A Crossing Lemma for Jordan Curves[*]

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Abstract

If two Jordan curves in the plane have precisely one point in common, and there they do not properly cross, then the common point is called a *touching point*. The main result of this paper is a Crossing Lemma for closed curves: Let X and T stand for the sets of intersection points and touching points, respectively, in a family of n simple curves in the plane, no three of which pass through the same point. If $|T| > cn$, for some fixed constant $c > 0$, then we prove that $|X| = \Omega(|T|(\log\log(|T|/n))^{1/504})$. In particular, if $|T|/n \to \infty$, then the number of intersection points is much larger than the number of touching points.

As a corollary, we confirm the following long-standing conjecture of Richter and Thomassen: The total number of intersection points between n pairwise intersecting simple closed (i.e., Jordan) curves in the plane, no three of which pass through the same point, is at least $(1-o(1))n^2$.

1 Introduction

1.1 Preliminaries

Arrangements of curves and surfaces. It was a fruitful and surprising discovery made in the 1980s that the Piano Mover's Problem and many other algorithmic and optimization questions in motion planning, ray shooting, computer graphics etc., boil down to computing certain elementary substructures (e.g., cells, envelopes, k-levels, or zones) in arrangements of curves in the plane and surfaces in higher dimensions [Ed87, KLPS86, PaS09, ShA95]. Hence, the performance of the most efficient algorithms for the solution of such problems is typically determined by the *combinatorial complexity* of a single cell or a collection of several cells in the underlying arrangement, that is, the total number of their faces of all dimensions.

The study of arrangements has brought about a renaissance of Erdős-type combinatorial geometry. For instance, in the plane, Erdős's famous question [Er46] on the maximum number of times the unit distance can occur among n points in the plane can be generalized as follows [CEGSW90]: *What is the maximum total number of sides of n cells in an arrangement of n unit circles in the plane?* In the limiting case, when k circles pass through the same point p (which is, therefore, at unit distance from k circle centers), p can be regarded as a degenerate cell with k sides.

Several beautiful paradigms have emerged as a result of this interplay between combinatorial and computational geometry, from the random sampling argument of Clarkson and Shor [CS89] through epsilon-nets (Haussler-Welzl [HW87]) to the discrepancy method (Chazelle [Cha00]). It is worth noting that most of these tools are restricted to families of curves and surfaces of *bounded description complexity*. This roughly means that a curve in the family can be given by a bounded number of reals (like the coefficients of a bounded degree polynomial). For the exact definition, see [ShA95].

Another tool that proved to be applicable to Erdős's questions on repeated distances is the *Crossing Lemma* of Ajtai, Chvátal, Newborn, Szemerédi and Leighton [ACNS82, Le83]. It states that no matter how we a draw a sufficiently dense graph $G = (V, E)$ in the plane or on a fixed surface, the number of crossings between its edges is at least $\Omega(|E|^3/|V|^2)$.

In particular, this implies that if G has a lot more edges than vertices (that is, $|E|/n \to \infty$), then its number of crossings is much larger than its number of edges. The best known upper bound on the k-set problem [De98], needed for the analysis of many important geometric algorithms, and the most elegant proofs of the Szemerédi-Trotter theorem [SzT83a], [SzT83b] on the maximum number of incidences between a set of points and a set of lines (or other, more complicated, curves) were also established using the Crossing Lemma [PaS98]. These proofs easily generalize from lines to pseudo-segments (i.e., curves with at most one intersection per pair).

Tangencies and lenses. The Circle Packing Theorem of Koebe, Andreev and Thurston [Koe36, An70, Thu97] implies that any n-vertex planar graph is isomorphic to a graph whose vertices correspond to n non-overlapping disks in the plane, two vertices being connected by an edge if and only if the boundary circles of the corresponding disks touch each other. Conversely, for any set of closed Jordan curves in general position in the plane, with the property that any two curves are either disjoint or touch at a single point, the corresponding touching graph is easily seen to be planar (see, e.g., [KLPS86]).

The present work furthers the above relation by showing, in analogy to the Crossing Lemma, that the number of proper crossing points among n Jordan curves in general position grows faster than the number τ of touching pairs, provided that $\tau/n \to \infty$.

Previously, the study of tangencies in arrangements of curves has been mostly restricted to special families of curves (e.g., boundaries of convex sets or curves of bounded description com-

plexity). Motivated by potential applications to motion planning, Tamaki and Tokuyama [TT98] extended the k-set bounds and incidence bounds from lines to more general curves, by trying to cut the curves into as few pseudo-segments as possible, and then applying the known bounds to them. In this context, the number of *tangencies (touchings)* between the original curves plays a special role. By locally perturbing two curves in a small neighborhood of their touching point, one can create two nearby crossings and a small "lens" between them. In order to decompose the curves into pseudo-segments, we have to make at least one cut on the boundary of each lens. In many scenarios, the number of cuts needed is roughly proportional to the number of touching points, more precisely, to the maximum number of non-overlapping lenses. This approach was later refined and extended in a series of papers [ArS02], [AgS05], [Ch1], [Ch2], [Ch3], [MaT06] and [ANPPSS04].

In particular, Agarwal *et al.* [ANPPSS04] studied arrangements of pseudo-discs (that is, closed Jordan curves with at most two intersections per pair) and used lenses to establish several fundamental results on geometric incidences and cell complexity. Their analysis crucially relied on the following claim: Any family of n pairwise intersecting pseudo-circles admits at most $O(n)$ tangencies. In the special case where the curves are algebraic, any incidence or tangency can be described by a polynomial equation. Following the pioneering work of Dvir [Dv10], Guth and Katz [GK10], [GK15], many of these problems have been revisited from an algebraic perspective.

The structure of tangencies between *convex sets* was addressed in [PST12]. It was shown that the number of tangencies between n members of any family of plane convex sets that can be obtained as the union of k packings (systems of disjoints sets) is at most $O(kn)$. The proof of this fact is somewhat delicate, because the boundaries of two convex sets can cross any number of times.

Richter-Thomassen Conjecture. Richter and Thomassen conjectured in 1995 [RiT95] that the total number of intersection points between n pairwise intersecting closed Jordan curves in general position in the plane is at least $(1 - o(1))n^2$.

Note that if there are no tangencies between the curves, then any two curves intersect at least twice, so that the number of intersection points is at least $2\binom{n}{2} = (1-o(1))n^2$. However, if touchings are allowed, the situation is more complicated.

The best known general lower bound is due to Dhruv Mubayi [Mu02], who showed that the number of intersection points is at least $(4/5-o(1))n^2$. If any pair of curves have at most a bounded number of points, then the conjecture follows from the Kővári–Sós–Turán Theorem [KST54] in extremal graph theory, as proved by Salazar [Sa99]. In an earlier paper [PRT15], the authors settled the special case where the curves are *convex* or, more generally, if each curve can be cut into a constant number of x-monotone arcs. (An arc is called *x-monotone* if every vertical line intersects it in at most one point.) The problem has remained open for general families of simple closed curves.

Algebraic techniques. As mentioned before, the polynomial technique of Guth and Katz [GK10, G15], which led to a spectacular breakthrough concerning Erdős's problem on distinct distances, has inspired a lot of recent research related to incidences between points, curves, and surfaces [G, G13, GK15, SSZ15]. For instance, Ellenberg, Solymosi and Zahl [ESZ16] have shown that any family of low-degree algebraic curves in the plane determines $O(n^{3/2})$ tangencies (where the constant of proportionality can depend on the maximum degree of the curves). Unfortunately, the new techniques only apply in an algebraic framework, where the curves and surfaces in question must be algebraic varieties of bounded degree. Since two algebraic curves of bounded degree that do not share a component have only a bounded number of points in common, restricting the Richter-Thomassen conjecture to such curves, reduces the question to the above mentioned result of Salazar [Sa99]. For many similar problems related to intersection patterns of curves, including the special

case of the Erdős-Hajnal conjecture[1] [EH89], our present techniques are not sufficient to handle the case when two curves may intersect an arbitrary number of times [FPT11, FP08, FP10]. There are only very few exceptional examples, when one is able to drop this assumption [Ma14, FP12]. The main result of this paper represents one of the rare exceptions.

1.2 Our results

The main result of this paper is a Crossing Lemma for a family of Jordan curves. We are going to show, roughly speaking, that the number of *proper* (i.e., transversal) crossings between the curves is much larger than the number of touching pairs of curves, provided that the number of touching pairs is super-linear in the number of curves.

To formulate this result more conveniently, we need to agree on the terminology. We say that two (open or closed) curves *intersect* if they have at least one point in common. An intersection point p is called a *touching point* (in short, a *touching*) if p is the *only* intersection point of the two curves, and they do not properly cross at p. Note that this definition is somewhat counterintuitive: we do not call a point of tangency between two curves a touching if the curves also intersect at another point. Without this restriction we cannot claim that there are much more crossings than touching points. Indeed, consider n lines in general position in the plane. Notice that one can slightly perturb them to turn each crossing into a proper crossing and a separate point of tangency. In such an arrangement, half of the $2\binom{n}{2}$ intersection points are tangencies. It is assumed throughout that all curves are in *general position*, that is, no three of them pass through the same point and no two share infinitely many points.

We state our Crossing Lemma in two forms. First, we formulate it for pairwise intersecting closed curves. In this formulation, we can prove a slightly better asymptotic gap between the number of intersections and the number of touchings:

Theorem 1 *Let $\mathcal{A}$ be a collection of n pairwise intersecting closed Jordan curves in general position in the plane. Let T denote the set of touching points and let X denote the set of intersection points between the elements of $\mathcal{A}$. We have*[2]

$$\frac{|X|}{|T|} = \Omega\left((\log\log n)^{1/12}\right).$$

We will see that Theorem 1 is an easy corollary to its bipartite version:

Theorem 2 *Let $\mathcal{F}$ and $\mathcal{G}$ be two disjoint collections of closed Jordan curves in the plane, each consisting of n curves. Suppose that $\mathcal{F} \cup \mathcal{G}$ is in general position and that any two curves from the same collection intersect. Let T denote the set of all touching points between the curves in $\mathcal{F}$ and the curves in $\mathcal{G}$, and let X denote the set of all intersection points between the elements of $\mathcal{F} \cup \mathcal{G}$. We have*

$$\frac{|X|}{|T|} = \Omega\left((\log\log n)^{1/12}\right).$$

Most of this paper is devoted to the proof of Theorem 2. Theorem 1 can be deduced from Theorem 2, as follows.

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[1]The conjecture, as applied to string graphs, claims that there exists $\epsilon > 0$ such that among n Jordan curves in the plane one always finds at least n^ϵ pairwise intersecting or at least n^ϵ pairwise disjoint ones.

[2]The preliminary version [PRT16] states a too optimistic (and, unfortunately, less accurate) estimate of $\Omega\left((\log\log n)^{1/8}\right)$.

Proof of Theorem 1 (using Theorem 2): Assume without loss of generality that n is even and consider a random partition of the curves of $\mathcal{A}$ into $n/2$-sized families $\mathcal{F}$ and $\mathcal{G}$. Notice that the expected number of the original touchings with the two touching curves ending up in distinct families $\mathcal{F}$ and $\mathcal{G}$ is at least $|T|/2$. The overall number of intersection points does not change, so applying Theorem 2) to the families $\mathcal{F}$ and $\mathcal{G}$ proves the statement of Theorem 1. ♠

We use Theorem 1 to settle the Richter-Thomassen conjecture [RiT95]:

Theorem 3 *The total number of intersection points between n pairwise intersecting closed curves in general position in the plane is at least $(1 - o(1))n^2$.*

Proof of Theorem 3 (using Theorem 1): It is enough to notice that if $|T| = o(n^2)$, then the statement follows from the trivial bound $|X| \geq 2\binom{n}{2} - |T|$. Otherwise, if $|T| \geq \varepsilon n^2$ for some $\varepsilon > 0$, Theorem 1 immediately implies that $|X| = \Omega(n^2(\log \log n)^{1/12})$, which is much better than required. ♠

The assumption that the curves are closed is crucial for Theorem 3, as a family of pairwise-intersecting segments in general position in the plane determines only $\binom{n}{2} < n^2/2$ intersections. However, both Theorem 1 and Theorem 2 readily extend to families of Jordan arcs. Indeed, slightly inflating the Jordan arcs to closed Jordan curves, while preserving all of the touching pairs, we increase the number of intersection points by a factor of at most 4.

Next, we formulate a version of Theorem 1 without the assumption that the curves are pairwise intersecting. Note, however, that one may draw n circles with up to $3n - 6$ touchings and no proper crossings [ANPPSS04, PaS09]. One might believe that some linear lower bound on the number of touching, like $|T| \geq 10n$ should be enough for us to prove a separation $|T| = o(|X|)$, but this is false as shown by the following example. Fix a large constant k and consider $n - k$ pairwise disjoint unit circles in the plane. It is easy to select k other closed curves in general position such that each of them touches every circle and any pair of them intersect at most $n - k + 1$ times. In this arrangement, $|T| = k(n - k)$ and $|X| \leq |T| + \binom{k}{2}(n - k + 1)$, so that we have $\frac{|X|}{|T|} \leq k$, a constant. This motivates to formulate our lower bound on $\frac{|X|}{|T|}$ not as a function of n assuming a lower bound on $|T|$, but rather as a function of $|T|/n$. We conjecture the following strong bound. We formulate it for Jordan arcs and not for closed Jordan curves.

Conjecture 1 *Let $\mathcal{A}$ be a collection of n Jordan arcs in general position in the plane. Let T denote the set of touching points and X the set of intersection points between the elements of $\mathcal{A}$. We have*

$$\frac{|X|}{|T|} = \Omega\left(\log \frac{|T|}{n}\right).$$

The conjectured logarithmic separation between X and T, if true, cannot be improved. Indeed, Fox *et al.* [FFPP10] constructed two n-sized families, $\mathcal{F}$ and $\mathcal{G}$, of pairwise intersecting x-monotone curves in the plane such that every curve in $\mathcal{F}$ touches every curve in $\mathcal{G}$, and the total number of intersections between the members of $\mathcal{F} \cup \mathcal{G}$ is $O(n^2 \log n)$. They also showed that for this setting one always has $\Omega(n^2 \log n)$ intersections.

Though we have been unable to verify Conjecture 1, we can deduce the following weaker bound from Theorem 2.

Theorem 4 *Let $\mathcal{A}$ be a collection of n Jordan arcs in general position in the plane. Let T denote the set of touching points and X the set of intersection points between the elements of $\mathcal{A}$. We have*

$$\frac{|X|}{|T|} = \Omega\left(\left(\log \log \frac{|T|}{n}\right)^{1/504}\right).$$

1.3 Organization and overview

The paper is organized as follows.

In Section 2, we prove our main technical tool, Theorem 2. Unlike many previous bounds on the crossing numbers of geometric structures, which relied on Euler's formula [ACNS82, Le83] or parity arguments from topology [Tut70], our analysis is based on a more local machinery of *charging schemes* – a powerful yet simple method developed in Computational Geometry to estimate the number of special features of bounded description complexity in arrangements of algebraic curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^d$; see [ShA95, Section 7] for a comprehensive demonstration of this technique.

In the most typical planar scenario, we are given an arrangement of n algebraic curves and seek a non-trivial upper bound on the number of "special" vertices which satisfy a certain topological condition (e.g., vertices that lie on the boundary of a given face). That is, we are to show that the concerned vertices are relatively scarce, and the vast majority of the intersection points do not possess the desired property. To this end, we assign each special vertex v to several other vertices v' in the arrangement. The assignment is fractional and specified by a rule in which v "receives" at least c_{in} units of charge from the other vertices v'. In most instances, the charging rule is of a fairly local nature and respects some natural criterion of proximity between v and v' within the arrangement. A successful charging scheme must guarantee that the total charge "sent" by any vertex v' is much smaller than c_{in}.

We adapt the above charging paradigm to show that only few of the intersection points can be touchings. Since the original machinery applies (with very few exceptions) only to objects of bounded description complexity, the adaptation requires extra care to avoid "overcharging" of intersection points.

In Section 3, we establish Theorem 4. The proof proceeds as follows. First, we use a simple sampling argument to replace the hypothesis of Theorem 1 with a somewhat weaker one – the intersection graph must be rather dense. To deduce Theorem 4 from the "dense" Crossing Lemma, we partition the arrangement into sufficiently dense pieces by repeatedly applying the separator result of Fox and Pach [FP08].

2 Proof of Theorem 2

This is the most complex part of the paper. We start with a brief and informal outline of the proof. First we bound the number of touching points $t \in T$ that are contained in an arc of arbitrary size whose "crossing to touching ratio" is high. We will call these "happy" touching points and bounding their number relative to the number $|X|$ of intersection points is simple. The rest of the touching points we call "sad" and denote their set by T'. We use the so-called charging method to bound $|T'|$: we send certain amounts of "charge" from points in X to points in $|T'|$. If we manage to make sure that the total charge sent by any point in X is at most c_{out} and the total charge received by any point $t \in T'$ is at least c_{in}, then we have established that $|X|/|T'| \geq c_{\mathrm{in}}/c_{\mathrm{out}}$. Note that we used the same method in our paper [PRT15] to prove certain special cases of the Richter–Thomassen conjecture. For most of the charging argument, there is no need to restrict our attention to sad touching points, but at one crucial point, namely in the proof of Claim 6, it helps us that we have already taken care of all happy ones.

The proof of Theorem 2 is organized as follows. In Section 2.1, we fix the arrangement of curves, define happy and sad touching points and bound the number of happy touching points relative to the number of intersection points.

In Section 2.2, we define our three charging rules by which intersection points send specified

amount of "charge" to sad touching points. These rules involve a parameter k that we call the "scale". The charging rules should be applied in many phases, each phase with a different scale.

In Section 2.3, we specify the number M of phases and the corresponding values of the scale parameter k. We establish a constant upper bound on the amount of charge sent by any point of X, averaged over the M phases. For the first and third charging rules, we have a constant upper bound in each individual phase, but for the second rule this is not the case, and here the appropriate choice of scales is important. In the rest of the proof, it is almost irrelevant how we set the value of the scale parameter k (within reasonable limits).

Section 2.4 is devoted to proving a lower bound of $\Omega((\log\log n)^{1/12})$ on the amount of charge received by a sad touching point in any given phase. We will use Claim 6 in this proof, whose proof is technically involved and is postponed to Section 2.6.

In Section 2.5, we finish the proof of Theorem 2 by bounding the number of sad touching points.

2.1 Happy and sad touchings

Let us fix the families $\mathcal{F}$ and $\mathcal{G}$ of n closed Jordan curves, as in the statement of Theorem 2. For the sake of brevity we write $\mathcal{A} = \mathcal{F} \cup \mathcal{G}$. Let $\alpha_1 = (\log\log n)^{1/12}/10$, where log denotes the binary logarithm. We need to prove $|X|/|T| = \Omega(\alpha_1)$. For this proof we assume that $|X| \leq \alpha_1 n^2$ as otherwise the statement follows from the trivial bound $|T| \leq n^2$. Note that as the statement we want to prove is asymptotic we can simplify our calculations by always assuming that n is large enough.

We call an arc a^* contained in one of the curves $a \in \mathcal{A}$ *happy* if $|X \cap a^*| \geq \alpha_1 |T \cap a^*|$. We say that a touching point $t \in T$ is *happy* if it is contained in a happy arc, otherwise we call t *sad* and denote the set of sad touching points by T'.
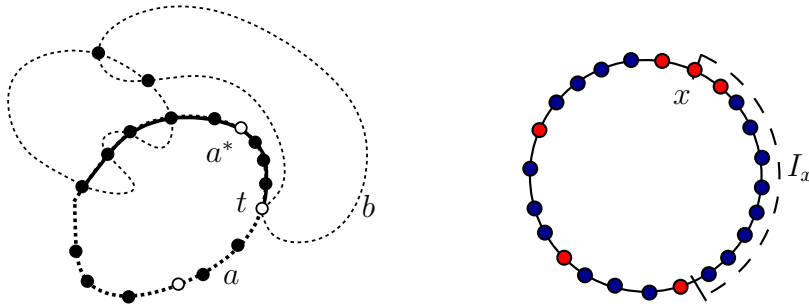

Figure 1: Left – The touching t is happy. The hollow points belong to $T \cap a$. The arc $a^* \subset a$ satisfies $|X \cap a^*| \geq \alpha_1 |T \cap a^*|$ with $\alpha_1 = 5$. Right – Lemma 1. Each red point $x \in R$ is contained in an arc I_x that satisfies $w(B \cap I_x) \geq \lambda |R \cap I_x|$. (In the depicted scenario $w(x) = 1$ for all x and we have $\lambda = 4$.)

To bound the number of happy touching points, we use the following simple lemma. See Figure 1 (right). Notice that the sets R and B may overlap. In the use of this lemma below the weight function w is constant. We still formulate the lemma with an arbitrary positive weight function w because we will use the same lemma a second time later in this section and there we will use a non-constant weight function.

Lemma 1 *Let a be a simple (open or closed) Jordan curve and let R and B be two finite subsets of points on a. Let λ be a positive constant and let $w : B \to \mathbb{R}$ be a positive weight function. For $S \subseteq B$ the weight of S is $w(S) = \sum_{x \in S} w(x)$. If every point $x \in R$ is contained in an arc $I_x \subseteq a$ that satisfies $w(B \cap I_x) \geq \lambda |R \cap I_x|$, then we have $w(B) \geq \lambda |R|/3$.*

Proof: We prove by induction on $|R|$. The claim trivially holds if R is empty, so we assume R is not empty and the statement of the lemma holds for R' and B' as long as $|R'| < |R|$.

Let us choose $x \in R$ to maximize $|R \cap I_x|$ breaking ties arbitrarily. Let $B' = B \setminus I_x$ and $R' = \{y \in R \mid I_y \cap I_x = \emptyset\}$. For $y \in R'$ we have $B' \cap I_y = B \cap I_y$ and $R' \cap I_y \subseteq R \cap I_y$, so the assumption of the lemma is satisfied for R' and B'. As $x \notin R'$ we have $|R'| < |R|$ and thus, by the inductive hypothesis, we have $w(B') \geq \lambda |R'|/3$. By the choice of x every $y \in R \setminus R'$ must either be in I_x or it is one of the $|I_x \cap R|$ next points in R in either side of the arc I_x. So we have $|R| - |R'| \leq 3|I_x \cap R|$. We further have $w(B) - w(B') = w(I_x \cap B) \geq \lambda |I_x \cap R| \geq \lambda(|R| - |R'|)/3$. Adding this inequality to the one obtained from the inductive hypothesis finishes the proof. ♠

Lemma 2 *Let T and X be the respective sets of touching points and intersection points as defined in Theorem 2, and let $T' \subset T$ be the set of sad touching points. Then we have $|T| - |T'| \leq 6|X|/\alpha_1$.*

Proof: We apply Lemma 1 for each curve $a \in \mathcal{A}$ with $\lambda = \alpha_1$, $B = B_a = a \cap X$ and $R = R_a$ being the set of touching points contained in a happy arc $a^* \subset a$. We use the uniform weight function $w(x) = 1$ for each $x \in B_a$. We obtain $|B_a| \geq \alpha_1 |R_a|/3$. Summing this for all $a \in \mathcal{A}$ we get $2|X|$ on the left hand side and at least $\alpha_1(|T| - |T'|)/3$ on the right hand side. ♠

2.2 The charging rules

As mentioned in the outline above we bound $|T'|$ using a charging scheme.

Each charging scheme describes a fractional assignment of the elements of a set A to the elements of another set B, and can be described as a weight assignment to the edges of the complete directed bipartite graph $B \times A$. In the language of charging schemes, $a \in A$ *receives* $w(b, a)$ units of charge from b, whilst b *sends* $w(a, b)$ units to a.

The eventual upper bound on $|A|$ in terms of $|B|$ depends on the minimal weighted indegree $c_{\text{in}} = \min_{a \in A} \sum_{b \in B} w(b, a)$ and the maximum weighted outdegree $c_{\text{out}} = \max_{b \in B} \sum_{a \in A} w(b, a)$. With these parameters, a standard double counting argument shows that $\frac{|A|}{|B|} \leq \frac{c_{\text{out}}}{c_{\text{in}}}$.

Our charging is done in phases: in each phase we fix the value of the parameter k ("the scale") and perform certain chargings with that scale. Our goal is to make sure that each point in X sends out a constant charge in each phase, while each touching in T' receives a charge of $\Omega(\alpha_1)$ in each phase. If we could do this, then a single phase would be enough to prove Theorem 2. But we will not quite achieve this goal. Some points in X will be overcharged in certain phases: they send out more than a constant amount of charge. This problem is solved by considering several phases at once. The exact values of the scale parameter k will be set in Section 2.3 to ensure that *on average* no intersection is overcharged.

Arcs and lenses. Before specifying the exact charging rules we introduce some notation. We orient each curve a in $\mathcal{F}$ so that all other curves from $\mathcal{G}$ touching a touches it on its right side. This is possible as if $a \in \mathcal{F}$ has a touching curve on either side, then these curves are not intersecting counter to our assumption that each pair of curves in $\mathcal{G}$ intersect. We similarly orient the curves of $\mathcal{G}$. We use the word *arc* for closed segments of the curves in $\mathcal{A}$. We will use lowercase letters with an asterisk to denote arcs. The arcs inherit their orientation from the curve of $\mathcal{A}$ containing them and this orientation distinguishes the *starting point* and the *end point* of an arc. For distinct points p and q in a curve $a \in \mathcal{A}$ we write *the arc of a from p to q* to refer to the single arc on a with p as its starting and q as its end point. We can simply refer to an arc as "the arc from p to q" unless $p, q \in X$ represent two intersections of the same two curves from $\mathcal{A}$. The orientation also makes

references like "the next k points of T along a after p", or "the last k points of T along a before p" unambiguous. By the *length* of an arc we mean the number of sad touching points it contains. Let $x \in X$ be a non-touching intersection point of the curves $a, b \in \mathcal{G}$, and let y be another intersection point of the same two curves, the next such point along a.

We call the arc of a from x to y a *lens*. (Note that most texts include both arcs from x to y in their definition of a lens, but for us it is simpler to focus on a single arc. We will only use the term lens for lenses determined by curves in $\mathcal{G}$.)

We set the following parameters: $\alpha = \alpha_1 + 2$, $v = 21000\alpha^{12}$ depending only on n and the parameter $w = w(k) = k^3/(4000\alpha^5 n^2)$ that also depends on the scale.

Let us consider the phase with scale k. We start with describing our three charging rules sending charges from intersection points in X to sad touchings in T'. See Figure 2.
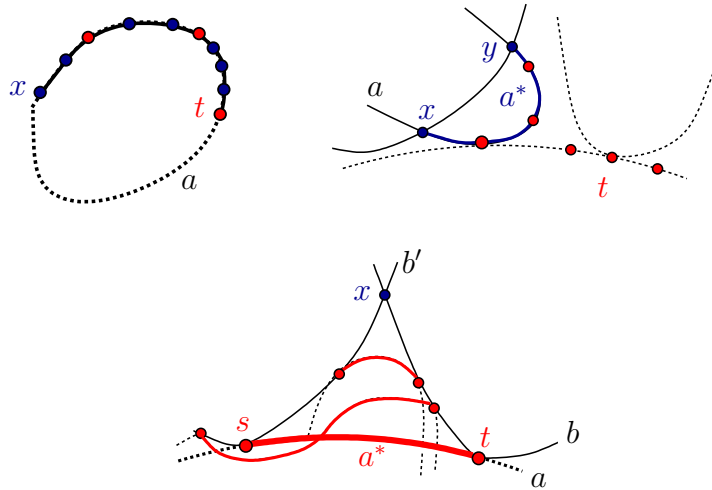

Figure 2: Left: First charging rule. A point $x \in X$ sends $1/k$ units to a sad touching $t \in T'$ if the interval from t to x (or vice versa) has length at most k. Right: Second charging rule. The sad touching t receives $v/(k(l+w))$ units from the lens a^* with endpoints x and y. Bottom: Third charging rule. The starting point t of the poor arc a^* receives $2a/k$ units from the apex $x \in X$ because there exist at most k/a other poor arcs with apex x that start between x and t.

First charging rule. A point $x \in X$ sends a charge of $1/k$ to a point $t \in T'$ if the two points are on a common curve of $\mathcal{A}$ and either the arc from x to t or the arc from t to x has length at most k.

Second charging rule. If the length l of a lens a^* satisfies $l \leq 3\alpha^3 k$, then a^* sends a charge of $v/(k(l+w))$ to all points $t \in T'$ that have an arc of length at most $k+1$ along a curve in $\mathcal{F}$ from t to a point in $a^* \cap T'$.

For accounting purposes, we consider a charge sent by a lens a^* to be sent by the starting point of a^*. Note that exactly two lenses starts at every non-touching intersection point between two curves from $\mathcal{G}$.

We call a point of T' *poor* in this phase if it receives less than a total charge of α from the first two charging rules. We call an arc *poor* if it starts at a poor touching point, ends at a sad touching point and has length at most $k+1$.

Let a^* be an arc of a curve $a \in \mathcal{F}$ starting at $t \in T'$ and ending at $s \in T'$. Let b and b' be the curves in $\mathcal{G}$ touching a in the points t and s, respectively. We define the *apex* of the arc a^* as the

first point on b' after s that also belongs to b. This is a well defined point in X as b and b' (as any pair of curves in $\mathcal{G}$) must intersect.

Third charging rule. Let a^* be a poor arc starting at $t \in T'$ and having $x \in X$ as its apex. The intersection point x sends a charge of $2\alpha/k$ to t in this phase unless there are more than k/α poor arcs, each starting at a point in the arc from x to t and having x as its apex.

2.3   Total charge sent

Lemma 3 *The total charge sent from a intersection point $x \in X$ in a phase according to the first and third rules is at most 8.*

Proof: The first rule sends a charge of $1/k$ from x to the first k sad touching points in each of four "directions" (in both directions of both curves containing x). That is at most $4k$ sad touching points for a total charge of at most 4.

The third rule sends a charge of $2\alpha/k$ to the first $\lfloor k/\alpha \rfloor$ touching points from x along either curves containing x satisfying a certain condition (namely being the starting point of a poor arc having x as its apex) for a total charge of at most 4. ♠

Note that both the first and the third rule charges an intersection point of two curves in $\mathcal{G}$ irrespective of the scale. In contrast, each lens has an intrinsic length value l, which roughly describes the scale k of the phase where this vertex can be charged via the second rule. A statement similar to Lemma 3 is false for the second charging rule because it severely overcharges the lenses whose length is approximately k. Observe, however, that the rule does not charge a lens longer than $3\alpha^3 k$ and charges it very lightly if the lens is much shorter than $w = w(k)$. This is enough for us to set up the scales of the different phases in such a way that no intersection point is overcharged on average.

For technical reasons, in phase k we overcharge lenses of length l within a fairly large interval, namely for $k^3/(n^2 poly(\alpha)) < l < k\, poly(\alpha)$. To avoid overcharging the same lens in many phases, we can only have $O(\log \log n)$ phases. This is the reason that our lower bound on $|X|/|T|$ in Theorem 2 (and as a consequence also in Theorems 1 and 4) is substantially weaker than the similar bound in [PRT15].

We use the following scales for the different phases of our charging: $k = 80\alpha^4 n/2^{3^u}$, where u is an integer satisfying $(\log \log n)/5 < u \le (\log \log n)/2$. We have $M = \lfloor (\log \log n)/2 \rfloor - \lfloor (\log \log n)/5 \rfloor$ phases.

Lemma 4 *For any intersection point $x \in X$, the charge leaving x by the second rule averaged over the M phases is at most 2.*

Proof: Each non-touching intersection point of two curves of $\mathcal{G}$ is the starting point of at most two lenses (no lens starts at a touching point). We bound the average charge sent by a fixed lens a^* by 1. Let l be the length of a^* and let k_0 be the smallest scale of a phase where the lens a^* is charged. We have $l \le 3\alpha^3 k_0$ and the total charge a^* sends in this phase is less than v. For phases with scale $k > k_0$, we bound the charge a^* sends by $vl/w(k) \le 3v\alpha^3 k_0/w(k)$. With the prior choice of scales k and the parameter $w(k)$, we have $w(k) \ge 3\alpha^3 k_0$ and the value of $w(k)$ grows by a factor greater than 2 every time we go from a scale to a larger scale. Thus, the total charge a^* sends in all the phases is at most $3v$. With our choice of the parameters, we have $M \ge 3v$ and this proves the estimate claimed. ♠

2.4 Total charge received.

Our goal in this section is to prove the following lemma.

Lemma 5 *Every sad touching point receives a total charge of at least α in every phase.*

We start with an informal summary of the argument.

Let $t \in T'$ be a sad touching point between a pair of curves $a \in \mathcal{F}$ and $b \in \mathcal{G}$. We consider the sequence $t_1, \ldots, t_k$ of the first k sad touching points that follow t along a.

We can assume that t is poor, as otherwise it receives enough charge by the first two rules. This implies, in particular, that the arc a_k^* of a from t to t_k (of length exactly $k+1$) contains fewer than αk intersection points of X, as all these points send a charge of $1/k$ to t by the first rule.

For each $1 \le i \le k$, we consider the apex x_i of the arc a_i^* of length $i+1$ from t to t_i; see Figure 3 (left). Notice that x_i is an intersection point of b and another curve $b_i \in \mathcal{G}$ touching a at t_i.
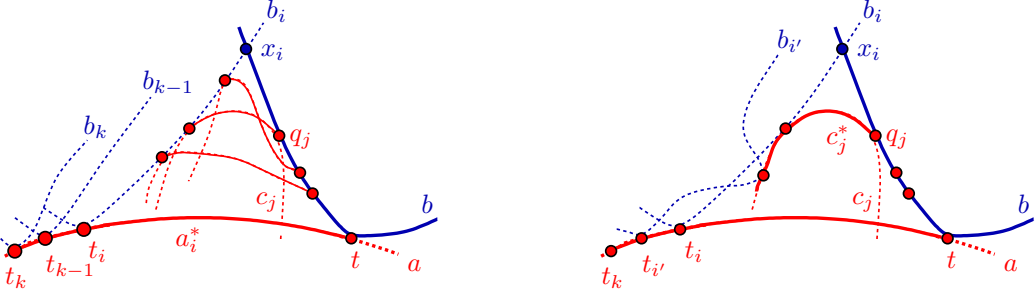

Figure 3: Left – The poor point t receives less than α units of charge through the third charging rule. For most $1 \le i \le k$ the arc a_i^* shares its apex x_i with k/α poor arcs whose underlying curves c_j meet a_i^*. Right – The curve $c_j \in Q_i \cap Q_{i'}$ is simultaneously tangent to both b_i and $b_{i'}$.

We further assume, for contradiction, that t receives less than α units of charge by the third rule from the apex points x_i. This implies that at least half of these points x_i do not send charge to t by the third rule. For each of these points x_i, not sending charge to t, there exist more than k/α poor arcs with apex x_i, each starting at a poor point q_j on the portion of b from x_i to t. We use Q_i to denote the set of such poor points q_j that are associated with x_i. For each of these poor arcs, its underlying curve c_j must intersect a within a_i^* or, else, it would be trapped in the region of $\mathbb{R}^2 \setminus (a \cup b \cup b_i)$ to the right of the arc a_i^* (and, thereby, remain disjoint from a). This implies that the total number of such curves c_j associated with at least one of the apexes x_i cannot exceed αk. Since the overall number of touchings between the curves b_i and c_j is at least $(k/2) \cdot (k/\alpha)$, the resulting bipartite graph of tangencies has density at least $1/(2\alpha^2)$.

All of the above touching points between $c_j \in \mathcal{F}$ and the curves $b_i \in \mathcal{G}$ must lie within an arc c_j^* of c_j of length $k+1$ which starts at q_j. Using that q_j is poor, the arc c_j^* contains at most αk points of X.

Since the graph of touchings between the curves $b_i \in \mathcal{G}$ and $c_j \in \mathcal{F}$ is dense, for an average pair $1 \le i, i' \le k$ there exist $|Q_i \cap Q_{i'}| = \Omega^*(k)$ curves c_j that are simultaneously tangent to both b_i and $b_{i'}$.[3] See Figure 3 (right).

Our parameters are fine-tuned so as to interpolate between the following extreme scenarios:

---

[3]The $O^*()$ and $\Omega^*()$ notation hides multiplicative factors of α. This notation is only used in this informal proof sketch.

(i) Any two points $q_j, q_{j'} \in Q_i \cap Q_{i'}$ are close along b in the following sense: the arc of b from q_j to $q_{j'}$, or the complementary arc from $q_{j'}$ to q_j, contains at most $2\alpha k$ points of T. Then the two curves $b_i, b_{i'} \in \mathcal{G}$ form a lens of length $l = O^*(k)$ as $b_{i'}$ enters the pocket formed by the touchings between b_i and each of the curves c_j and $c_{j'}$, or vice versa. See Figure 4 (left). The resulting lens of b_i and $b_{i'}$ can send, by our second rule, $\Omega^*(1/k^2)$ units of charge to one of the touchings $q_j, q_{j'}$. Repeating this argument for $\Omega(k^2)$ pairs $1 \leq i, i' \leq k$ would eventually contradict the choice of q_j as poor touching points.
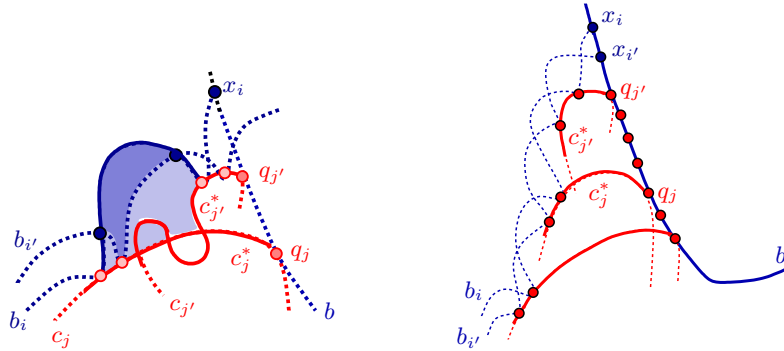

Figure 4: Left – In the first charging scenario, the poor touchings $q_j, q_{j'} \in Q_i \cap Q_{i'}$ are close along b^* (so they can meet one another). As $b_{i'}$ enters the pocket formed by the touchings between b_i and each of the curves $c_j, c_{j'}$, the curves b_i and $b_{i'}$ define a lens of length $l = O^*(k)$ which can send $\Omega^*(1/k^2)$ units to q_j. Right – In the second charging scenario, the pairwise disjoint arcs $c_j \in Q_i$ form the teeth of a comb-like arrangement.

(ii) No two points $q_j, q_{j'} \in Q_i \cap Q_{i'}$ are close along b. We argue that, for any $q_j, q_{j'} \in Q_i \cap Q_{i'}$, the respective short arcs $c_j^* \subset c_j$ and $c_{j'}^* \subset c_{j'}$ are disjoint and, therefore, they constitute the "teeth" of the comb-like arrangement Γ of these arcs together with b; see Figure 4 (right). It then follows that b_i and $b_{i'}$ experience at least $\Omega^*(k)$ intersections, as they touch the neighboring pairs of the teeth of Γ. Repeating this for $\Omega(k^2)$ pairs $1 \leq i, i' \leq k$, would contradict the initial assumption that the total number of intersection points satisfies $|X| \leq \alpha_1 n^2$.

We make the above argument formal and prove Lemma 5 through a series of small claims. We start with a simple observation that will allow us to speak about "the next k sad points" after a poor point on a curve:

Claim 1 *If a curve $a \in \mathcal{A}$ contains at most k sad points, then none of them is poor.*

Proof: Clearly, if $|a \cap T'| \leq k$, then every intersection point in $X \cap a$ sends a charge of $1/k$ to every point in $T' \cap a$ according to the first rule. The claim follows as there are at least $n - 1$ intersection points on a. ♠

For the proof of Lemma 5 we fix the phase with scale k and we also fix a single sad touching point $t \in T'$. We assume for contradiction that t receives a total charge of less than α. Note first that our assumption implies that t is poor.

Let the curves touching at t be $a \in \mathcal{F}$ and $b \in \mathcal{G}$. Let $t_1, t_2, \ldots, t_k$ be the first k sad touching points after t along a. By Claim 1, these exist. For $1 \leq i \leq k$, let a_i^* be the arc of a from t to t_i, let x_i be the apex of a_i^* and let b_i be the curve in $\mathcal{A}$ that touches a at t_i.

Defintion. We call a poor arc i-*fast* if it starts at a point in the arc from x_i to t and has x_i as its apex, see Figure 5 (left). We call an arc *fast* if it is i-fast for some $1 \leq i \leq k$.

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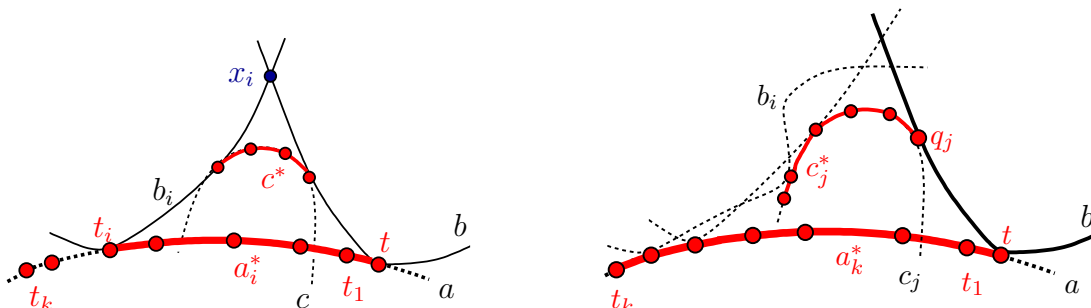By the third charging rule, each apex point x_i either transfers $2\alpha/k$ units of charge to t or gives rise to k/α i-fast arcs.

Figure 5: Left – A poor arc c^* is i-fast if it has apex x_i and starts on the arc of b from x_i to t. Right – A good point q_j together with the adjacent arc $c_j^* \subset c_j$. Notice that c_j must meet a within a_k^*, and c_j^* contains all the fast arcs that start at q_j.

Claim 2 *There are more than $k^2/(2\alpha)$ fast arcs.*

Proof: Note that a_i^* itself is i-fast and, therefore, t receives a charge of $2\alpha/k$ from x_i according to the third charging rule, unless there are more than k/α i-fast arcs. As t receives a total charge of less than α, we must have more than k/α i-fast arcs for each of more than $k/2$ different values of i. This proves the claim. ♠

Note that all fast arcs start at a sad touching point on b, and k of them start at t. We call a point *good* if at least $k/(4\alpha^2)$ fast arcs start there. Let us name the good points $q_1, \ldots, q_L$ in the order they appear on b starting at $q_1 = t$ and going along b in the reverse direction. For $1 \leq j \leq L$, let $c_j \in \mathcal{A}$ be the curve that touches b at q_j and let c_j^* be the unique arc from q_j to a point in $c_j \cap T'$ of length exactly $k + 1$. The existence follows from Claim 1. In particular, we have $c_1^* = a_k^*$. See Figure 5 (right).

Claim 3 *With the previous notation, the following is true.*

  (i) *All good points are poor.*

 (ii) *We have $|c_j^* \cap X| < \alpha k$ for all $1 \leq j \leq L$.*

(iii) *The number of good points is $L \leq \alpha k$.*

(iv) *At least $k^2/(4\alpha)$ fast arcs start at a good point.*

 (v) *Any i-fast arc that starts at one of the good points q_j ends at the point where b_i touches c_j, and it is contained in c_j^*.*

Proof: The first statement holds, because any fast arc starts at a poor point, by definition.

The second statement follows, as each point in $c_j^* \cap X$ sends a charge of $1/k$ to the poor point q_j, by the first rule.

We prove the third statement in a stronger form: the same bound holds for the number L' of all starting points of fast arcs. Let c^* be an i-fast arc starting at $q \neq t$. The curve $c \in \mathcal{A}$ containing c^* must intersect a and, thus, it must escape the triangle like region bounded by the arc of b from x_i

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to t, a_i^* and the arc of b_i from t_i to x_i. It cannot cross b or b_i, so it must leave through (or touch) a_i^*, "using up" at least one of the at most αk intersection points on a_k^*, as a_i^* is contained in a_k^*. Therefore, we have $L \leq L' \leq \alpha k$.

To see the fourth statement, note that there are at least $k^2/(2\alpha)$ fast arcs by Claim 2, but fewer than $L'k/(4\alpha^2) \leq k^2/(4\alpha)$ fast arcs start in points that are not good.

For the final statement, note that any i-fast arc has length at most $k + 1$, by definition. So, if such an arc starts at q_j, then it must be contained in c_j^*. The curve b_i must touch c_j at the end point of the i-fast arc, because the apex of the arc is x_i. ♠

We call an arc $z^* \subset b$ *short* if $|z^* \cap T| \leq 2\alpha k$. Note that while the length counts sad touching points on an arc, in this definition we count all touching points. We say that the good points q and q' are *close*, if either the arc of b from q to q' or the arc from q' to q is short.

Claim 4 *Let q and q' be good points. If the arc b^* from q' to q is short, then $|b^* \cap X| \leq 2\alpha\alpha_1 k$. If q_j and $q_{j'}$ are not close, then the arcs c_j^* and $c_{j'}^*$ are disjoint.*

Proof: The first claim holds, because q is a sad touching point, so the arc b^* ending there must have crossing-to-touching ratio below α_1.

For the second claim, assume that c_j^* and $c_{j'}^*$ intersect and let W be a Jordan curve connecting q_j to $q_{j'}$ along part of c_j^* and $c_{j'}^*$. Consider the two arcs b^* and b'^* that b is cut by q_j and $q_{j'}$. By our assumption, neither of these arcs is short, so each has more than $2\alpha k$ distinct curves of $\mathcal{F}$ touching it. As $W \cap X \leq 2\alpha k$, we must have a curve $z \in \mathcal{F}$ touching b^* that is disjoint from W. Analogously, we have another curve $z' \in \mathcal{F}$ touching b'^* and also disjoint from W. Now b and W separate z and z', contradicting the fact that they (as any two curves in $\mathcal{F}$) must intersect. This contradiction completes the proof of the claim. ♠

We call the distinct good points q_j and $q_{j'}$ *mingled* if $|c_j^* \cap c_{j'}^*| > \alpha^2 k/w$.

Claim 5 *Mingled points are close. A good point is mingled with at most w/α other good points.*

Proof: The first statement follows directly from Claim 4. The second statement follows from the statement of Claim 3 that c_j^* contains at most αk intersection points in total. ♠

For a good point q let I_q stand for the set of indices $1 \leq i \leq k$ with an i-fast arc starting at q. Similarly, for $1 \leq i \leq k$, let Q_i stand for the set of good points q at which an i-fast arc starts.

Claim 6 *Let $1 \leq j < j' \leq L$ be such that the arc from $q_{j'}$ to q_j is short, but q_j and $q_{j'}$ are not mingled. Then $|I_{q_j} \cap I_{q_{j'}}| < 6\alpha^2 k/\sqrt{v}$.*

Proof sketch. As the proof of Claim 6 is fairly involved, we only sketch it here, while postponing the full details to Section 2.6.

To simplify the presentation, let us first assume that the arc c_j^* and $c_{j'}^*$ are disjoint. Denote $I = I_{q_j} \cap I_{q_{j'}}$. Assume for a contradiction that $|I| \geq 6\alpha^2 k/\sqrt{v}$. The key observation is that at least $\binom{|I|}{2} = \Omega(\alpha^4 k^2/v)$ pairs of curves $b_i, b_{i'}$ with $i, i' \in I$ determine a lens of size $O(\alpha^3 k)$ each. As a result, q_j receives at least α units of charge from such lenses, by the second rule, contrary to its choice as a poor point.

Indeed, let b^* be the short arc of b from q_j to $q_{j'}$. Consider the curve $W = c_j^* \cup c_{j'}'^* \cup b^*$ which contains at most $O(\alpha^2 k)$ intersection points with the curves of $\mathcal{A}$; see Figure 6 (left). As each curve b_i, with $i \in I$, touches W at a pair of points $s_i \in c_j^*$ and $s_i' \in c_{j'}^*$, they determine a pocket F_i which is bounded by (i) the portion W_i of W between s_i to s_i', and (ii) the arc b_i^* of b_i between the same
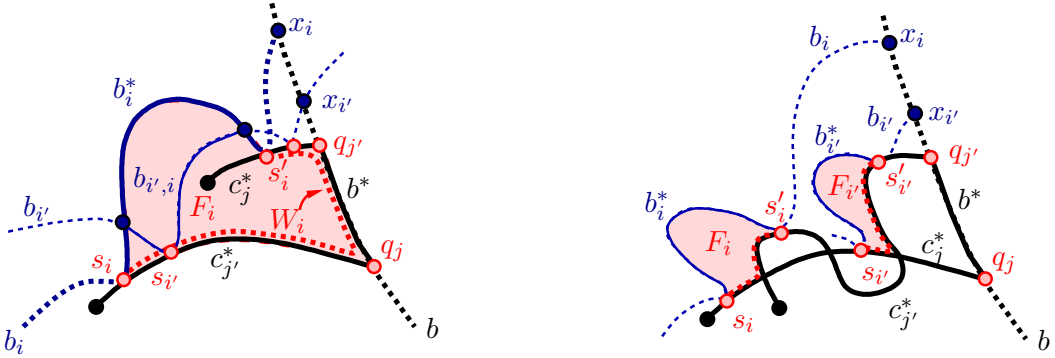
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Figure 6: Proof of Claim 6. Left: If c_j^* and $c_{j'}^*$ are disjoint, then any pair of curves $b_i, b_{i'}$ with $i, i' \in I_{q_j} \cap I_{q_{j'}}$ yields a lens $b_{i',i}$ or $b_{i,i'}$. Right: If the arcs c_j^* and $c_{j'}^*$ have multiple intersections, then a pair of curves b_i and $b_{i'}$ can determine non-overlapping pockets F_i and $F_{i'}$. Thus, b_i and $b_{i'}$ do not necessary determine a lens within F_i or $F_{i'}$.

two points and is to the right of b_i^*. For any other curve $b_{i'}$ with $i' \in I \setminus \{i\}$, which touches c_j within W_i, we define $b_{i',i}$ to be the shortest arc of $b_{i'}$ between two points of b_i that contains $s_{i'}$. Note that $b_{i',i}$ is a lens inside F_i.

To bound the size of the lens $b_{i',i}$, we argue that each curve that touches b_i^* must exit F_i through W_i (or, else, it will not meet one of the curves c_j or $c_{j'}$). Hence, the overall number of such curves does not exceed $O(\alpha^2 k)$. Since b_i^* is adjacent to a sad point s_i, we obtain that $|b_i^* \cap X| = O(\alpha^3 k)$. Finally, each curve that touches the lens $b_{i',i}$ must also leave F_i through $W_i \cup b_i^*$, so their number is also $O(\alpha^3 k)$, in fact, at most $3\alpha^3 k$. This means that, by the second charging rule, $b_{i',i}$ sends some charge to q_j.

The contradiction comes from q_j being poor despite the fact that any pair of distinct indices $i, i' \in I$ determine a lens (either $b_{i',i}$ or $b_{i,i'}$) sending charge to q_j.

If c_j^* and $c_{j'}^*$ intersect (possibly many times), then the above argument fails, as the curves b_i may determine smaller-size pockets F_i amidst $c_j^* \cup c_{j'}^*$, which do not necessarily overlap (see Figure 6 (right)). As a result, the number of lenses $b_{i',i}$ can be substantially smaller than $\binom{|I|}{2}$, and it generally depends on the number of intersections between c_j^* and $c_{j'}^*$. Nevertheless, since q_j and $q_{j'}$ are not mingled, the arcs c_j^* and $c_{j'}^*$ have at most $\alpha^2 k/w$ intersections, which enables to extend the previous analysis by finding somewhat fewer lenses sending charges to q_j or $q_{j'}$, but noticing that these lenses tend to be shorter and therefore send more charge. For the precise accounting (see Section 2.6), we use Lemma 1 again. ♠

Claim 7 *Let $1 \le j < j' \le L$ be such that the arc b^* from $q_{j'}$ to q_j is short. The number of good points in b^* is at most $50\alpha^3 w$.*

Proof: Let S be the set of good points in b^*. We have $|S|\dfrac{k}{4\alpha^2} \le \sum_{q \in S} |I_q| = \sum_{i=1}^{k} |Q_i \cap S|$ and, by

the Cauchy–Schwarz inequality, $\dfrac{k|S|^2}{16\alpha^4} \le \sum_{i=1}^{k} |Q_i \cap S|^2 = \sum_{q,q' \in S} |I_q \cap I_{q'}|$. We use the trivial bound

$|I_q \cap I_{q'}| \le k$ if $q = q'$ or if q and q' are mingled. By Claim 5, there are at most $(w/\alpha + 1)|S|$

such terms. The remaining terms can be bounded by $6\alpha^2 k/\sqrt{v}$ using Claim 6. We obtain $\dfrac{k|S|^2}{16\alpha^2} \le$

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$k(w/\alpha + 1)|S| + \dfrac{6|S|^2\alpha^2 k}{\sqrt{v}}$. Substitute $v = 21000\alpha^{12}$ and the claim follows. ♠.
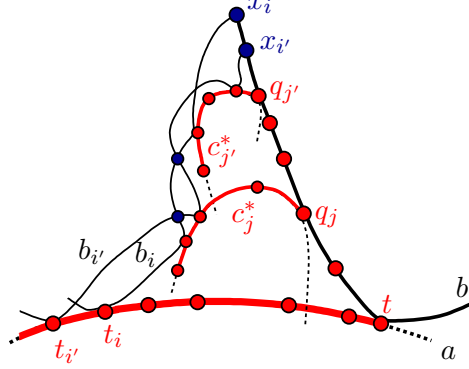

Figure 7: Proof of Claim 8. The good points $q_j, q_{j'} \in S$ are not close, and their arcs c_j^* and $c_{j'}^*$ form a pair of neighboring "teeth" in the comb Γ.

Claim 8 *For distinct indices $1 \le i, i' \le k$ the curves b_i and $b_{i'}$ have at least $|Q_i \cap Q_{i'}|/(100\alpha^3 w) - 1$ intersection points.*

Proof: Let $Q = Q_i \cap Q_{i'}$. We select a subset $S \subseteq Q$ with no two close points greedily: we consider the elements $q \in Q$ along b in reverse direction starting from t, and include them in S unless q is close to a point already in S. No short arc of b avoiding t contains more than $50\alpha^3 w$ good points by Claim 7. Thus, we have $|S| \ge |Q|/(50\alpha^3 w) - 1$. Let b^* be the arc of b from the point we put in S last to the point t we put there first. By Claim 4, the arcs c_j^* corresponding to the points $q_j \in S$ are pairwise disjoint. Let Γ be the comb-like arrangement of these arcs together with b^*, see Figure 7. Let b_i^* be the maximal arc on b_i from a touching point where an i-fast arc ends to the apex x_i of the i-fast arcs. Clearly, b_i^* touches all the "teeth" of the comb Γ, but it does not intersect its spine b^*. This implies that b_i^* touches the teeth in the same order as the arc b^*. This is also true for the analogously defined arc $b_{i'}^*$ of $b_{i'}$. Consider two neighboring teeth of the comb Γ. Obviously, either the part of b_i^* between the corresponding touching points is crossed by $b_{i'}$, or the part of $b_{i'}^*$ between the corresponding touching points is crossed by b_i. As the segments of b_i^* between touchings of consecutive teeth are disjoint, they represent at least $(|S| - 1)/2$ intersections between b_i and $b_{i'}$. ♠.

Having proved these claims, we return to the proof of Lemma 5. We started the proof by assuming that the lemma fails, so we need to arrive at a contradiction to finish the proof.

By Claim 3, we have many fast arcs starting at good points, namely $\sum_{j=1}^{L} |I_{q_j}| \ge \frac{k^2}{4\alpha}$. Using $L \le \alpha k$ (Claim 3 again) and the Cauchy–Schwarz inequality, we get

$$\frac{k^3}{16\alpha^3} \le \sum_{j=1}^{L} |I_{q_j}|^2 = \sum_{1 \le i, i' \le k} |Q_i \cap Q_{i'}|.$$

Subtracting the contribution of the case $i = i'$ and dividing by 2, we obtain

$$\sum_{1 \le i < i' \le k} |Q_i \cap Q_{i'}| \ge \frac{k^3}{40\alpha^3}. \tag{1}$$

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Claim 8 shows that this lower bound on the left-hand side of (1) provides a lower bound on the number of the intersection points between the curves b_i. We find that the number of such intersection points is at least $k^3/(4000\alpha^6 w) - k^2/2$. With the prior choice of parameters, this contradicts the assumption that the total number of intersections satisfies $|X| < \alpha_1 n^2$. This contradiction proves Lemma 5.

2.5 Wrapping up the proof of Theorem 2

Finishing the proof of Theorem 2 is simple once we have Lemmas 2, 3, 4 and 5. Considering all the charges in all the M phases of our scheme, every sad touching point $t \in T'$ receives a charge of at least $c_{\text{in}} = \alpha M$ by Lemma 5. For an intersection point $x \in X$, the total charge sent out is at most $c_{\text{out}} = 10M$ by Lemmas 3 and 4. Comparing the total charges sent and received we obtain

$$\frac{|X|}{|T'|} \geq \frac{c_{\text{in}}}{c_{\text{out}}} = \alpha/10.$$

We have $|T'| \leq 10|X|/\alpha$ from the line above and $|T| - |T'| \leq 6|X|/\alpha_1$ from Lemma 2. In total, we have $|T| \leq 16|X|/\alpha_1$, and the statement of the Theorem 2 follows. ♠

2.6 Proof of Claim 6

Proof: For simplicity, we write q and q' for q_j and $q_{j'}$, respectively. Analogously, we write c, c', c^* and c'^* for c_j, $c_{j'}$, c_j^* and $c_{j'}^*$, respectively. We write b^* for the short arc from q' to q.

Refer to Figure 8. Consider the arrangement of the curves c and c'. The curve b touches both of these curves, so it must be contained in a single face F^0 of the arrangement, and this face is to the right of both c and c'. As c and c' intersect, the boundary of F^0 is a simple closed Jordan curve which we denote by W^0. Clearly, W^0 consists of alternating arcs of c and c' each consistently oriented with F^0 to the right of them. The arcs of $c \cap W^0$ appear in the same cyclic order along c and W^0, and a similar statement is true for the segments of $c' \cap W^0$. Note, however, that outside F^0 the curves c and c' can behave wildly and all sorts of extra intersections can occur even between c^* and c'^*.
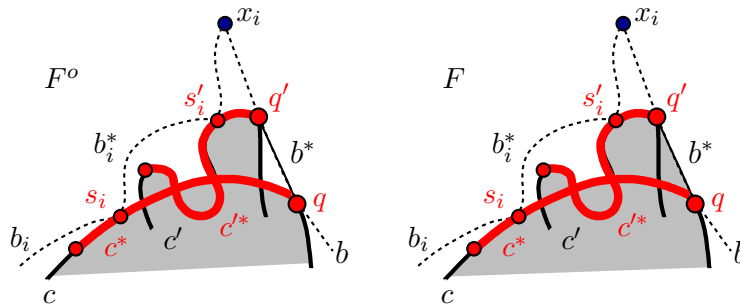

Figure 8: Proof of Claim 6. The curve b lies in the single face F^0 of the arrangement of c and c'. The cell F^0 contains all the curves b_i with $i \in I$ (left). The arc b^* from q' to q splits F^0 into two sub-faces, with all the touching points s_i and s_i' lying on the boundary the sub-face $F \subset F^0$ to the left of b^* (right).

The arc b^* splits this face in two, let F stand for the side of F^0 containing $b \setminus b^*$ (that is, on the left from b^*). Let W^1 stand for the part of W^0 on the boundary of F. Clearly, this is a Jordan curve connecting q' to q and all its segments coming from c and c' are consistently oriented from q' to q.

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Let us write $I = I_q \cap I_{q'}$. In what follows, we can assume that I is not empty. For each $i \in I$, the curve b_i touches both c and c' and intersects b, so it is confined to the face F^0. Let $s_i \in c^*$ and $s_i' \in c'^*$ be the points where b_i touches the boundary of F^0. Recall that x_i is an intersection point of b_i and b. Let b_i^* be the arc of b_i from s_i to s_i' or vice versa, whichever does not contain the apex x_i.

Proposition 1 *With the previous notation the following holds.*

(i) b_i^* does not intersect b, and the apex x_i is the first point on b_i after b_i^* that belongs to b.

(ii) We have $x_i \notin b^*$. (However, b^* can still intersect b_i at points other than x_i.)

Proof: Part (i) follows from the definition of the apex x_i, and using that both q and q' are starting points of i-fast arcs.

For part (ii), recall that $q = q_j$ and $q' = q_{j'}$ are starting points of i-fast arcs, so both of them must lie in the interval of b from x_i to $t = q_1$. As the good points $t = q_1, \ldots, q_j, \ldots, q_{j'}, \ldots, q_L$ are listed in the reverse order of b, the four points $x_i, q_{j'}, q_j, t = q_1$ must appear in this order along b. ♠

Proposition 1 implies that the interior of b_i^* is inside the face F, and its endpoints s_i and s_i' are on the boundary of F, and, consequently, also on W^1. In what follows, we define a simple open Jordan curve W that contains $(c^* \cup c'^*) \cap W^1$. In order for W to remain simple and connected, it may include segments outside c^* and c'^*. Nevertheless, W has at most $O(\alpha^2 k)$ intersection points with the curves of $\mathcal{A}$.

Tracing W. For the definition of the curve W, we consider the following cases:

In case c'^* does not intersect c, it lies entirely on W^0. In particular, c^* and c'^* are disjoint. Hence, we can simply take W to be the union of c^*, c'^* and b^*; see Figure 9. We call this the *the disjoint case*.

In case c'^* intersects c, we consider the first such intersection point ζ along c'^*, at which c'^* leaves W^0, and notice that ζ must belong to c^*; see Figure 10. Indeed, assume for a contradiction that ζ belongs to $c \setminus c^*$. Since (i), the order of the segments of $c \cap W^0$ along W^0 is consistent with their order along c, (ii) W^1 ends at the starting point q of c^*, and (iii) W^1 begins at q', the last appearance of c^* along W^0 is also contained in $W^0 \setminus W^1$. However, in that case, c^* can never show up on W^1, contrary to $I \neq \emptyset$ (and, thus, to the choice of s_i on $W^1 \cap c^*$ for $i \in I$). We can assume, then that the first intersection ζ of c'^* with c lies on c^*, and it is the first appearance of c and c^* along W^1.

As c'^* starts at q', there is a shortest segment U' of W^1 starting at q' with $c'^* \cap W^1 = c' \cap U'$. Although c^* starts at q, we similarly define the shortest interval $U \subset W^1$ starting at q' with the property that $c^* \cap W^1 = c \cap U$. Indeed, $c^* \cap W^0$ is confined to an interval of W^0 starting at q and we can choose U to be the intersection[4] of the shortest such interval with W^1.

We call the case when U is shorter than U' the *first intersecting case* (see Figure 10 (left)). In this case, we have $c^* \cap W^1 \subseteq U \subseteq c^* \cup c'^*$. If the end point u of U is in c'^*, we let the curve W start with U followed by the part of c'^* after u. If $u \notin c'^*$, then we let W start with U again, but we cannot directly add a part of c'^*, so instead we follow U by a curve retracing the last segment of U (which is contained in c^*) very close, but slightly outside F till we meet c'^*, and then add the remaining part of c'^*. Assuming that the last segment of U is retraced sufficiently close to c^*, any curve that meets the "reversed" segment γ must also meet c^*.

In case U' is shorter than U (the *second intersecting case*), we construct W symmetrically: We start with U' and add the remaining part of c^*, possibly using a reverse segment γ slightly outside F to connect the two parts. See Figure 10 (right).

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[4]This intersection is indeed a connected interval of W^1 since W^1 ends at q.
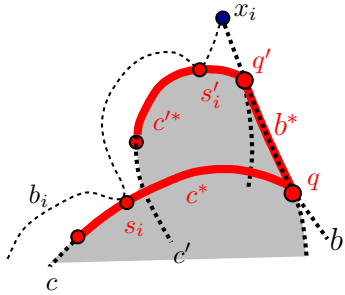
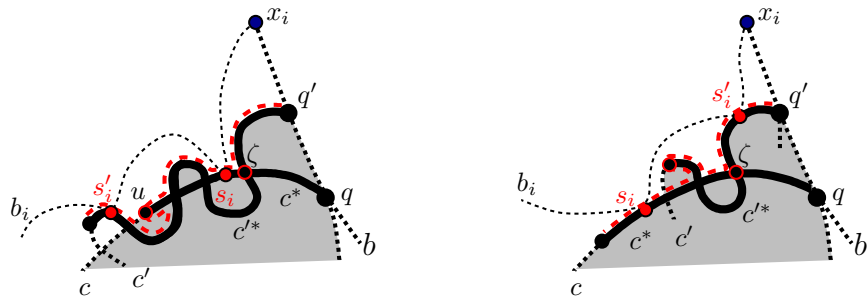Figure 9: Constructing the curve W. In the disjoint case, W includes the arc b^* from q' to q.

Figure 10: Constructing the curve W – the two intersecting scenarios. Notice that the first intersection ζ of c'^* with c must belong to c^*. Left: In the first intersecting scenario, W may have to retrace the last segment of U. Notice that $c' \cap W$ consists only of segments of c'^* (while c can meet W at a point outside c^*). Right: In the second intersecting case, W may have to retrace the last segment of U', and $c \cap W$ consists only of segments of c^*.

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Note that the only case when this construction does not work is when U' consists of a single segment so we could follow it backward all the way and still not meet c. But this cannot happen, because configurations like that are treated separately in the disjoint case.

The properties of W are summarized in the following proposition.

Proposition 2 *W is a simple open Jordan curve. It contains $(c^* \cup c'^*) \cap W^1$ and consists only of segments of c^*, c'^*, and possibly one additional segment. Specifically, W satisfies the following properties:*

(a) *In the disjoint case (when c'^* does not intersect c), we have $W = c^* \cup c'^* \cup b^*$, $b \cap W = b^*$ and $c \cap W = c^*$.*

(b) *In both intersecting cases, (b1) W lies outside the interior of F, and is composed of segments of c^* and c'^*, and possibly of an additional segment γ closely retracing[5] the previous segment of W from the outside of F, (b2) the order of the segments of $c^* \cap W$ and $c'^* \cap W$ along W is consistent with the respective orientations of c and c', and (b3) $b \cap W = \{q'\}$.*

(c) *In the first intersecting case, $c' \cap W$ consists of segments of c'^*. When following W past the end point of any of these segments, the curve W continues on the right of c'.*

(d) *In the second intersecting case, $c \cap W$ consists of segments of c^*. When following W past the end point of any of these segments, the curve W continues on the right of c.*

Proof: Most of the statements follow directly from our construction of W. Note, however, that in the intersecting case c and c' or even c^* and c'^* can intersect in unexpected ways outside (the closure of) F^0. The curve W is still simple, as it consists of a segment (namely, either U or U') of the (simple) boundary of F followed by a segment of the simple curve c^* or c'^*, possibly with a "retracing" curve γ in between. In the first intersecting case, c'^* contains this final segment, so the remaining part of $c' \cap W$ is all from $c'^* \cap W^1$. Property (b3) follows from (b2) and since c^* is a proper subarc of c, so none of the segments of $c^* \cap W^1$ can be adjacent to q. Part (c) holds as F^0 is to the right of all the segments on its boundary. Note, however, that $c \cap W$ may contain several "unintended" intersection points of c and (this last part of) c'^*. Furthermore, γ (if it exists) is to the left of the segment it retraces. Part (d) can be seen similarly, with the roles of c and c' reversed. ♠

Let us define B as the set of points where W is intersected by a curve in $\mathcal{A}$ (other than the curve W follows at that segment). We have $X \cap W \subseteq B$, but B may contain further crossing points along the reverse segment γ, if such a segment exists. Still, in the intersecting case we have $|B| \le 3\alpha k$ by Claim 3(iii), while $|B| \le 2\alpha\alpha_1 k + 2\alpha k$, where we use also Claim 4 to estimate the size of $B \cap b^* = X \cap b^*$. We will use the bound $|B| \le 2\alpha^2 k$ that holds in both cases and apply Lemma 1 to B with a non-uniform weight function w_0. We set $w_0(x) = w + 1$ for $x \in B \cap c^* \cap c'^*$ or (in the disjoint case) if $x = q$. We set $w_0(x) = 1$ otherwise. For the total weight we have $w_0(B) \le 3\alpha^2 k$ as q and q' are not mingled.

For $i \in I$, let W_i be the portion of W between the touching points s_i and s'_i, and let us write $l_i = w_0(B \cap W_i)$. Let $R_0 = \{s_i \mid i \in I\}$ and set $\lambda = 3\sqrt{v}$. Let $R = \{s_i \mid i \in I, |W_i \cap R_0| \le l_i/\lambda\}$ and $\hat{I} = \{i \in I \mid s_i \notin R\}$. We have $|I| = |R| + |\hat{I}|$. In what follows, we bound $|R|$ and $|\hat{I}|$ separately.

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[5]Namely every intersection of γ with a curve $\sigma \in \mathcal{A}$ corresponds to a unique crossing (non-touching intersection) of σ with the retraced segment of W and vice versa. γ does not not contain points of $X \cup T$.

Bounding $|R|$. We apply Lemma 1 for the curve W, the sets R and B, the weight function w_0 and the parameter λ. The condition is satisfied, as for the interval W_i ending at $s_i \in R$ we have $|W_i \cap R| \leq |W_i \cap R_0| \leq l_i/\lambda = w_0(W_i \cap B)/\lambda$. From Lemma 1, we conclude that

$$|R| \leq \frac{3w_0(B)}{\lambda} \leq \frac{9\alpha^2 k}{\lambda}. \tag{2}$$

Bounding $|\hat{I}|$. Fix $i \in \hat{I}$, whose respective arc W_i contains at least l_i/λ points $s_i \in R_0$. Consider $W_i \cup b_i^*$ and let F_i be the side of this closed Jordan curve to the right of b_i^*; see Figure 11.
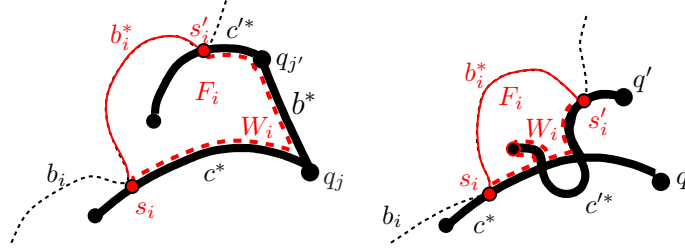

Figure 11: The portion W_i of W between $s_i' = b_i \cap c'$ and $s_i = b_i \cap c$ is traced. The face F_i of $\mathbb{R}^2 \setminus (b_i \cup W_i)$ lies to the right of b_i. Left: The arcs c^* and c'^* are disjoint so both W and W_i must include b^*. Right: The scenario where c^* and c'^* intersect.

First we sketch our argument for our bound on $|\hat{I}|$. The rigorous calculation is after Propositions 3 and 4. We will show that any curve $b_{i'}$ whose respective touching $s_{i'}$ lies on W_i, determines within F_i a lens $b_{i',i}$ of length at most $3\alpha^3 k$; see Figure 12. Each of these lenses will send some charge to q via the second rule. The final bound on $|\hat{I}|$ follows from q being poor despite all these incoming charge.
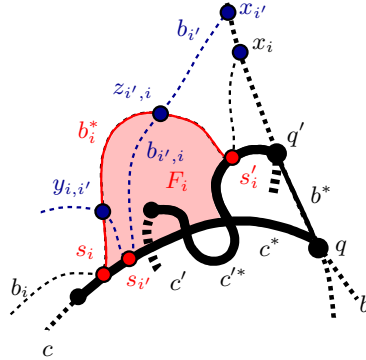

Figure 12: The segment W_i contains a touching $s_{i'}$ between c with $b_{i'}$, for $i' \in I \setminus \{i\}$. Each such $b_{i'}$ yields a lens $b_{i',i}$ of length at most $\alpha(l_i - w)$.

The crux of our argument is showing that each lens $b_{i',i}$ has length at most $\alpha(l_i - w)$. To this end, we first bound the quantities $|b_i^* \cap T|$ and $|b_i^* \cap X|$ in terms of the overall weight l_i of W_i.

Proposition 3 *With the previous assumptions, b_i^* contains at most $l_i - w$ points of T, and at most $\alpha_1(l_i - w)$ points of X.*

The proof of Proposition 3 relies on the following topological property. We will use this observation to argue that no curve in $\mathcal{A}$ can stay inside F_i. Indeed, curves in $\mathcal{G}$ have to intersect b and curves in $\mathcal{F}$ have to intersect both c and c'.

Proposition 4 *The curve b does not intersect the interior of F_i. Furthermore, at least one of the curves c and c' does not intersect the interior of F_i.*

Proof: The proof is based on controlling where b and one of c and c' intersects the boundary $W_i \cup b_i^*$ of F_i.

In the disjoint case, b^* separates along W the points s_i on c^* and s_i' on c'^*, so $b^* \subset W_i$. This means $b \cap W_i = b^*$ by Proposition 2(a). Note that the curve b continues outside F_i after it leaves the boundary of F_i at q. Since $b \cap b_i^* = \emptyset$ by Proposition 1 (and b cannot cross $c^* \cap W_i$ or $c'^* \cap W_i$), the arc $b \setminus b^*$ too cannot enter the interior of F_i. We similarly argue that $c \setminus c^*$ cannot enter the interior of F_i through either of the boundary arcs b_i^* and $c'^* \cap W_i$.

For the rest of the proof, we consider the intersecting cases. We still have $b \cap b_i^* = \emptyset$ by Proposition 1. Now we have $b \cap W = \{q'\} \notin W_i$ by Proposition 2(b), so b never intersects the boundary of F_i. Following W past s_i and s_i', we see that it leaves W_i and F_i in one of these directions and reaches the point $q' \in b \setminus F_i$. This implies that b is again outside F_i.

In the first intersecting case, we consider the curve c'. It touches b_i, so we have $c' \cap (W_i \cup b_i^*) = c' \cap W_i \subseteq c' \cap W$ and this is covered by Proposition 2(c). Recall that the part of c' on the boundary of F_i consists of segments of c'^*. It is easy to see that F_i is on the right of each of these segments, just as W_i continues to the right of c' after the end points of any of these segments (again, by Proposition 2(c)). This shows that c' never enters the interior of F_i.

A similar argument in the second intersecting case shows that c never enters the interior of F_i. ♠

Proof of Proposition 3. The bound on $|b_i^* \cap T|$ follows from the fact that any curve $\sigma \in \mathcal{F}$ touching b_i^* must intersect W_i. Indeed, any such curve σ is in F_i in a small neighborhood around the point where it touches b_i^*. Since the curve σ intersects *each* of the curves $c, c' \in \mathcal{F}$ (and at least one of c and c' is disjoint from the interior of F_i), σ must meet ∂F_i at a point of $B \cap W_i$. We have $|B \cap W_i| \le l_i - w$, as W_i contains at least one of the heavy points with weight $w + 1$. The bound on $|b_i^* \cap X|$ follows since the endpoint s_i of b_i^* is sad. ♠

We are now ready to bound the cardinality of $\hat{I}$ and thus complete the proof of Claim 6. Let us consider $i, i' \in I$ with $i \ne i'$ and $s_{i'} \in W_i$. Follow $b_{i'}$ from $s_{i'}$ in both directions. It starts out inside F_i and eventually has to reach b that is disjoint of the interior of F_i. As the part of $b_{i'}$ around $s_{i'}$ is in F, the first intersection with b in either direction is outside b^*. Proposition 4 implies that $b_{i'}$ must properly cross the boundary of F_i to meet b. The first intersection point in either direction must be on b_i^*, for it cannot be on b^*, and $b_{i'}$ touches both c_j and $c_{j'}$. Let us call these intersection points $y_{i',i}$ and $z_{i',i}$ such that the arc $b_{i',i}^*$ from $y_{i',i}$ to $z_{i',i}$ along $b_{i'}$ is a lens, is inside F_i and it contains $s_{i'}$; see Figure 12.

Note that $b_{i',i}^*$ is a lens. The length $l_{i',i}$ of this lens is at most $\alpha(l_i - w)$. Indeed, Proposition 4 implies that any curve touching $b_{i',i}^*$ must intersect the boundary of F_i so as to be able to intersect *both* c and c'. However, by Proposition 3, the total number of points on this boundary at which a curve of $\mathcal{A}$ intersects it, is at most $|B \cap W_i| + |b_i^* \cap X| \le (\alpha_1 + 2)(l_i - w) = \alpha(l_i - w)$.

Each such lens $b_{i',i}$ sends a charge of $v/((l_{i',i} + w)k)$ to q, by the second charging rule. Indeed, this rule applies to $b_{i',i}$, because its length is $l_{i',i} \le \alpha l_i \le \alpha |B| \le 3\alpha^3 k$, and the arc from q to $s_{i'}$ satisfies the requirements. The amount of the charge sent is

$$\frac{v}{(l_{i',i} + w)k} \ge \frac{v}{(\alpha(l_i - w) + w)k} \ge \frac{v}{\alpha(l_i - \lambda)k}.$$

If we further assume that $i \in \hat{I}$, then we have more than l_i/λ choices of $i' \in I$ with $s_{i'} \in W_i$. One of these choices is $i' = i$, but more than $l_i/\lambda - 1$ other choices will give rise to lenses $b_{i',i}$, each sending a charge of at least $v/(\alpha(l_i - \lambda)k)$ to q. The total of these charges for a fixed $i \in \hat{I}$ is at least $v/(\alpha\lambda k)$, and for all $i \in \hat{I}$ this is at least $|\hat{I}|v/(\alpha\lambda k)$.

We know that q_j is poor, so this charge does not reach the threshold of α. As a consequence, we have $|\hat{I}| \le \alpha^2\lambda k/v$. To finish the proof of Claim 6, we use this last estimate, Equation (2), the fact $|I| = |R| + |\hat{I}|$ and substitute $\lambda = 3\sqrt{v}$. ♠

3 Proof of Theorem 4

To prove Theorem 4, we have to get rid of the assumption in Theorem 1 that the curves are pairwise intersecting. We achieve this in two steps. First, in Subsection 3.1, we state and prove a separation result between the number of touchings and the number of intersections that does not assume strict pairwise intersection, but still assumes a very dense intersection graph. Then, in Subsection 3.2, we apply planar separation arguments to get rid of this milder assumption. In both of these steps, we lose in the separating function, namely, the exponent of the log log function decreases. We did not attempt to optimize for this exponent, because we believe that even a logarithmic separation should hold, as stated in Conjecture 1.

3.1 Sampling

In this subsection, we prove the following lemma. Note that, like Theorem 4, it is about open Jordan curves, not closed ones.

Lemma 6 *Let $\mathcal{A}$ be a family of n simple open Jordan curves in general position in the plane. Let T be the set of touching points between curves of $\mathcal{A}$ and let X be the set of intersection points. With $h = n^2/|T|$ and $f = |X|/|T|$ we have $f^{72}h^{144} = \Omega(\log\log n)$.*

Proof: As the statement of the lemma is asymptotic, we may assume in the calculations below that n is sufficiently large.

We select a pair of distinct curves $a_0, b_0 \in \mathcal{A}$. We try to select them so as to satisfy these conditions (see Figure 13):

(a) For $m_0 = |a_0 \cap b_0|$ we want $m_0 \le 120fh$.

(b) For $m_1 = |(a_0 \cup b_0) \cap X|$ we want $m_1 \le 80nfh$.

(c) Let m_2 be the number of touchings $t \in T$ between two curves in $\mathcal{A} \setminus \{a_0, b_0\}$ with both of these curves intersecting $a_0 \cup b_0$ (see Figure 13). We want $m_2 \ge |T|/(20h^2)$.

By selecting the pair (a_0, b_0) uniformly at random, the expectation of m_0 is $E[m_0] = |X|/\binom{n}{2} < 3f/h$. We have $E[m_1] \le 2|X|/n = 2nf/h$. For the expectation of m_2 notice that at most $|T|/2$ touchings are contained in a curve $a \in \mathcal{A}$ with $|a \cap T| \le |T|/(2n)$. The remaining elements of T (at least $|T|/2$ of them) are counted in m_2 with probability at least $|T^2|/(5n^4)$ each, yielding $E[m_2] \ge |T|^3/(10n^4) = |T|/(10h^2)$.

By Markov's inequality condition (a) fails with probability less than $1/(40h^2)$ and the same holds for condition (b). Using Markov's inequality again and the fact that $m_2 \le |T|$ we have that condition (c) is satisfied with probability at least $1/(20h^2)$. Thus, all three conditions are simultaneously satisfied with some positive probability. We fix such a choice of the curves a_0 and b_0 and call them the *ground curves*.

Let the family $\mathcal{A}' \subseteq \mathcal{A} \setminus \{a_0, b_0\}$ consist of the curves that intersect at least one of a_0 or b_0. By property (c), these curves create $m_2 \ge |T|/(20h^2)$ touchings.
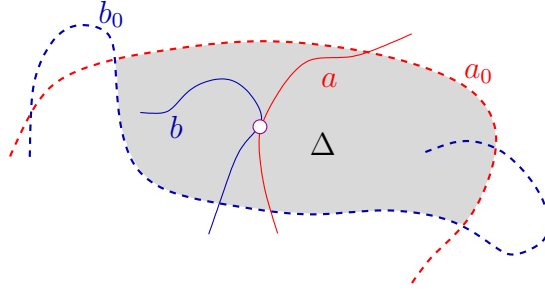
Figure 13: Proof of Lemma 6. We select a pair $a_0, b_0 \in \mathcal{A}$ satisfying three properties including that at least $|T|/(20h^2)$ of the touchings involve a pair of curves $a, b \in \mathcal{A} \setminus \{a_0, b_0\}$ both intersecting $a_0 \cup b_0$. We then choose an open cell $\Delta \subset \mathbb{R}^2 \setminus (a_0 \cup b_0)$ which contains at least $|T|/(2400fh^3)$ of these touchings.

Let us consider the arrangement of a_0 and b_0. In case the ground curves are disjoint there is a single cell of this arrangement and its boundary is $a_0 \cup b_0$. We will treat this somewhat peculiar case later. Otherwise, the arrangement has $m_0 \leq 120fh$ cells, each with a connected boundary. For each open cell $\Delta \subset \mathbb{R}^2 \setminus (a_0 \cup b_0)$, let $T_\Delta \subset T \cap \Delta$ denote the set of touching points between the curves of $\mathcal{A}'$ within Δ (again, see Figure 13). By the pigeon-hole principle, there exists an open cell $\Delta \subset \mathbb{R}^2 \setminus (a_0 \cup b_0)$ with

$$|T_\Delta| \geq |T|/(2400fh^3).$$

We fix such a cell Δ. We consider each connected component of $a \cap \Delta$ for curves $a \in \mathcal{A}'$. These are simple Jordan curves in Δ with at least one end point on the boundary. We make the curves slightly shorter to make sure each has exactly one endpoint on the boundary but they still determine the same set T_Δ of touchings. We denote by $\mathcal{A}''$ the resulting family of $m \leq n + m_1 \leq (80fh + 1)n$ curves; see Figure 14 (left).

We can slightly inflate the boundary of Δ to a simple closed Jordan curve $c \subset \Delta$ with c intersecting each curve $a \in \mathcal{A}''$ exactly once, close to the end point of a on the boundary of Δ and with all the touching points T_Δ on the side Δ' of c contained in Δ. Let us enumerate the curves in $\mathcal{A}''$ as $\mathcal{A}'' = \{a_1, a_2, \ldots, a_m\}$ such that the intersection points $p_i = a_i \cap c$ appear on c in this cyclic order.

We pick a parameter $1 \leq \ell \leq m = |\mathcal{A}''|$ and form the family $\mathcal{F}$ by slightly modifying the curves a_i for $1 \leq i \leq \ell$ and form $\mathcal{G}$ by slightly modifying a_i for $\ell < i \leq m$. The slight modification consists of keeping $a_i \cap \Delta'$ and attaching a curve b_i to it that starts at p_i and is disjoint from Δ'; see Figure 14 (right). We choose these additional curves such that (i) the curves b_i and $b_{i'}$ are disjoint if $i \leq \ell < i'$, (ii) the distinct curves b_i and $b_{i'}$ intersect exactly once if $i, i' \leq \ell$ or $i, i' > \ell$ and (iii) the curves b_i are in general position. Clearly, such curves b_i exist.

The family $\mathcal{F}$ consist of at most m pairwise intersecting Jordan curves and the same is true for $\mathcal{G}$. With adding dummy curves we can actually assume that both families consist of $m + n$ curves.

Let us choose ℓ uniformly at random. The touching between the curves a_i and $a_{i'}$ with $i < i'$ will remain a touching between the corresponding modified curves (one in $\mathcal{F}$, the other one in $\mathcal{G}$) if we have $i \leq \ell < i'$. This happens with probability $(i' - i)/m$. Among the $|T_\Delta| \geq |T|/(2400fh^3)$ such touchings at most half can be between curves a_i and $a_{i'}$ with $i < i' < i + x$ for $x = |T|/(4800fh^3m)$. Each touching point of the other half of T_Δ remains a touching points between a curve in $\mathcal{F}$ and a curve in $\mathcal{G}$ with probability at least x/m. Thus, the expected number of touchings between a curve in $\mathcal{F}$ and curve in $\mathcal{G}$ is at least $(|T_\Delta|/2) \cdot (x/m)$. We choose ℓ such that the actual number of these touchings is at least this expectation.
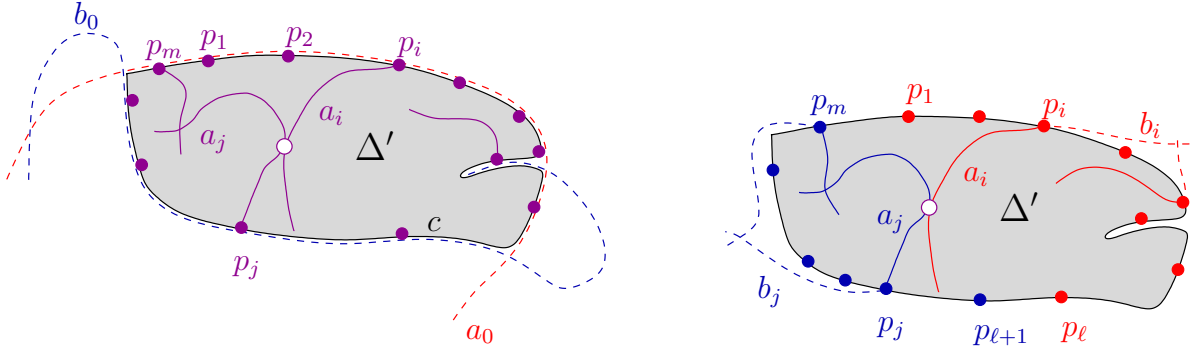
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Figure 14: Proof of Lemma 6 – constructing the families $\mathcal{F}$ and $\mathcal{G}$. Left: We obtain a new family $\mathcal{A}''$ by trimming each curve of $\mathcal{A}$ to Δ. We then slightly inflate the boundary of Δ to a simple closed Jordan curve c which encloses a region $\Delta' \subset \Delta$ and meets each $a_i \in \mathcal{A}''$ at a single point p_i. Right: We choose a random index $1 \leq \ell \leq m$ and augment each $a_i \in \mathcal{A}''$ with a curve b_i outside Δ', with the property that b_i and b_j intersect at a single point if and only if $1 \leq i, j \leq \ell$ or $\ell < i, j \leq m$, and otherwise they are disjoint.

Some of the intersection points between curves in $\mathcal{F} \cup \mathcal{G}$ come from X. We have one remaining intersection point between any two curves in $\mathcal{F}$ and also between any two curves in $\mathcal{G}$ for a total of $O(m^2 + n^2)$ additional intersection points.

We are almost ready to apply Theorem 2 to finish the proof. The only hurdle yet to clear is to pass from open Jordan curves to closed ones. This can simply be done by slightly inflating each curve. The process can be done in such a way that (i) touching curves remain touching, (ii) intersecting curves remain intersecting, (iii) the general position property is preserved and (iv) the number of intersection points is multiplied by at most 4. With this we obtain families $\mathcal{F}'$ and $\mathcal{G}'$, each consisting of $m + n$ pairwise intersecting simple closed Jordan curves with the total number of intersections between curves in $\mathcal{F} \cup \mathcal{G}$ being $O(|X| + m^2 + n^2) = O(f^2 h^2 n^2)$ and with the number of tangencies between a curve in $\mathcal{F}$ and a curve in $\mathcal{G}$ at least $(|T_\Delta|/2) \cdot (x/m) = \Omega(f^{-4} h^{-10} n^2)$. Applying Theorem 2 to these families $\mathcal{F}'$ and $\mathcal{G}'$ yields the statement of the lemma.

Finally, we have to consider the special case when the ground curves a_0 and b_0 chosen in the first step of our proof are disjoint. In this case the arrangement of the ground curves has a single cell $\Delta = \mathbb{R}^2 \setminus (a_0 \cup b_0)$. We define $\mathcal{A}''$ exactly as in the general case, so $\mathcal{A}''$ consists of curves contained in Δ with one end point on one of the ground curves. We distinguish "type-a" or "type-b" curves in $\mathcal{A}''$ according to whether it has an end point on a_0 or on b_0.

If at least one third of the touchings between two curves of $\mathcal{A}''$ are between two type-a curves, then we simply ignore b_0 and the type-b curves and consider the cell $\Delta^* = \mathbb{R}^2 \setminus a_0$ and the type-a curves. We can finish the proof as in the general case as Δ^* has a connected boundary.

If at least one third of the touchings between two curves of $\mathcal{A}''$ are between two type-b curves, then we proceed analogously.

If none of the above two cases hold, then we concentrate on the touchings between a type-a and a type-b curve: at least one third of all touchings between curves of $\mathcal{A}''$ must be like this. The situation is even simpler in this case with no need for any random choice. We obtain $\mathcal{F}$ by modifying slightly the type-a curves and we obtain $\mathcal{G}$ by modifying slightly the type-b curves. For this we have to separately inflate the two ground curves. We finish the proof as in the general case. ♠

3.2 Separation

The main ingredient we need for the proof of Theorem 4 is the following separator theorem of Fox and Pach [FP08] for *intersection graphs* of families of Jordan arcs.

Theorem 5 *For any collection $\mathcal{A}$ of n Jordan arcs in the plane in general position with a total of x intersection points, there is a subset $\mathcal{B} \subseteq \mathcal{A}$ of cardinality $O(\sqrt{x})$ so that $\mathcal{A} \setminus \mathcal{B}$ can be divided into disjoint subsets $\mathcal{A}_1$ and $\mathcal{A}_2$ of cardinality at most $2n/3$ each with the property that no arc of $\mathcal{A}_1$ meets an arc of $\mathcal{A}_2$.*

Repeated application of this result yields the following.

Lemma 7 *Let $\mathcal{A}$ be a collection of n simple Jordan curves in general position in the plane. Let d be maximum number of intersection points in a curve of $\mathcal{A}$ and let T be an arbitrary subset of the intersection points with $|T|^2 \geq nd^3$. There exist a subset $\mathcal{A}_0 \subseteq \mathcal{A}$ of $\Theta(n^2 d^3/|T|^2)$ curves such that $\Omega(nd^3/|T|)$ of the points in T are intersection points of curves of $\mathcal{A}_0$.*

Proof: Let us set a threshold parameter $1 \leq M \leq n$. We split the $\mathcal{A}$ by finding a small separator subset $\mathcal{B} \subset \mathcal{A}$ and partitioning $\mathcal{A} \setminus \mathcal{B}$ into $\mathcal{A}_1$ and $\mathcal{A}_2$, as described in Theorem 5. We recursively apply this procedure to the families $\mathcal{A}_1$ and $\mathcal{A}_2$, stopping only when we obtain subsets of size less than M. Let $\mathcal{B}'$ denote the set of all separator arcs that are removed at any invocation of the recursion and let $\mathcal{A}'_i$ for $i \in I$ stand for the final partition of $\mathcal{A} \setminus \mathcal{B}'$. Note that, by the properties of the separation, the curves in $\mathcal{A}'_i$ do not intersect curves from other parts $\mathcal{A}'_j$, $j \neq i$.

When we split a set $\mathcal{A}' \subseteq \mathcal{A}$ of size $|\mathcal{A}'| = m \geq M$, we find a subset $\mathcal{B}' \subseteq \mathcal{A}'$ with size $|\mathcal{B}'| = O(\sqrt{dm})$, as the curves in $\mathcal{A}'$ determine at most dm intersection points. The resulting parts of $\mathcal{A}' \setminus \mathcal{B}'$ have size at most $2m/3$.

The final parts produced by the above partitioning satisfy $|\mathcal{A}'_i| < M$, since this was our halting condition. Taking into account that these parts were obtained by splitting a subset of size at least M, they also satisfy $|\mathcal{A}'_i| \geq M/3 - O(\sqrt{Md})$.

Consider all the different parts $\mathcal{A}'$ obtained in intermediate steps of the above partitioning process. Those sets of size $|\mathcal{A}'| = m$ in an interval $(3/2)^i M \leq m < (3/2)^{i+1} M$ for some fixed integer i are clearly pairwise disjoint, so their number is at most $n/((3/2)^i M)$. We find a separator of size $O(\sqrt{(3/2)^i Md})$ for each one of them. The total contribution of this interval to the size of $\mathcal{B}'$ is $O((3/2)^{-i/2} M^{-1/2} n d^{1/2})$. We sum these contributions for the integers $0 \leq i \leq \log_{3/2} n$, and obtain $|\mathcal{B}'| = O(n\sqrt{d/M})$.

We set $M = Cn^2 d^3/|T|^2$ with a large constant C. We may assume this yields $M \leq n$, for otherwise the choice $\mathcal{A}_0 = \mathcal{A}$ satisfies the requirements of the lemma. It is clear from our bound on $|\mathcal{B}'|$ that if C is large enough, we have $|\mathcal{B}'| \leq |T|/(2d)$. Analogously, from $|\mathcal{A}'_i| \geq M/3 - O(\sqrt{Md})$ that holds for all $i \in I$, we get $|\mathcal{A}'_i| \geq M/4$ if C is large enough.

During the partition, we lose at most $|\mathcal{B}'|d \leq |T|/2$ intersections between the curves of $\mathcal{A}$, that is, for at least half of the points $t \in T$, neither of the two curves of $\mathcal{A}$ through t are in $\mathcal{B}'$. Both of these curves are therefore in the same part $\mathcal{A}'_i$ as curves from distinct parts are disjoint. By the pigeonhole principle, there is a part $\mathcal{A}'_{i_0}$ with at least $|T|/(2|I|)$ of the points of T, showing up as intersection points between curves of $\mathcal{A}'_{i_0}$. The choice $\mathcal{A}_0 = \mathcal{A}'_{i_0}$ satisfies the requirements of the lemma since $|A'_{i_0}| = \Theta(M) = \Theta(n^2 d^3/|T|^2)$ and $|T|/(2|I|) = \Omega(nd^3/|T|)$, where we use that $|A'_i| \geq M/4$ and therefore $|I| \leq n/(M/4) = O(|T|^2/(nd^3))$. ♠

Proof of Theorem 4: As the statement of the theorem is asymptotic in nature we may assume that $|T|/n$ is sufficiently large. Note that $|T|/n < n$, so this also means that n is sufficiently large. We introduce the notation $f = |X|/|T|$.

We first reduce the maximum number of intersection points on a curve in $\mathcal{A}$ to $d = \lfloor |X|/n \rfloor$. To this end, we break each arc $a \in \mathcal{A}$ into sub-arcs $a_1, \ldots, a_h$ all of which, with the possible exception of the last one, contain exactly d intersection points. This splitting yields a family $\mathcal{A}'$ of $n' = \Theta(n)$ Jordan arcs without modifying the set X of intersection points or the set T of touching points.

If $|T|^2 < n'd^3$ we apply Lemma 6 to $\mathcal{A}'$. The lemma claims that $f^{72}h^{144} = \Omega(\log \log n')$ for $h = n'^2/|T|$. We further have $|T|^2 < n'd^3 = O(|X|^3/n^2)$ yielding $h < O(f^3)$ and thus $f^{72}h^{144} = O(f^{504})$. The statement of the theorem follows.

If $|T|^2 \geq n'd^3$, then we apply Lemma 7 to the collection $\mathcal{A}'$ and the set of touching points T. Let $\mathcal{A}'_0 \subseteq \mathcal{A}'$ be the collection whose existence is claimed by this lemma. The size $|\mathcal{A}'_0|$ of this collection is $n'_0 = \Theta(n'^2 d^3/|T|^2) = \Theta(f^2|X|/n)$. The family determines $x'_0 \leq n'_0 d = O(f^2|X|^2/n^2)$ intersections among which $t'_0 = \Omega(n'd^3/|T|) = \Omega(f|X|^2/n^2|)$ are touchings. We apply Lemma 6 to the family $\mathcal{A}'_0$. With $f'_0 = x'_0/t'_0 = O(f)$ and $h'_0 = n'^2_0/t'_0 = O(f^3)$, the lemma states $f'^{72}_0 h'^{144}_0 = \Omega(\log \log n'_0)$. Here $f'^{72}_0 h'^{144}_0 = O(f^{504})$ and $n'_0 = \theta(f^2|X|/n) = \Omega(|X|/n) = \Omega(|T|/n)$. The statement of the theorem follows again. ♠

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