delta}(1)$$

*uniformly for all* $x \in I$, $|z| \leqq Z$. *In the second formula* $o_{m,\delta}(1)$ *tends to* 0 *if* $m \to \infty$ *and* $\delta \to 0$, *where* $\delta = |nm^{-\frac{1}{2}}\rho^m(x) - 1|$; *moreover in this formula*

i) *if* $\xi_1$ *is non-lattice valued, then* $\tilde{\alpha} = \alpha$ *and* $z$ *is arbitrary*;

ii) *if* $\xi_1$ *is lattice-valued, then* $\tilde{\alpha} = \Delta^{-1}(1 - e^{-\alpha\Delta})$ *and* $z$ *has to be chosen in such a way that* $mx + z \in L_m$.

THEOREM 5. *For arbitrary* $x \in X^+$,

$$P\left(\lim_{m\to\infty} \frac{1}{m} \log \tau_m(x) = -\log \rho(x)\right) = 1 .$$

$$P\left(\lim_{n\to\infty} \frac{\nu_n(x)}{\log n} = -\frac{1}{\log \rho(x)}\right) = 1 , \quad \text{and}$$

$$P(\lim_{n\to\infty} \zeta_n(m) = x) = 1 .$$

*In the third case* $m = m(n)$ *is an arbitrary sequence with the property* $\lim_{n\to\infty} 1/m \log n = -\log \rho(x)$.

The proof of Theorem 4 is straightforward. The proof of Theorem 5 is based on Theorem 3 and is similar to the proof given by Erdös and Rényi (1970) so we omit the details. Actually the third statement of Theorem 5 is a slight generalization of their theorem. The second statement of Theorem 5 is a generalization of the original theorem of Rényi (1970) on the "pure heads" given in his book. Now the function $h(c)$ mentioned in the introduction is the following: for $1/c = -\log \rho(x)$ the number $h(c)$ is the same as our $x$. Note that we have no theorem on the limit distribution of $\nu(x)$.

According to Theorem 4 the order of $\tau_m(x)$ is $\rho^{-m}m^{\frac{1}{2}}$, which is also reflected in Theorem 5. This approximation can be refined as follows:

$$P\left(\limsup_{m\to\infty} \sup_{x\in I} \frac{\log \tau_m(x) + m \log \rho(x) - \frac{1}{2}\log m}{\log \log m} \leqq 1\right) = 1 ,$$

$$P\left(\liminf_{m\to\infty} \inf_{x\in I} \frac{\log \tau_m(x) + m \log \rho(x) - \frac{1}{2}\log m}{\log m} \geqq -2\right) = 1 .$$

Note that the term $\log \tau_m(x) + m \log \rho(x) - \frac{1}{2} \log m$ is stochastically bounded by Theorem 4 and hence after division by an arbitrary function tending to infinity, the ratio tends to 0 stochastically.

**5. The sequence of indices of large blocks.** Fix $x \in I \subset X^+$ and write $\tau_m(x, k)$ for the first index of the $k$th large block, i.e. $\tau_m(x, k)$ is the first index of the $k$th of those blocks which have an average greater than or equal to $x$:

$$\begin{aligned} \eta_m(i) &\geqq x \qquad \text{for} \quad i = \tau_m(x, 1), \tau_m(x, 2), \cdots \\ \eta_m(j) &< x \qquad \text{for} \quad \tau_m(x, i-1) < j < \tau_m(x, i)\,; \qquad i = 1, 2, \cdots \end{aligned}$$

where $\tau_m(x, 0) = 0$. In particular $\tau_m(x, 1) = \tau_m(x)$.

Divide the $\tau$'s into groups in the following way. Starting with $\gamma_1 = 1$ we define a sequence of random variables recursively, as follows: $\gamma_k (k \geqq 2)$ is the smallest integer for which $\tau_m(x, \gamma_k) - \tau_m(x, \gamma_{k-1}) \geqq m$. Denote the set of differences $\{\tau_m(x, j) - \tau_m(x, \gamma_k); j = \gamma_k + 1, \gamma_k + 2, \cdots, \gamma_{k+1} - 1\}$ by $\Gamma_m(x, k)$($\Gamma_m(x, k)$ is empty if $\gamma_{k+1} = \gamma_k + 1$), and, for the sake of brevity, denote $\Gamma_m(x, 1)$ by $\Gamma_m(x)$. The set $\Gamma_m(x)$ is a random subset of the set of the first $m - 1$ integers: $\Gamma_m(x) \subset \{1, 2, \cdots, m-1\}$. In other words, starting from the first large block and going $j (j < m)$ steps forward we get at a large block if and only if $j \in \Gamma_m(x)$. For determining the limit distribution of the random set $\Gamma_m(x)$ first we present a modification of Theorem 1, in which we investigate the joint distribution of the first variables and the "surplus variable" in the first large block, rather than in an arbitrary large block as done in Theorem 1.

THEOREM 6. *If $I \subset X^+$ is a closed interval, and $k$ is fixed, then*

$$\begin{aligned} \lim_{m\to\infty} P\left(\textstyle\sum_{i=\tau_m(x)}^{\tau_m(x)+m-1} \xi_i < L_m(mx) + y_0, \xi_{\tau_m(x)} < y_1, \cdots, \xi_{\tau_m(x)+k-1} < y_k\right) \\ = G^*(y_0) \textstyle\prod_{i=1}^{k} F_\alpha(y_i) \end{aligned}$$

*uniformly for $x \in I$, where $L_m(u)$ is the same as in Theorem 1, and for $u \geqq 0$*

$$G^*(u) = 1 - e^{-\alpha u} \cdot \frac{P(\sum_{i=1}^{j} \tilde{\xi}_i > u + \tilde{\xi}_0 + \sum_{i=1}^{j} \xi_i; j = 1, 2, \cdots)}{P(\sum_{i=1}^{j} \tilde{\xi}_i > \tilde{\xi}_0 + \sum_{i=1}^{j} \xi_i; j = 1, 2, \cdots)} .$$

*If $\xi_1$ is lattice-valued, then we have to replace $y_0$ in $G^*(y_0)$ by $\tilde{y}_0 = \inf\{j\Delta : j\Delta \geqq y_0\}$.*

The proof of this theorem is a combination of the methods used in proving Theorems 1 and 2 and so is omitted.

Let $\xi_0^*$ be a random variable with distribution function $G^*(u)$ and let $\xi_0^*$ be independent of the random variables $\{\tilde{\xi}_i\}$, and $\{\xi_i\}$. Define the random subset $\Gamma$ of the natural numbers as follows: $j \in \Gamma$ if and only if the inequality

$$\sum_{i=1}^{j} \tilde{\xi}_i \leqq \xi_0^* + \sum_{i=1}^{j} \xi_i$$

holds.

COROLLARY. *If $I \subset X^+$ is a closed interval and $H$ is an arbitrary subset of natural numbers, then*

$$\lim_{m\to\infty} P(\Gamma_m(x) = H) = P(\Gamma = H)$$

*uniformly for $x \in I$.*

THEOREM 7. *If $x \in I \subset X^+$ and $k$ is fixed, then the random sets $\Gamma_m(x, 1)$, $\Gamma_m(x, 2), \cdots, \Gamma_m(x, k)$ have the same asymptotic distribution and they are asymptotically independent. The limit distribution of the normed differences*

$$\frac{p(x)\rho^m(x)}{b_m(x)m^{\frac{1}{2}}} [\tau_m(x, \gamma_k) - \tau_m(x, \gamma_{k-1})]$$

*is an exponential distribution with parameter $\lambda = 1$.*

PROOF. The random variables $\tau_m(x, 1) + m - 1, \tau_m(x, \gamma_2) - \tau_m(x, \gamma_1), \cdots, \tau_m(x, \gamma_k) - \tau_m(x, \gamma_{k-1})$ are independent and indentically distributed. Applying Theorem 4 we get the second statement of the theorem. Starting with $\tilde{\gamma}_1 = \gamma_1^* = 1$ define the sequence of random variables $\tilde{\gamma}_k, \gamma_k^*$ recursively as follows: $\tilde{\gamma}_k (k \geqq 2)$ is the smallest integer for which $\tau_m(x, \tilde{\gamma}_k) - \tau_m(x, \tilde{\gamma}_{k-1}) \geqq 2m$, and $\gamma_k^* (k \geqq 2)$ is the smallest integer for which $\tau_m(x, \gamma_k^*) - \tau_m(x, \tilde{\gamma}_{k-1}) \geqq m$. Denote by $\tilde{\Gamma}_m(x, k)$ the set of differences $\{\tau_m(x, j) - \tau_m(x, \tilde{\gamma}_k); j = \tilde{\gamma}_k + 1, \tilde{\gamma}_k + 2, \cdots, \gamma_{k+1}^* - 1\}$. The random sets $\tilde{\Gamma}_m(x, 1), \tilde{\Gamma}_m(x, 2), \cdots, \tilde{\Gamma}_m(x, k)$ are independent and identically distributed. The probability $P(\Gamma_m(x, j) = \tilde{\Gamma}_m(x, j); j = 1, 2, \cdots, k) \geqq 1 - \sum_{i=2}^{k} P(\tau_m(x, \gamma_i) < \tau_m(x, \gamma_{i-1}) + 2m)$ tends to 1, and this proves our theorem.

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