Disjointness graphs of short polygonal chains

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Abstract

The *disjointness graph* of a set system is a graph whose vertices are the sets, two being connected by an edge if and only if they are disjoint. It is known that the disjointness graph G of any system of segments in the plane is χ-*bounded*, that is, its chromatic number $\chi(G)$ is upper bounded by a function of its clique number $\omega(G)$.

Here we show that this statement does not remain true for systems of polygonal chains of length 2. We also construct systems of polygonal chains of length 3 such that their disjointness graphs have arbitrarily large girth and chromatic number. In the opposite direction, we show that the class of disjointness graphs of (possibly self-intersecting) 2-*way infinite* polygonal chains of length 3 is χ-bounded: for every such graph G, we have $\chi(G) \leq (\omega(G))^3 + \omega(G)$.

1   Introduction

Ramsey theory has many applications to other parts of mathematics and computer science [R04], including complexity theory [MW87], approximation algorithms, [MS85], coding [KSV05], geometric data structures [MN07], graph drawing and representation [BEF98]. Constructing nearly optimal Ramsey graphs is a notoriously difficult combinatorial problem [F77]. The few efficient constructions that we have are far from optimal, but they can come in handy in those areas where we have interesting theorems, but lack nontrivial constructions. Here we provide two examples from combinatorial geometry, based on two classical constructions of Erdős and Hajnal [EH66, EH68]. We close this paper with a result pointing in the opposite direction.

For any graph G, let $\chi(G)$ and $\omega(G)$ denote the *chromatic number* and the *clique number* of G, respectively. Clearly, we have $\chi(G) \geq \omega(G)$, and if equality holds for every induced subgraph of G, then G is called a *perfect graph*. Following Gyárfás and Lehel [GL83], [GL85], [G85], [G87], a class of graphs $\mathcal{G}$ is said to be χ-*bounded* if there is a function f such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$.

Which classes of graphs are χ-bounded? Or, reversing the question, if a graph has small clique number, how can its chromatic number be large? These questions are related to the some of the deepest unsolved problems in graph theory. There are two different approaches that have yielded spectacular results in recent years.

One can investigate what kind of substructures must necessarily occur in graphs of high chromatic number. According to Hadwiger's conjecture [H43], if the chromatic number of a graph is at least t, then it must contain a K_t-minor. (We now know that it contains a K_s-minor with $s = \Omega(t/(\log \log t))$; cf. [DP21].) Gyárfás [G75] proved that if a graph has bounded clique number and its chromatic number is sufficiently large, then it must contain a long induced path; see also [GHM03]. According to the (still open) Gyárfás-Sumner conjecture [S81], the same is true for any fixed tree instead of a path. Scott and Seymour proved that the class of graphs with no induced odd cycle of length at least 5 is χ-bounded. For many beautiful recent results of this kind, see the survey [SS20].

The second fruitful research direction was initiated by Asplund and Grünbaum [AG60]: Find geometrically defined classes of graphs that are χ-bounded. Given a set S of geometric objects, their *intersection graph (resp., disjointness graph)* is a graph on the vertex set S, in which two vertices are connected by an edge if and only if the corresponding objects have a nonempty intersection (resp., are disjoint). It was proved in [AG60] that the class of intersection graphs of axis-parallel rectangles in the plane is χ-bounded (see also [ChW21]). The corresponding statement is false for boxes in 3 and higher dimensions [B65], and even for segments in the plane [PKK14].

For *disjointness* graphs G of systems of segments in the plane, we have $\chi(G) \leq (\omega(G))^4$ [LMPT94]. The same is true for systems of *x-monotone curves*, that is, for continuous curves in the plane with the property that every vertical line intersects them in at most one point. It was shown in [PT20] that, in this generality, the order of magnitude of this bound cannot be improved. On the other hand, we proved [PTT17] that the class of disjointness graphs of *strings* (continuous curves in the plane) is not χ-bounded. Improving our construction, Mütze, Walczak, and Wiechert [MWW18] exhibited systems of polygonal curves consisting of *three* segments such that their disjointness graphs are triangle-free ($\omega = 2$), yet their chromatic numbers can be arbitrarily large.

The above results leave open the case of polygonal curves consisting of *two* segments. Our first result settles this case. A polygonal curve consisting of k segments is called a *polygonal k-chain*.

Theorem 1. *There exist arrangements of polygonal 2-chains in the plane whose disjointness graphs are triangle-free and have arbitrarily large chromatic numbers.*

We do not know if Theorem 1 can be strengthened by requiring that the disjointness graph of the curves has large girth.

Problem 2. *Do there exist arrangements of polygonal 2-chains in the plane whose disjointness graphs have arbitrarily large girth and chromatic number?*

Our next result shows that the answer to the above question is in the affirmative if, instead of 2-chains, we are allowed to use polygonal 3-chains.

Theorem 3. *For any integers g and k, there is an arrangement of non-selfintersecting polygonal 3-chains in the plane whose disjointness graph has girth at least g and chromatic number at least k.*

A 1-*way infinite* polygonal 2-chain is the union of a half-line and a segment that share an endpoint. In our proof of Theorem 1, we actually construct arrangements of 1-way infinite polygonal 2-chains whose disjointness graphs are triangle free and have arbitrarily large chromatic number. Doubly tracing these 1-way infinite 2-chains and slightly perturbing the resulting curve, we obtain an arrangement of 2-*way infinite* 4-chains, *i.e.*, 4-chains whose first and last pieces are half-lines. Hence, we obtain the following

Corollary 4. *There exist arrangements of non-selfintersecting 2-way infinite polygonal 4-chains in the plane whose disjointness graphs are triangle-free and have arbitrarily large chromatic numbers.*

Our next theorem shows that Corollary 4 is optimal: the class of disjointness graphs of (possibly self-intersecting) 2-way infinite polygonal 3-chains is χ-bounded.

Theorem 5. *Let G be the disjointness graph of an arrangement of 2-way infinite polygonal 3-chains in the plane. Then we have $\chi(G) \leq (\omega(G))^3 + \omega(G)$.*

In fact, we will establish Theorem 5 in a somewhat stronger setting: for arrangements of 2-way infinite curves that consist of three x-monotone pieces; see Theorem 7. With more work, the bound in Theorem 5 and Theorem 7 can be improved to $\chi(G) \leq (\omega(G))^3$.

In the polygonal case, our proof is algorithmic. There is a polynomial time algorithm in the number of the polygonal chains, which, for every k, either finds k pairwise disjoint chains or produces a coloring of their disjointness graph with at most k^3 colors.

In Sections 2 and 3, we establish Theorems 1 and 3, respectively. Section 4 contains the proof of Theorem 5. We end this note with a few remarks and open problems.

In what follows, we informally call a polygonal 2-chain a *V-shape* and a polygonal 3-chain a *Z-shape*.

2   Shift graphs—Proof of Theorem 1

For every $n > 1$, Erdős and Hajnal [EH68] defined the *shift graph S_n*, as follows. The vertex set of S_n consist of all pairs (a, b) with $1 \leq a < b \leq n$, where two vertices, (a, b) and (a', b'), are connected by an edge if and only if $b = a'$ or $b' = a$. It is easy to see that S_n is triangle-free and that $\chi(S_n) = \lceil \log_2 n \rceil$.

Order the vertices (a, b) of S_n according to the *co-lexicographic order*, that is, let $(a, b) \prec (a', b')$ if $b < b'$, or if $b = b'$ and $a < a'$. Let $v_1, \ldots, v_{\binom{n}{2}}$ denote the vertices of S_n, listed in this order.

Let $v_i = (a, b)$ be a vertex. Its neighbors having a smaller index are (a', b') with $b' = a$. No such neighbor exist if and only if $a = 1$. Notice that, for any i,

1. *either* v_i has no neighbor v_j with a smaller index $j < i$,

2. *or* there exist integers $c(i), d(i)$ with $1 \leq c(i) \leq d(i) < i$ such that for every $j < i$,

$$v_j v_i \in E(S_n) \iff c(i) \leq j \leq d(i).$$

Recall that a 1-*way infinite V-shape* is the union of a half-line and a segment that share an endpoint. In the rest of this proof, for simplicity, we call a 1-way infinite V-shape *long*.

Our goal is to assign a long V-shape to each vertex of S_n so that two V-shapes are disjoint if and only if the corresponding vertices are adjacent in S_n. This will prove Theorem 1, because in any finite collection of long V-shapes, we can cut the half-lines short so that the resulting (bounded) V-shapes have the same intersection structure. Hence, we obtain a collection of V-shapes with S_n as its disjointness graph, and the graphs S_n are triangle-free and their chromatic numbers tend to infinity, as $n \to \infty$.

We assign the long V-shape V_i to the vertex v_i of S_n recursively starting at V_1. Let h_i and s_i denote the half-line and the straight-line segment, respectively, comprising V_i and let us denote their common endpoint by $p_i = (x_i, y_i)$. We write q_i for the other endpoint of s_i.

During the recursive process, we will maintain the following properties:

(i) p_i is the left end point of both h_i and s_i;

(ii) both h_i and s_i have positive slopes;

(iii) s_i is above h_i, *i.e.*, the slope of s_i is larger than the slope of h_i;

(iv) for any $i > j$, the slope of h_i will be smaller than the slope of h_j;

(v) for any $i > j$, we have $x_i < x_j$ and $y_i < y_j$.

Let V_1 be any long V-shape satisfying the above conditions. Let $i > 1$, and assume recursively that we have already constructed the long V-shapes $V_1, \ldots, V_{i-1}$ satisfying the above requirements. Next, we define V_i. We distinguish two case;

Case A: *The vertex* $v_i = (a, b)$ *has no neighbor with a smaller index,* i.e., *we have* $a = 1$.

Let ℓ be a horizontal line passing above p_1. It will intersect every V_j with $1 \leq j < i$. Slightly rotate ℓ about any fixed point of the plane so that the resulting line ℓ' has a very small positive slope, smaller than the slope of h_{i-1} and it still intersects all V_j for $j < i$. Choose a point $p_i = (x_i, y_i) \in \ell$, very far to the left, so that $x_i < x_{i-1}$ and $y_i < y_{i-1}$. Let h_i be the part of ℓ' to the right of p_i, and let q_i be a point to the right of p_i which lies above h_i. One can choose q_i such that the segment $s_i = p_i q_i$ does not intersect any of the earlier V_j.

Case B: *The vertex* $v_i = (a, b)$ *has at least one neighbor of smaller index,* i.e., $a > 1$.

Let $c(i)$ and $d(i)$ be the constants satisfying property (2) above and let ℓ be a horizontal line that passes below $p_{d(i)}$ and above $p_{d(i)+1}$. In case $d(i) + 1 = i$ we could simply choose ℓ to be an arbitrary horizontal line below $p_{d(i)}$, but the careful reader may notice that this case never occurs as no vertex v_i in S_n is adjacent to v_{i-1}.
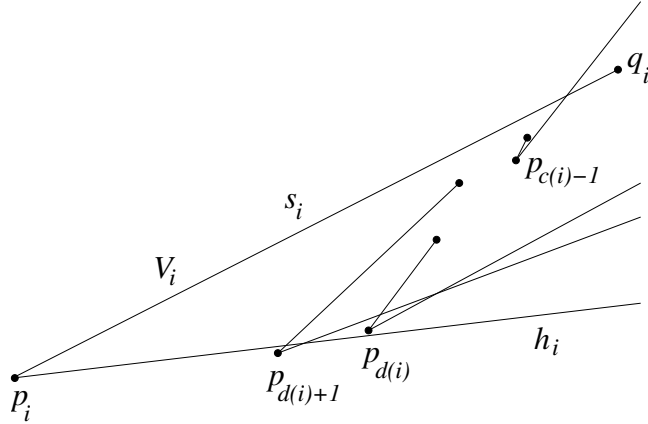
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Figure 1: Inserting V_i.

The line ℓ intersects every V_j with $d(i) < j < i$ and is disjoint from all V_j with $j \leq d(i)$. Slightly rotate ℓ about any fixed point in the plane so that the resulting line ℓ' has a very small positive slope, smaller than that of h_{i-1} and it still intersects the same previously defined long V-shapes V_j. Select a slope α which is larger than the slope of $h_{c(i)}$, but smaller than the slope of $h_{c(i)-1}$, if $h_{c(i)-1}$ exists, that is, if $c(i) > 1$.

For any $j < i$, let ℓ_j and ℓ'_j denote the lines of slope α through p_j and q_j, respectively. Choose a point $p_i = (x_i, y_i) \in \ell'$ so far to the left that we have $x_i < x_{i-1}$, $y_i < y_{i-1}$ and p_i lies above the lines ℓ_j and ℓ'_j, for all $j \leq i$.

Let h_i be the part of ℓ' to the right of p_i. Let f be the half-line of slope α, whose left endpoint is p_i. Then f goes strictly above all s_j for $j < i$ and also of all h_j with $c(i) \leq j < i$, but will intersect all h_j with $1 \leq j < c(i)$. Choose q_i on f to the right of these intersection points, then the segment $s_i = p_i q_i$ also intersects all h_j with $1 \leq j < c(i)$.

Notice that the long V-shape V_i consisting of h_i and s_i constructed above satisfies the conditions (i)–(v) listed above, further it intersects exactly those other long V-shapes V_j ($j < i$) for which v_j and v_i are not adjacent in S_n. See Fig. 1. This means that the disjointness graph of the collection of the $\binom{n}{2}$ long V-shapes constructed above is exactly S_n. This completes the proof of Theorem 1. $\qquad\square$

In the above proof, we have constructed a collection of 1-way infinite V-shapes in which each pair intersects at most twice. With a little additional care (namely, by insisting that each q_i is higher than p_1), we can achieve the following. For $1 \leq i < j \leq \binom{n}{2}$, with $v_i = (a, b)$ and $v_j = (a', b')$, we have

- if $a' < b$, then V_i and V_j intersect once;

- if $a' = b$, then V_i and V_j are disjoint;

- if $a' > b$, then V_i and V_j intersect twice.

3   Hypergraphs of large girth—Proof of Theorem 3

A hypergraph H is a pair (V, E), where V is a finite vertex set, E is the set of hyperedges, that is, a collection of subsets of V. It is called *n-uniform* if each of its hyperedges has n vertices. In a *proper coloring* of H, every vertex is assigned a color in such a way that none of the hyperedges is monochromatic. The *chromatic number* of H is the smallest number of colors used in a proper coloring of H. A *Berge-cycle* in H consists of a sequence of distinct vertices $v_1, \ldots, v_k$ and a sequence of distinct hyperedges $e_1, \ldots, e_k \in E$ with $v_i, v_{i+1} \in e_i$ for $1 \leq i < k$ and $v_k, v_1 \in e_k$. Here k is the *length* of the Berge-cycle and it is assumed to be at least 2. The *girth* of a hypergraph is the length of its shortest Berge-cycle (or infinite if it has no Berge-cycle).

For the proof, we need the following classical result.

Erdős-Hajnal Theorem [EH66] (Corollary 13.4). *For any integers $n \geq 2$, $g \geq 3$, and $k \geq 2$, there exists an n-uniform hypergraph with girth at least g and chromatic number at least k.*

Theorem 3 is a direct consequence of part (5) of the following statement.

Lemma 6. *For any integers $g \geq 3$, $k \geq 2$, there is a natural number $n = n(g, k)$ such that for every set P of n points on the x-axis in $\mathbf{R}^2$ and for every real $c > 0$, there is an arrangement $Z = Z(P)$ of n Z-shapes satisfying the following conditions.*

(1) Each point in P is the endpoint of exactly one Z-shape in Z.

(2) Apart from a single endpoint in P, every Z-shape in Z lies strictly above the x-axis.

(3) No Z-shape in Z is self-intersecting and any two cross at most twice.

(4) For any Z-shape $z = pqrs \in Z$ whose vertices p, q, r, s have x-coordinates x_p, x_q, x_r, x_s, and $p \in P$, we have $x_q + c < x_p < x_s < x_r - c$.

(5) The disjointness graph of the Z-shapes in Z has girth at least g and chromatic number at least k.

Proof. For each g, we prove the lemma by induction on k. We fix $g \geq 3$. For $k = 2$, $n(g, 2) = 2$ is a good choice. For any two points on the x-axis and any $c > 0$, we can take two disjoint Z-shapes satisfying the requirements. Their disjointness graph is K_2, its chromatic number 2 and it has infinite girth. See Fig. 2.
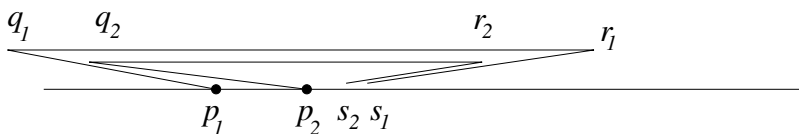

Figure 2: The case $k = 2$.

Suppose now that $k \geq 2$ and that we have already proved the statement for k. Now we prove it for $k + 1$. Let $n = n(g, k)$.

By the Erdős-Hajnal Theorem stated above, there exists an n-uniform hypergraph H whose girth is at least g and chromatic number at least $k + 1$. Let $v_1, v_2, \ldots, v_m$ denote the vertices of

H and $e_1, e_2, \ldots, e_M$ the hyperedges of H. Let $N = nM + m$. We show that $n(g, k+1) = N$ satisfies the requirements of the lemma.

Let P be an arbitrary set of N points on the x-axis and let $c > 0$. For any $v_i \in V(H)$, let d_i denote the *degree* of v_i, that is, the number of hyperedges that contain v_i. Obviously, we have

$$\sum_{i=1}^{m}(d_i + 1) = nM + m = N.$$

Choose m disjoint open intervals, $I_1, \ldots, I_m$, such that each I_i contains precisely $d_i + 1$ points of P. For every i, $1 \le i \le m$, we associate the interval I_i with vertex v_i of H. Let p_i denote the leftmost point in $P \cap I_i$. For every i and j ($1 \le i \le m$, $1 \le j \le M$) for which $v_i \in e_j$, assign a distinct point $p_i^j \in (P \cap I_i) \setminus \{p_i\}$ to the pair (v_i, e_j).

Next, we construct a set of N Z-shapes that satisfy conditions (1)–(5) of the lemma with parameters $g, k+1$, and c. We construct subsets Z_j of our eventual set of Z-shapes for $1 \le j \le M$. We construct these sets one by one starting at Z_1 and using the inductive hypothesis for various subsets of P of size n and with a parameter c' that we choose to be larger than c plus the diameter of P.

For $j = 1$, consider the $n = n(g, k)$-element point set $P_1' = \{p_i^1 : v_i \in e_1\}$. By the induction hypothesis, there is a set Z_1 of Z-shapes such that one of their endpoints belongs to P', and they satisfy conditions (1)–(5) with parameter c'.

Suppose that $j > 1$ and that we have already constructed the sets of Z-shapes $Z_1, \ldots, Z_{j-1}$. Now let $P_j' = \{p_i^j : v_i \in e_j\}$. By the induction hypothesis, there is a set Z_j' of Z-shapes with one of their endpoints in P' which satisfy conditions (1)–(5) with parameter c'. Apply an affine transformation $(x, y) \to (x, y/K_j)$ to all Z-shapes in Z_j', where K_j is a very large constant to be specified later. The resulting family of Z-shapes, Z_j, still satisfies all defining conditions and, by choosing K_j large enough, we can achieve that every element of Z_j intersects every Z-shape in $\bigcup_{h<j} Z_h$ exactly once or twice.

The set $\bigcup_{j=1}^{M} Z_j$ contains exactly one Z-shape starting at each point p_i^j. We still need to add one Z-shape $z_i = p_i q_i r_i s_i$ starting at each point p_i, $1 \le i \le m$. We define them recursively for $i = 1, \ldots, m$. We make sure that each $z_i = p_i q_i r_i s_i$ satisfies the following properties.

(i) *The segment $q_i r_i$ is horizontal and the y-coordinate of its points is larger than the y-coordinate of any point of any Z-shape in $(\bigcup_{j=1}^{M} Z_j) \cup \{z_h : 1 \le h < i\}$.*

(ii) *The slope of $p_i q_i$ is $-\varepsilon_i$, the slope of $r_i s_i$ is ε_i, for a sufficiently small constant $\varepsilon_i > 0$, to be specified later.*

(iii) *The x-coordinate of s_i is equal to the x-coordinate of the right endpoint of I_i, and the y-coordinate of s_i is ε_i.*

Clearly, if we choose $\varepsilon_i > 0$ sufficiently small, then z_i is disjoint from all Z-shapes in $\bigcup_{j=1}^{M} Z_j$ that start in I_i, but it intersects exactly once all other Z-shapes already defined. Also, z_i satisfies conditions (2) and (3), and it satisfies condition (4), too, provided that ε_i is sufficiently small. See Fig. 3.
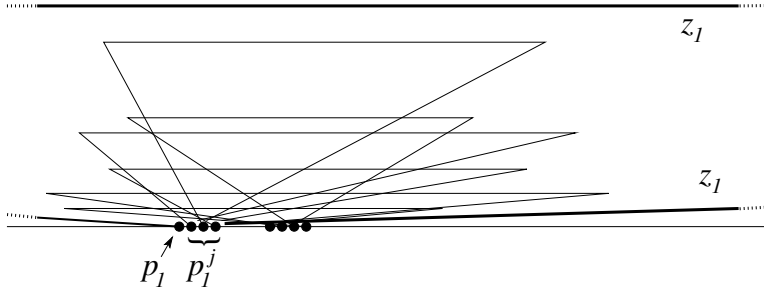
Figure 3: Inserting z_1.

As we maintained conditions (1)–(4) throughout the construction, it remains only to prove that the disjointness graph G of Z satisfies condition (5) with $k + 1$ in place of k.

To this end, let us explore the structure of G. The vertices of G can be partitioned into the sets Z_j for $1 \leq j \leq M$ and the independent set $W = \{z_i : 1 \leq i \leq m\}$. Further, there is no edge between two distinct sets Z_j and $Z_{j'}$. There is a single edge from z_i to Z_j if $v_i \in e_j$, and there is no edge from z_i to Z_j otherwise. Finally, each vertex in Z_j is adjacent to exactly one of the vertices z_i, and it satisfies $v_i \in e_j$.

The structure above implies that each cycle C of G is either contained in a single set Z_j, or it passes through several sets Z_j and several vertices in W. In the former case, by our assumption on the disjointness graph of Z_j, the length of C is at most g. In the latter case, let us record the vertices of W and the sets Z_j as the cycle passes through them: $z_{i_1}, Z_{j_1}, z_{i_2}, Z_{j_2}, \ldots, z_{i_h}, Z_{j_h}$. Here, the vertices $v_{i_1}, \ldots, v_{i_h}$ are all distinct and, if the same is true for the hyperedges $e_{j_1}, \ldots, e_{j_h}$, then they form a Berge-cycle of length h in the hypergraph H. If the hyperedges are not all distinct, then an even shorter Berge-cycle is formed by any repetition-free interval between two occurrences of the same hyperedge. By our assumption on the girth of H, we have $h \geq g$ in both cases, so all cycles of G have length at least g, as required.

Suppose now that there is a proper k-coloring of G. Restricting it to the set W (and identifying each $z_i \in W$ with the vertex v_i of H), we obtain a k-coloring of the vertices of the hypergraph H. By our assumption, this cannot be a proper coloring. Therefore, there is a monochromatic hyperedge e_j. In this case, no vertex in Z_j can receive the common color of the vertices of e_j, so we have a proper $(k-1)$-coloring of Z_j. This contradicts our assumption on the disjointness graph of Z_j and, thus, proves that G has no proper k-coloring. This concludes the proof of Lemma 6 and, hence, of Theorem 3. $\qquad\square$

James Davies [D21] used a very similar construction to show that there are intersection graphs of axis-parallel boxes and intersection graphs of lines in 3-space with arbitrarily large girths and chromatic numbers.

4 Two-way infinite polygonal chains—Proof of Theorem 5

As we pointed out at the end of Section 2, the class of disjointness graphs of 1-way infinite V-shapes is not χ-bounded. But if we require both ends of a V-shape to be long, the situation will change.

A 2-*way infinite polygonal k-chain* is a continuous curve in the plane consisting of two half-lines connected by an (ordinary) polygonal $(k-2)$-chain. We can relax this definition by requiring only that each of the k pieces are x-monotone, and the first and the last pieces have unbounded projections to the x-axis. In this case, the curve is called a 2-way infinite *k-monotone chain*.

According to this definition, a 2-way infinite polygonal 2-chain (V-shape)) whose half-lines are not vertical is a 2-way infinite 2-*monotone* chain. It can also be regarded as a degenerate 2-way infinite 3-*monotone* chain. Note that by performing a suitable rotation, if necessary, we can always assume that none of the half-line pieces of a finite arrangement of 2-way infinite polygonal k-chains is vertical. Therefore, the following theorem implies Theorem 5.
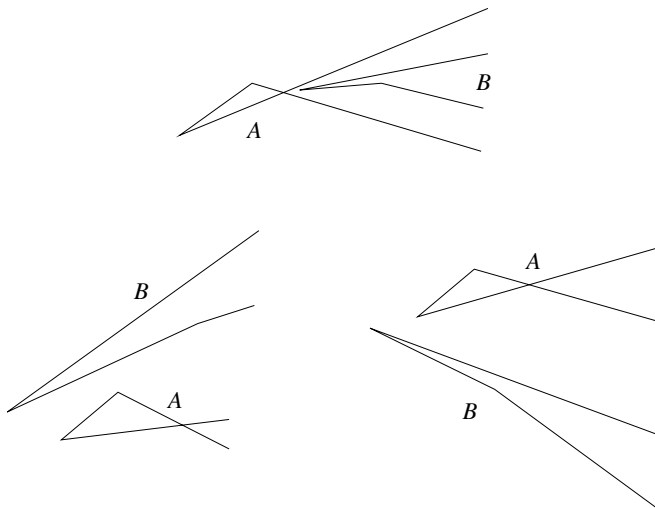

Figure 4: The three partial orders: A is to the left of B, below B, and above B.

Theorem 7. *The disjointness graph G of a finite arrangement of 2-way infinite 3-monotone chains satisfies $\chi(G) \leq (\omega(G))^3 + \omega(G)$.*

Proof. We call a (possibly self-intersecting) 2-way infinite k-monotone chain A *wide* if it intersects every vertical line. A chain A with this property divides the plane into (open) connected components, exactly one of which contains a vertical half-line pointing upwards. We call this component the *upside of A*. For any two wide 2-way infinite k-monotone chains, A and B, we say that A is *higher* than B if A is contained in the upside of A. In this case, the upside of B is also contained in the upside of A. Therefore, the relation "higher" defines a partial

order on any arrangement of wide k-monotone chains. According to this partial order, only *disjoint* pairs are comparable. Since any two disjoint *wide* 2-way infinite k-monotone chains are comparable, the disjointness graph of any collection of wide 2-way infinite k-monotone chains is a comparability graph.

Now we turn our attention to the non-wide case. The complement of a non-wide 2-way infinite k-monotone chain A has precisely one connected component which contains a vertical line. We call this component the *large component*. The chain A is said to be a *right chain* if A is to the right of the vertical lines in the large component, otherwise it is a *left chain*. If A is a right chain, we call its large component the *left side* of A. On the other hand, if A is a left chain, we call the union of *all* connected components of the complement of A, other than its large component, the *left side* of A.

For any two non-wide 2-way infinite k-monotone chains, A and B, we say that A is *to the left* of B if both A and its left side are contained in the left side of B. Obviously, this relation also defines a partial order, with respect to which only disjoint non-wide chains are comparable. It is not true that any two disjoint non-wide 2-way infinite 3-monotone chains are comparable. Therefore, we need to introduce two further partial orders.

For any two subsets of the plane, A and B, we say that A is *below* B (A is *above* B, resp.), if the following two conditions are satisfied:

1. every vertical line that intersects A also intersects B;

2. if $a \in A \cap \ell$ and $b \in B \cap \ell$ for a vertical line ℓ, then the y-coordinate of a is strictly *lower* (*higher*, resp.) than the y-coordinate of b.

Note that "above" and "below" are two separate partial orders and not the inverses of each other. It is clear that both of these relations are partial orders on arbitrary planar sets and that any two comparable sets are disjoint. See Fig. 4.

Lemma 8. *Any two disjoint non-wide 2-way infinite 3-monotone chains, A and B, are comparable by one of the three relations "below", "above", or "to the left".*

To establish the lemma, note that non-wide 2-way infinite 3-monotone chains must be, in fact, 2-way infinite 2-*monotone* chains. A left chain with this property is the union of the graphs of two continuous functions $f_1, f_2 : (-\infty, a] \to \mathbb{R}$, where $f_1(a) = f_2(a)$. Let B be another left chain obtained as the union of the graphs of two continuous functions $g_1, g_2 : (-\infty, b] \to \mathbb{R}$, and assume that A and B are disjoint. We can assume, by symmetry, that $b \leq a$. Consider $g_1(b) = g_2(b)$. It is easy to see that if it is below both $f_1(b)$ and $f_2(b)$, then B is below A. If it is above both $f_1(b)$ and $f_2(b)$, then B is above A. Finally, if $g_1(b)$ is between $f_1(b)$ and $f_2(b)$, then B is to the left of A. A similar argument applies if both A and B are right chains. Finally, if a left chain is disjoint from a right chain, then the left chain is always to the left of the right chain. This completes the proof of Lemma 8.

Now we return to the proof of Theorem 7. Fix a family F of 2-way infinite 3-monotone chains, and let G denote their disjointness graph. Let F_1 and F_2 consist of the wide and non-wide elements of F, respectively. We have seen that the disjointness graph $G[F_1]$ of F_1 is a

comparability graph. Comparability graphs are perfect, so we have $\chi(G[F_1]) = \omega(G[F_1])$. We also proved that the comparability graph $G[F_2]$ of F_2 is the union of three comparability graphs. This implies that $\chi(G[F_2]) \leq (\omega(G[F_2]))^3$.

For the entire graph G, we have

$$\chi(G) \leq \chi(G[F_1]) + \chi(G[F_2]) \leq \omega(G[F_1]) + (\omega(G[F_2]))^3 \leq \omega(G) + (\omega(G))^3,$$

as required. This completes the proof of the theorem. $\qquad\square$

In [PT20], for every $k \geq 2$, we constructed arrangements of x-monotone curves such that their left endpoints lie on the y-axis and their disjointness graphs have clique number k and chromatic number $\binom{k+1}{2}$. We can extend these curves to the left by adding horizontal half-lines without changing their intersection structure. Traversing the resulting curves twice, we obtain families of 2-way infinite 2-monotone chains such that their disjointness graphs satisfy $\chi(G) = \binom{\omega(G)+1}{2}$. We do not know whether the order of magnitude of the bound in Theorem 7 is best possible.

We were unable to improve on the bound in Theorem 7 even for 2-way infinite polygonal 3-chains. The best lower bound we have in this case is $\omega(G)^{(\log 5/\log 2)-1} \approx \omega(G)^{1.32}$, and it follows from a construction in [LMPT94].

5   Concluding remarks.

A. Given an arrangement $\mathcal{C}$ of curves in the plane and a line ℓ, we say that $\mathcal{C}$ is *grounded on* ℓ if every member $c \in \mathcal{C}$ lies in the same closed half-plane bounded by ℓ, and c has precisely one point in common with ℓ, which is one of its endpoints.

The chromatic number of intersection graphs of grounded curves has been extensively studied (see [DK21], for a survey), but less is known about the corresponding problem for disjointness graphs. In the proof of Theorem 1, we constructed arrangements of 1-way infinite V-shapes whose disjointness graphs are triangle-free and whose chromatic numbers are arbitrarily large. Applying a suitable projective transformation, these arrangements can be turned into arrangements of *grounded* V-shapes.

B. In Problem 2, we asked whether the disjointness graph of an arrangement of V-shapes can have simultaneously arbitrarily high chromatic number and girth. The following statement provides an affirmative answer to a relaxed version of this question. The *odd-girth* of a graph is the length of the shortest odd cycle in it (or infinite if the graph is bipartite).

Proposition 9. *There exist arrangements of polygonal 2-chains in the plane whose disjointness graphs have arbitrarily large odd-girths and chromatic numbers.*

Proof. The proof is based on the same idea as the Proof of Theorem 1, where we represented the shift graph S_n as the disjointness graph of an arrangement of V-shapes. The vertices of S_n are pairs (a, b) of integers $1 \leq a < b \leq n$, so they can be associated with the edges of the complete graph K_n. Thus, the vertices of S_n associated with the edges of a subgraph $G \subseteq K_n$ induce a subgraph $G^* \subseteq S_n$. It is easy to verify that for any $G \subseteq K_n$, we have

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(1) $\chi(G^*) \geq \log(\chi(G))$ *and*

(2) *the odd-girth of G^* is strictly larger than the odd-girth of G.*

For any integers g and k, there exist $n = n(g,k)$ and a subgraph $G \subset K_n$ with girth (and, hence, odd-girth) at least g and chromatic number at least k. By properties (1) and (2), the odd-girth of the corresponding induced subgraph G^* of S_n will be larger than g, and its chromatic number will be at least $\log k$. The graph G^* inherits from S_n a representation as a disjointness graph of V-shapes. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Unfortunately, getting rid of short *even* cycles, even 4-cycles, looks impossible by using this simple trick.

C. The arrangements of polygonal curves proving Theorems 1 and 3 have the property that any two of them have at most *two* points in common. It would be interesting to decide whether these theorems remain true if we insist that the curves are *single-crossing*, that is, any two curves have at most one point in common at which they properly cross.

Conjecture 10. *The class of disjointness graphs of single-crossing polygonal 2-chains is χ-bounded.*

Mütze *et al.* [MWW18] proved that the same statement is false for polygonal 3-chains.

D. To prove Theorem 1, we established that the shift graph S_n, a triangle-free graph of unbounded chromatic number, can be obtained as the disjointness graph of V-shapes. However, the *fractional chromatic number* of S_n is bounded: it is smaller than 4 for every n. Do there exist triangle-free disjointness graphs of V-shapes with arbitrarily large fractional chromatic number?

Analogously, our construction for Theorem 3 gives disjointness graphs with bounded fractional chromatic number. Do there exist disjointness graphs of Z-shapes with arbitrarily large girth and fractional chromatic number?

E. It was shown in [PTT17] that the class disjointness graphs of segments in the space is χ-bounded. On the other hand, it follows from Theorem 1, that the class of disjointness graphs of 2-chains (or k-chains, for $k \geq 2$) in the space is not χ-bounded.

For 1-way infinite chains the situation is very similar: the class of disjointness graphs of half-lines in the space is χ-bounded and by Theorem 1, the class of disjointness graphs of 1-way infinite 2-chains (or k-chains, for $k \geq 2$) in the space is not χ-bounded.

By Theorem 1, there is a system of 1-way infinite 2-chains in the plane whose disjointness graphs are triangle-free and have arbitrarily large chromatic number. In the space, we can extend each of them by a half-line which is disjoint from the other chains. This implies that the class of disjointness graphs of 2-way infinite 3-chains is not χ-bounded. The same holds for 2-way infinite k-chains for any $k \geq 3$.

So, the only remaining case is the 2-way infinite 2-chains. We were unable to decide if their disjointness graphs are χ-bounded.

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