

Construction of locally plane graphs with many edges

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Abstract

A graph drawn in the plane with straight-line edges is called a geometric graph. If no path of length at most k in a geometric graph G is self-intersecting we call G k -locally plane. The main result of this paper is a construction of k -locally plane graphs with a super-linear number of edges. For the proof we develop randomized thinning procedures for edge-colored bipartite (abstract) graphs that can be applied to other problems as well.

1 Introduction

A *geometric graph* G is a straight-line drawing of a simple, finite (abstract) graph (V, E) , i.e., we identify the vertices $x \in V$ with distinct points in the Euclidean plane, and we identify any edge $\{x, y\} \in E$ with the straight line segments xy in the plane. We assume that the edge xy does not pass through any vertex of G besides x and y . We call (V, E) the abstract graph *underlying* G . We say that the edges $e_1, e_2 \in E$ *cross* if the corresponding line segments cross each other, i.e., if they have a common interior point. We say that a subgraph of G is *self-intersecting* if it contains a pair of crossing edges.

Geometric graphs without crossing edges are plane drawings of planar graphs: they have at most $3n - 6$ edges if $n \geq 3$ is the number of vertices.

*Reserach partially supported by NSERC grant 329527 and by OTKA grants T-046234, AT048826 and NK-62321.

Avital and Hanani [3], Erdős, and Perles initiated in the mid 1960s the systematic study of similar questions for more complicated *forbidden configurations*: Let H be set of forbidden configurations (geometric subgraphs). What is the maximal number of edges of an n vertex geometric graph not containing any configuration belonging to H ? This problem can be regarded as a geometric version of the fundamental problem of extremal graph theory: What is the maximum number of edges that an abstract graph on n vertices can have without containing subgraphs of a certain kind.

Many questions of the above type on geometric graphs have been addressed in recent years. In a number of papers linear upper bounds have been established for the number of edges of a graph, under various forbidden configurations. They include the configurations of three pairwise crossing edges [2], four pairwise crossing edges [1], the configurations of an edge crossed by many edges [9], or even two large stars with all edges of one of them crossing all edges of the other [13].

For a constant number of 5 or more pairwise crossing edges Pavel Valtr has the best result [11]: a geometric graph on n vertices avoiding this configuration has $O(n \log n)$ edges. Adam Marcus and the present author [4] building on an earlier result of Pinchasi and Radoičić [10] prove an $O(n^{3/2} \log n)$ bound on the number of edges of an n vertex geometric graph not containing self-intersecting cycles of length four. No construction is known beating the $\Omega(n^{3/2})$ edges of an abstract graph having no cycles of length four.

For surveys on geometric graph theory, consult [5], [6] and [8].

In this paper we consider forbidding self-intersecting paths. For $k \geq 3$ we call a geometric graph *k-locally plane* if it has no self-intersecting subgraph (whose underlying abstract graph is) isomorphic to a path of length at most k .

Pach et al. [7] consider 3-locally plane graphs, i.e., the case of geometric graphs with no self-intersecting paths of length three. They prove matching lower and upper bounds of $\Theta(n \log n)$ on the maximal number of edges of a 3-locally plane graph on n vertices.

We extend the the lower bound result of [7] by forbidding all self-intersecting drawings of longer paths. Technically *k-locally plane* graphs are defined by forbidding self intersecting paths of length k or shorter, but forbidding only self-intersecting paths of length *exactly* k would lead to almost the same extremal function. Indeed, one can delete at most nk edges from any graph on n vertices, such that all the non-zero degrees in the remaining graph are larger than k . This ensures that all shorter paths can be extended to a path of length k . It is possible, but not likely, that if one only forbids paths of length k with *the first and last edges crossing* significantly higher number of

edges is achievable.

For even k a geometric graph is k -locally plane if and only if the $k/2$ -neighborhood of any vertex x is intersection free. Note that this requirement is much stronger than the similar condition on abstract graphs, namely that the $k/2$ neighborhood of any point is planar. One can construct graphs with girth larger than k and $\Omega(n^{\frac{k}{k-1}})$ edges. In such a graph the $k/2$ -neighborhood of any vertex is a tree, still by [7] the graph does not even have 3-locally plane drawing.

Extending the lower bound result in [7] we prove in Theorem 3 that for arbitrary fixed $k \geq 3$ there exist k -locally plane graphs on n vertices with $\Omega(n \log^{(\lfloor k/2 \rfloor)} n)$ edges. Here $\log^{(t)}$ denotes t times iterated logarithm and the hidden constant in Ω depends on k . Given two arbitrarily small disks in the plane we can even ensure that all edges of the constructed graph connect a vertex from one disk with another vertex from the second. This ensures that all the edges of the constructed geometric graph are arbitrarily close to each other in length and direction. In the view of the author this makes the existence of a high average degree (or for that matter high minimum degree) 100-locally plane graphs even more surprising.

As a simple corollary we can characterize the abstract graphs H such that any geometric graph having no self-intersecting subgraph isomorphic to H has a linear number of edges. These graphs H are the forests with at least two nontrivial components. To see the linear bound for the number of edges of a geometric graph avoiding a self-intersecting copy of such a forest H first delete a linear number of edges from an arbitrary geometric graph G until all non-zero degrees of the remaining geometric graph G' are at least $|V(H)|$. If G' is crossing free the linear bound of the number of edges follows. If you find a pair of crossing edges in G' they can be extended to a subgraph of G' isomorphic to H . On the other hand, if H contains a cycle, then even an abstract graph avoiding it can have a super-linear number of edges. If H is a tree of diameter k , then a k -locally plane geometric graph has no self-intersecting copy of H . Notice that the extremal number of edges in this case (assuming $k > 2$) is $O(n \log n)$ by [7], thus much smaller than the $\Omega(n^\alpha)$ edges ($\alpha > 1$) for forbidden cycles.

The main tool used in the proof of the above result is a randomized thinning procedure that takes a d edge colored bipartite graph of average degree $\Theta(d)$ and returns a subgraph on the same vertex set with average degree $\Theta(\log d)$ that does not have a special type of colored path (walk) of length four. The procedure can be applied recursively to obtain a subgraph avoiding longer paths of certain types. We believe this thinning procedure

to be of independent interest. In particular it can be used to obtain optimal 0-1 matrix constructions for certain avoided submatrix problems, see the exact statement in Section 4 and the details in [12].

In Section 2 we define two thinning procedures for edge colored bipartite graphs and prove their main properties. This is the most technical part of the paper. While these procedures proved useful in other setting too (and the author finds the involved combinatorics appealing) this entire section can be skipped if one reads the definition of k -flat graphs (the two paragraphs before Lemma 2.9) and is willing to accept Corollary 2.1 at the end of the section (we also use the simple observation in Lemma 2.10). In fact, in order to understand the main ideas behind the main result of this paper it is recommended to skip Section 2 on the first reading and to go straight to Section 3 where we use Corollary 2.1 to construct locally plane graphs with many edges. In Section 4 we comment on the optimality of the thinning procedures and have some concluding remarks.

2 Thinning

In this section we state and prove combinatorial statements about edge colored abstract graphs, i.e., we do not consider here geometric graphs at all. The connection to locally plane geometric graphs will be made clear in Section 3.

A *bipartite graph* is a triple $G = (A, B, E)$ with disjoint vertex sets A and B (called *sides*) and edge set $E \subseteq A \times B$. In particular, all graphs considered in this paper are *simple*, i.e., they do not have multiple edges or loops. The edge connecting the vertices x and y of G is denoted by xy or (x, y) . The latter notation is only used if $x \in A$ and $y \in B$. By a *d -edge coloring* of a graph we mean a mapping $\chi : E \rightarrow \{1, 2, \dots, d\}$ such that adjacent edges receive different colors. When we do not specify d we call such coloring a *proper edge coloring* but we always assume that the “set of colors” are linearly ordered. The degree of any vertex in G is at most the number d of colors, and our results are interesting if the average degree is close to d . Unless stated otherwise the subgraphs of an edge colored graph are considered with the inherited edge coloring. Our goal is to obtain a subgraph of G with as many edges as possible without containing a certain type of colored path or walk.

2.1 Heavy paths

A simple example of the above concept is the following. We call a path $P = v_0v_1v_2v_3$ of length 3 *heavy* if $v_0 \in B$ and the colors $c_1 = \chi(v_0v_1)$, $c_2 = \chi(v_1v_2)$, $c_3 = \chi(v_2v_3)$, satisfy $c_2 < c_1 \leq c_3$. The next lemma describes a thinning procedure that gets rid of heavy paths. Although we do not need this lemma in our construction, we present it as a simple analogue of our results for more complicated forbidden walks.

Lemma 2.1. *Let $G = (A, B, E)$ be a bipartite graph with a proper edge coloring $\chi : E \rightarrow \{1, 2, \dots, d\}$. Then there exists a subgraph $G' = (A, B, E')$ of G that does not contain a heavy path with $|E'| \geq |E|/(3\lceil\sqrt{d}\rceil)$.*

The constant 3 in the lemma could be replaced by the base of the natural logarithm. Notice that if G had average degree $\Theta(d)$, then the average degree of G' is $\Omega(\sqrt{d})$.

Proof: Let $t = \lceil\sqrt{d}\rceil$ and select a uniform random value $i_y \in \{1, 2, \dots, t\}$ independently for each vertex $y \in B$. We say that an edge $e \in E$ is of *class* $\lceil\chi(e)/t\rceil$. We call an edge $e = (x, y) \in E$ *eligible* if its class is i_y . Let the subgraph $G' = (A, B, E')$ consists of those eligible edges $e = (x, y) \in E$ for which there exists no other eligible edge $(x, y') \in E$ of the same class. Note that the words “class” and “eligible” will be used in a different meaning when defining the two thinning procedures in the next subsection.

By the construction, all edges incident to a vertex $x \in A$ have different classes and all edges incident to a vertex $y \in B$ have the same class. Let e_1 , e_2 and e_3 form a path in G' starting in B . Then e_2 and e_3 are of the same class, while the class of e_1 is different. For the colors $c_i = \chi(e_i)$ this rules out the order $c_2 < c_1 \leq c_3$. Thus, G' does not contain a heavy path. Note that another order, $c_3 \leq c_1 < c_2$ is also impossible.

The number of edges in G' depends on the random choices we made. Any edge $(x, y) \in E$ is eligible with probability $1/t$, and this is independent for all the edges incident to a vertex $x \in A$. As edges of a fixed color form a matching, there are at most t edges of any given class incident to x . Thus, we have

$$\Pr[(x, y) \in E'] \geq \frac{(1 - 1/t)^{t-1}}{t} > \frac{1}{3t}.$$

The expected number of edges in G' is

$$\text{Exp}[|E'|] \geq \frac{|E|}{3t}.$$

It is possible to choose the random variables i_y so that the size of E' is at least as large as its expected value. This proves the lemma. \square

2.2 Fast and slow walks

Next we turn to more complicated forbidden subgraphs. For motivation we mention that self-crossing paths of length 4 in the 3-locally plane graphs of [7] (considered with their natural edge coloring) are exactly the *fast walks* (to be defined below). For technical reasons, it will be more convenient to consider walks, i.e., to permit that a vertex is visited more than once, but we will not allow *backtracking*, i.e., turning back on the same edge immediately after it was traversed. Thus, for us a *walk* of length k is a sequence v_0, v_1, \dots, v_k of vertices in the graph such that $v_{i-1}v_i$ is an edge for $1 \leq i \leq k$ and $v_{i-2} \neq v_i$ for $2 \leq i \leq k$. The χ -*coloring* (or simply coloring) of this walk is the sequence $(\chi(v_0v_1), \chi(v_1v_2), \dots, \chi(v_{k-1}v_k))$ of the colors of the edges of the walk. If χ is a proper edge coloring, then any two consecutive elements of the coloring sequence are different.

We use \log to denote the binary logarithm. We introduce the notation $P(a, b)$ for two non-equal strings $a, b \in \{0, 1\}^t$ to denote the first position $i \in \{1, 2, \dots, t\}$, where a and b differ. We consider the set $\{0, 1\}^t$ to be ordered lexicographically, i.e., for $a, b \in \{0, 1\}^t$ we have $a < b$ if a has 0 in position $P(a, b)$ (and thus b has 1 there).

The following trivial observation is used often in this paper. We state it here without a proof.

Lemma 2.2. *Let $t \geq 1$ and let a, b and c be distinct binary strings of length t with $P(a, b) < P(a, c)$. We have $P(b, c) = P(a, b)$. Furthermore $a > b$ implies $c > b$, and $a < b$ implies $c < b$.*

A walk of length 4 with coloring (c_1, c_2, c_3, c_4) is called a *fast walk* if $c_2 < c_3 < c_4 \leq c_1$. Note that a fast walk may start in either class A or B . We call a walk of length 4 a *slow walk* if it starts in the class B and its coloring (c_1, c_2, c_3, c_4) satisfies $c_2 < c_3 < c_4$ and $c_2 < c_1 \leq c_4$. Note that either the color c_1 or c_3 can be larger in a slow walk, or they can be equal.

The two *thinning procedures* below find a random subgraph of an edge-colored bipartite graph. One is designed to avoid slow walks, the other is designed to avoid fast walks.

Lexicographic thinning Let $G = (A, B, E)$ be a bipartite graph and let $\chi : E \rightarrow \{1, \dots, d\}$ be a proper edge coloring with $d \geq 2$. *Lexicographic thinning* is a randomized procedure that produces a subset $E' \subseteq E$ of the edges and the corresponding subgraph $G' = (A, B, E')$ of G as follows:

Let $t = \lceil \frac{\log d}{2} \rceil + 1$. Let H be the set of triplets (a, i, z) , where $a \in \{0, 1\}^t$, $i \in \{2, 3, \dots, t\}$, $z \in \{1, 2, 3, \dots, 2^i\}$, and a has 0 in position i . Straightforward calculation gives that $|H| = 2^{2t} - 2^{t+1} \geq 2d$.

We order H lexicographically, i.e., $(a, i, z) < (b, j, s)$ if $a < b$, or $a = b$ and $i < j$, or $(a, i) = (b, j)$ and $z < s$.

Consider the following random function $F : \{1, \dots, d\} \rightarrow H$. We select uniformly at random the value $F(1) = (a, i, z) \in H$ with the property that the first bit of a is 0. We make $F(2)$ to be the next element of H larger than $F(1)$, and in general $F(k)$ is the next element of H larger than $F(k-1)$ for $2 \leq k \leq d$. As $|H| \geq 2d$ and $F(1)$ is chosen from the first half of H , this defines F . In what follows we simply identify the color k with the element $F(k) \in H$ without any reference to the function F .

We say that $(a, i, z) \in H$ and any edge with this color is of *class* a and *type* i , while z will play no role except in counting how many values it can take.

We choose an independent uniform random value $a_x \in \{0, 1\}^t$ for each vertex $x \in A \cup B$. Let $e = (x, y) \in E$ be an edge of class a and type i . We say that e is *eligible* if $a = a_y < a_x$ and $P(a, a_x) = i$. Let the subgraph $G' = (A, B, E')$ contain those edges $e \in E$ that are eligible but not adjacent to another eligible edge e' of the same type as e .

Reversed thinning Let $G = (A, B, E)$ be a bipartite graph and let $\chi : E \rightarrow \{1, \dots, d\}$ be a proper edge coloring with $d \geq 2$. *Reversed thinning* is a randomized procedure that produces a subset $E' \subseteq E$ of the edges and the corresponding subgraph $G' = (A, B, E')$ of G .

Reversed thinning is almost identical to the lexicographic thinning, the only difference is in the ordering of the set H . We define t and H as in the case of lexicographic thinning. Recall that H is the set of triplets (a, i, z) where $a \in \{0, 1\}^t$, $i \in \{2, 3, \dots, t\}$, $z \in \{1, 2, \dots, 2^i\}$, and a has 0 in position i . We still order $\{0, 1\}^t$ lexicographically, but now we reverse the lexicographic order of H in the middle term i . That is, we have $(a, i, z) < (b, j, s)$ if $a < b$, or $a = b$ and $i > j$ or $(a, i) = (b, j)$ and $z < s$.

We define the function $F : \{1, \dots, d\} \rightarrow H$, the *types* and *classes* of colors and edges, *eligible* edges and the subset E' of edges the same way as for the lexicographic thinning, but using this modified ordering of H .

Note that we associated a type in $\{2, \dots, t\}$ and a class in $\{0, 1\}^t$ to each edge in either procedure and they satisfy that

- an edge with a smaller class has smaller color;
- among edges of equal class an edge with smaller type has smaller color in the case of lexicographic thinning and it has larger color in the case of reversed thinning;

- among the edges incident to a vertex at most 2^i have the same class a and the same type i .

Proving most of the properties of the thinning procedures this is all we need to know about how classes and types are associated to the edges and we could use a deterministic scheme for F . But for Lemma 2.6 we need that all types are well represented and the randomization in the identification function F (as well as the dummy first bit of the class) is introduced to ensure this on the average. This randomization is not needed if one assumes all color classes have roughly the same size.

The next lemmas state the basic properties of the thinning procedure. Lemma 2.3 lists common properties of the two procedures, while Lemmas 2.4 and 2.5 state the result of the thinning satisfies its “design criteria” avoiding slow or fast walks. Finally Lemma 2.6 shows that enough edges remain in the constructed subgraphs on average. Note that Lemmas 2.4 and 2.5 are special cases of the more complex Lemmas 2.7 and 2.8 proved independently. We state and prove the simple cases separately for clarity, but these proofs could be skipped.

Lemma 2.3. *Let $G = (A, B, E)$ be a bipartite graph with a proper edge coloring $\chi : E \rightarrow \{1, \dots, d\}$. If $G' = (A, B, E')$ is the result of either the lexicographic or the reversed thinning then we have*

- Adjacent edges in G' have distinct types.*
- If two edges of G' meet in B , they have the same class.*
- Suppose two distinct edges e and e' of G' meet in A . Let their classes and types be a, a' and i, i' , respectively. If $i < i'$ then $a < a'$ and $P(a, a') = i$.*
- G' has no heavy path.*

Proof: The definition of E' immediately gives a).

For b) note that all eligible edges incident to $y \in B$ have a_y for class.

For c) let $x \in A$ be the common vertex of the two edges and apply Lemma 2.2 for a_x, a , and a' .

Finally d) follows since if a walk of G' starts in B then its coloring (c_1, c_2, c_3) must satisfy that c_1 and c_2 have different class by c) but c_2 and c_3 have the same class by b), so $c_2 < c_1 \leq c_3$ is impossible. \square

Lemma 2.4. *Let $G = (A, B, E)$ be a bipartite graph with a proper edge coloring. Lexicographic thinning produces a subgraph G' with no slow walk.*

Proof: Suppose $v_0v_1v_2v_3v_4$ is a walk in G' starting at $v_0 \in B$ and let its coloring be (c_1, c_2, c_3, c_4) . Assume $c_1 > c_2 < c_3 < c_4$. We show that $c_1 > c_4$, so this walk is not slow. By Lemma 2.3/a,b, as c_2 and c_3 are colors of edges incident to $v_2 \in B$, they have the same class, but they have different types: $c_2 = (a, i, z)$, $c_3 = (a, j, s)$ with $i \neq j$. We use lexicographic ordering, so $c_2 < c_3$ implies $i < j$. Both c_1 and c_2 are colors of edges incident to $v_1 \in A$, so by Lemma 2.3/c their classes are different. Since $c_1 > c_2$ we have $b > a$ for the class b of c_1 . Still by Lemma 2.3/c $P(a, b) = i$. Similarly, c_3 and c_4 are colors of distinct edges in E' incident to $v_3 \in A$, so they have different classes. As $c_3 < c_4$ we have $c > a$ for the class c of c_4 . We have $P(a, c) = j$. By Lemma 2.2 we have $b > c$. This proves $c_1 > c_4$ as claimed. \square

Lemma 2.5. *Let $G = (A, B, E)$ be a bipartite graph with a proper edge coloring. Reversed thinning produces a subgraph G' with no fast walk.*

Proof: Suppose $v_0v_1v_2v_3v_4$ is a walk in G' with coloring (c_1, c_2, c_3, c_4) . Assume $c_1 > c_2 < c_3 < c_4$. We show that $c_1 < c_4$, so this walk is not fast. First assume the walk starts at $v_0 \in A$. As G' does not contain a heavy path, $v_3v_2v_1v_0$ is not heavy, so $c_1 < c_3$. This implies $c_1 < c_4$ as claimed.

Now assume $v_0 \in B$. Just as in the proof of the previous lemma, c_2 and c_3 are colors of edges incident to $v_2 \in B$, so they have the same class, but they have different types: $c_2 = (a, i, z)$, $c_3 = (a, j, s)$ with $i \neq j$. We use the reversed ordering, so $c_2 < c_3$ implies $i > j$. Both c_1 and c_2 are colors of edges incident to $v_1 \in A$, so their classes are different. Since $c_1 > c_2$ we have $b > a$ for the class b of c_1 and $P(a, b) = i$. Similarly, c_3 and c_4 are colors of distinct edges in E' incident to $v_3 \in A$, so they have different classes. As $c_3 < c_4$ we have $c > a$ for the class c of c_4 and $P(a, c) = j$. By Lemma 2.2 we have $c > b$. This proves $c_1 < c_4$ as claimed. \square

Below we estimate the number of edges in E' . Recall that both thinning procedures are randomized. We can show that the subgraphs they produce have a large expected number of edges. We did not make any effort to optimize for the constant in this lemma.

Lemma 2.6. *Let $G = (A, B, E)$ be a bipartite graph with a d edge coloring. Let $G' = (A, B, E')$ be the result of either the lexicographic or the reversed thinning. We have*

$$\text{Exp}[|E'|] \geq \frac{t-1}{240d}|E| \geq \frac{\log d}{480d}|E|.$$

Proof: We compute the probability for a fixed edge $e = (x, y) \in E$ to end up in E' . For this we break down the random process producing E'

into three phases. In the first phase we select F . With F the color $\chi(e)$ is identified with an element of H , most importantly, the type of e is fixed. In the second phase we select a_x and a_y uniformly at random. These choices determine if e is eligible. If e is not eligible then $e \notin E'$. So in the third phase we consider F , a_x , and a_y fixed and assume e is eligible. We select the random values a_z for vertices $z \neq x, y$. This effects if other edges are eligible and if $e \in E'$. Here is the detailed calculation:

Let $e \in E$ have the color $\chi(e) = k \in \{1, 2, \dots, d\}$. The choice of F in the first phase determines $F(k) = (a, i, z) \in H$. By the construction of F , if we call a' the last $t - 1$ bits of a then (a', i, z) is uniformly distributed among all its possible values. In particular, the probability that e becomes a type i edge is exactly

$$\Pr[e \text{ is of type } i] = \frac{2^{t+i-2}}{2^{2t-1} - 2^t} = \frac{2^i}{2^{t+1} - 4}.$$

For phase two we consider the function F identifying colors with elements of H fixed. Consider an edge $e = (x, y)$ of color $(a, i, z) \in H$. This edge is eligible if $a_y = a < a_x$ and $P(a, a_x) = i$. This determines a_y and the first i bits of a_x . Recall that by the definition of H the string a has 0 in position i . Thus, the probability that the edge e of type i is eligible is exactly 2^{-t-i} .

Assume for the third phase that e is eligible. Consider another edge $e' = (x, y') \in E$ with color $\chi(e') = (a', i, z')$ of type i . If a and a' do not agree in the first i positions, then e' is not eligible. If they agree in the first i positions, then e' is eligible if and only if $a_{y'} = a'$, so with probability 2^{-t} . Let k'_e be the number of edges (x, y') of type i with the first i digits of their class agreeing with a , but not counting e itself. We have $k'_e < 2^t$.

Consider now an edge $e'' = (x'', y) \in E$ with color (a'', i, z'') such that $e'' \neq e$. As e is eligible, e'' can only be eligible if $a'' = a$. If $a'' = a$ then e'' is eligible if and only if a_x and $a_{x''}$ agree in the first i digits. This happens with probability 2^{-i} . For the number k''_e of the edges $(x'', y) \neq e$ of type i and class a we have $k''_e < 2^i$.

In phase three the eligibility of all these edges e' and e'' adjacent with e are independent events.

Still consider the function F fixed. The total probability for an edge e of type i to be in E' is

$$\begin{aligned} \Pr[e \in E' | F] &= 2^{-t-i} (1 - 2^{-t})^{k'_e} (1 - 2^{-i})^{k''_e} \\ &\geq 2^{-t-i} (1 - 2^{-t})^{2^t-1} (1 - 2^{-i})^{2^i-1} \\ &> 2^{-t-i} / 7.5. \end{aligned}$$

The total probability of $e \in E'$ can be calculated from the distribution of its type and the above conditional probability depending on its type:

$$\Pr[e \in E'] > \sum_{i=2}^t \frac{2^i}{2^{t+1} - 4} \cdot \frac{2^{-t-i}}{7.5} > \frac{t-1}{15 \cdot 2^{2t}} > \frac{t-1}{240d}.$$

The expected number of edges in E' is then

$$\text{Exp}[|E'|] > \frac{t-1}{240d}|E| \geq \frac{\log d}{480d}|E|.$$

□

Theorem 1. *Let $G = (A, B, E)$ be a bipartite graph with a proper edge coloring. There exists a subgraph $G' = (A, B, E')$ of G without a slow walk and with $|E'| > \frac{\log d}{480d} \cdot |E|$. Similarly, there exists a subgraph $G'' = (A, B, E'')$ of G without a fast walk and with $|E''| > \frac{\log d}{480d} \cdot |E|$.*

Proof: By Lemmas 2.4 and 2.5 the results of the lexicographic and reversed thinnings avoid the slow and fast walks, respectively. There exists an instance of the random choices with the size of E' being at least its expectation given in Lemma 2.6. This proves the theorem. □

2.3 Longer forbidden walks

Here we generalize the concept of fast and slow walks to longer walks. Consider a bipartite graph $G = (A, B, E)$ with a proper edge coloring. For $k \geq 2$ we call a walk of length $2k$ in G a k -fast walk if its coloring (c_1, \dots, c_{2k}) satisfies $c_1 > c_2 > \dots > c_k < c_{k+1} < c_{k+2} < \dots < c_{2k}$ and $c_1 \geq c_{2k}$. For $k \geq 2$ we call a walk of length $2k$ in G a k -slow walk if its coloring (c_1, \dots, c_{2k}) satisfies the following: $c_{2j-1} > c_{2j}$ for $1 \leq j \leq k/2$; $c_{2j} < c_{2j+1}$ for $1 \leq j < k/2$; $c_{2j-1} < c_{2j}$ for $k/2 < j \leq k$; $c_{2j} > c_{2j+1}$ for $k/2 \leq j < k$; and finally $c_1 \geq c_{2k}$. If a k -slow walk starts in the vertex set B we call it a (k, B) -slow walk, otherwise it is a (k, A) -slow walk.

Notice that 2-fast walks are the fast walks and $(2, B)$ -slow walks are the slow walks with their orientation reversed. For the coloring (c_1, \dots, c_{2k}) of a k -fast walk c_j is in between c_{j-1} and c_{j+1} for all $1 < j < 2k$, $j \neq k$, while c_k is the smallest color on this list. For the coloring (c_1, \dots, c_{2k}) of a k -slow walk the situation is reversed: the only index $1 < j < 2k$ with c_j being in between c_{j-1} and c_{j+1} is the index $j = k$.

In order to apply the lexicographic and reversed thinning recursively we have to change the coloring of the subgraph. Let $G = (A, B, E)$ be a

bipartite graph with a proper edge coloring given by $\chi : E \rightarrow \{1, \dots, d\}$. Let $G' = (A, B, E')$ the result of the lexicographic or the reversed thinning of G . Recall that the edges in E' have a type $2 \leq i \leq t$ with $t = \lceil (\log d)/2 \rceil + 1$. The *type edge coloring* of G' is the map $\chi' : E' \rightarrow \{1, \dots, t-1\}$ defined by $\chi'(e) = t+1-i$ for an edge $e \in E'$ of type i . By Lemma 2.3/a χ' is a proper edge coloring of G' .

Lemma 2.7. *Let $G = (A, B, E)$ be a bipartite graph with proper edge coloring given by $\chi : E \rightarrow \{1, \dots, d\}$. Let $G' = (A, B, E')$ the result of the lexicographic thinning of G . Let χ' be the type edge coloring of G' and let $k \geq 2$. If a subgraph $G'' = (A, B, E'')$ of G' with its edge coloring given by χ' has no (k', A) -slow walk for $2 \leq k' < k$ then G'' with its edge coloring given by χ has no (k, B) -slow walk.*

Notice that the $k = 2$ case of this lemma gives a second proof of Lemma 2.4.

Proof: Let $W = v_0 v_1 \dots v_{2k}$ be a walk in G'' starting at $v_0 \in B$ and let its χ -coloring be $(c_1, c_2, \dots, c_{2k})$. Assume that $c_i > c_{i+1}$ or $c_i < c_{i+1}$ for $1 \leq i < 2k$ as required in the definition of a (k, B) -slow walk. We need to show $c_1 < c_{2k}$.

We identify the colors of χ with the triplets $(a, i, z) \in H$ as in the definition of lexicographic thinning. We let $c_j = (a_j, i_j, z_j)$. The χ' -coloring of W is $(t+1-i_1, \dots, t+1-i_{2k})$. We have $i_j \neq i_{j+1}$ for $1 \leq j < 2k$.

For $1 \leq j < k$ the colors c_{2j} and c_{2j+1} are colors of distinct edges incident to $v_{2j} \in B$, so by Lemma 2.3/b their class is the same: $a_{2j} = a_{2j+1}$. We consider lexicographic thinning, so the order between c_{2j} and c_{2j+1} is the same as the order between their types: i_{2j} and i_{2j+1} . For $1 \leq j < k/2$ we have $i_{2j} < i_{2j+1}$ but for $k/2 \leq j < k$ we have $i_{2j} > i_{2j+1}$.

For $1 \leq j \leq k$ the colors c_{2j-1} and c_{2j} are colors of edges incident to $v_{2j-1} \in A$. By Lemma 2.3/c the classes of these colors do not agree, and the ordering between the classes, between the types, and between the colors themselves are the same. Thus, for $1 \leq j \leq k/2$ we have $a_{2j-1} > a_{2j}$ and $i_{2j-1} > i_{2j}$. For $k/2 < j \leq k$ we have $a_{2j-1} < a_{2j}$ and $i_{2j-1} < i_{2j}$. Also by Lemma 2.3/c for all $1 \leq j \leq k$ we have $P(a_{2j-1}, a_{2j}) = \min(i_{2j-1}, i_{2j})$.

The sequence a_1, a_2, \dots, a_k is monotone decreasing and it changes only in every other step. The first positions of change between distinct consecutive elements are $i_2, i_4, \dots, i_{2\lfloor k/2 \rfloor}$. So we have $a_1 > a_k$ and $P(a_1, a_k) = \min(S_1)$ for the set $S_1 = \{i_2, i_4, \dots, i_{2\lfloor k/2 \rfloor}\}$.

Similarly, $a_k, a_{k+1}, \dots, a_{2k}$ is monotone increasing and it changes only in every other step. The first positions of change between distinct con-

secutive elements are $i_{2\lfloor k/2\rfloor+1}, \dots, i_{2k-3}, i_{2k-1}$. So we have $a_k < a_{2k}$ and $P(a_k, a_{2k}) = \min(S_2)$ for the set $S_2 = \{i_{2\lfloor k/2\rfloor+1}, \dots, i_{2k-3}, i_{2k-1}\}$.

Let us consider an arbitrary value $1 \leq j < k/2$ and let $2 \leq k' = k - 2j + 1 < k$. Consider the $2k'$ long middle portion W' of the walk W : let $W' = v_{2j-1}v_{2j} \dots v_{2k-2j+1}$. This is a walk of length $2k'$ in G'' starting at $v_{2j-1} \in A$. By our assumption on G'' this is not a (k', A) -slow walk if considered with the coloring χ' . But the χ' -coloring of W' is $(t + 1 - i_{2j}, t + 1 - i_{2j+1}, \dots, t + 1 - i_{2k-2j+1})$ and the consecutive values in this list compare as required for a (k', A) -slow walk. Therefore, we must have $t + 1 - i_{2j} < t + 1 - i_{2k-2j+1}$.

We have just proved $i_{2j} > i_{2k-2j+1}$ for $1 \leq j < k/2$. For even k and $j = k/2$ the same formula compares the types of two consecutive edges of W and we have already seen its validity in that case too. For every element of the set S_1 we have just found a smaller element of the set S_2 . Therefore, $\min(S_1) > \min(S_2)$. Using that $a_{2k} > a_k$ and $P(a_1, a_k) = \min(S_1) > \min(S_2) = P(a_k, a_{2k})$ Lemma 2.2 gives $a_1 < a_{2k}$. This implies $c_1 < c_{2k}$ and finishes the proof of the lemma. \square

Lemma 2.8. *Let $G = (A, B, E)$ be a bipartite graph with proper edge coloring given by $\chi : E \rightarrow \{1, \dots, d\}$. Let $G' = (A, B, E')$ the result of the reversed thinning of G . Let χ' be the type edge coloring of G' and let $k \geq 2$. If a subgraph $G'' = (A, B, E'')$ of G' with its edge coloring given by χ' has no (k', A) -slow walk for $2 \leq k' < k$ then G'' with its edge coloring given by χ has no k -fast walk.*

Notice that the $k = 2$ case of this lemma gives a second proof of Lemma 2.5.

Proof: The proof of this lemma is very similar to that of Lemma 2.7.

Let $W = v_0v_1 \dots v_{2k}$ be a walk in G'' and let its χ -coloring be $(c_1, c_2, \dots, c_{2k})$. Assume that $c_1 > c_2 > \dots > c_k < c_{k+1} < c_{k+2} < \dots < c_{2k}$ as required in the definition of a k -fast walk. We need to show $c_1 < c_{2k}$. Instead, we prove the slightly stronger statement that the class of c_1 is smaller than the class of c_{2k} . We first do that for walks starting in B : assume that $v_0 \in B$.

We identify the colors of χ with the triplets $(a, i, z) \in H$ as in the definition of reverse thinning. We let $c_j = (a_j, i_j, z_j)$. Note that the χ' -coloring of W is $(t + 1 - i_1, \dots, t + 1 - i_{2k})$. We have $i_j \neq i_{j+1}$ for $1 \leq j < 2k$.

For $1 \leq j < k$ the colors c_{2j} and c_{2j+1} are colors of distinct edges incident to $v_{2j} \in B$, so by Lemma 2.3/b their classes are the same: $a_{2j} = a_{2j+1}$. Thus the order between c_{2j} and c_{2j+1} is determined by the order between i_{2j} and i_{2j+1} , but as we use the reversed ordering in H the order between c_{2j} and

c_{2j+1} is reversed compared to the order between i_{2j} and i_{2j+1} . Specifically, for $1 \leq j < k/2$ we have $i_{2j} < i_{2j+1}$ and for $k/2 \leq j < k$ we have $i_{2j} > i_{2j+1}$. For $1 \leq j \leq k$ the colors c_{2j-1} and c_{2j} are colors of edges incident to $v_{2j-1} \in A$. By Lemma 2.3/c the classes of these colors do not agree, and the ordering between the classes, between the types, and between the colors themselves are the same. Thus, for $1 \leq j \leq k/2$ we have $a_{2j-1} > a_{2j}$ and $i_{2j-1} > i_{2j}$. For $k/2 < j \leq k$ we have $a_{2j-1} < a_{2j}$ and $i_{2j-1} < i_{2j}$. Also by Lemma 2.3/c for $1 \leq j \leq k$ we have $P(a_{2j-1}, a_{2j}) = \min(i_{2j-1}, i_{2j})$.

At this point we have the same ordering of the classes and types of the coloring of W as in the proof of Lemma 2.7. We also have the same assumption that G'' with the edge coloring χ' has no (k', A) -slow walk for $2 \leq k' < k$. So we arrive to the same conclusion $a_1 < a_{2k}$ with an identical proof.

We finish the proof of the lemma by considering the alternative case when W starts in A . Now c_1 and c_2 are colors of edges sharing a vertex $v_1 \in B$, so by Lemma 2.3/b their classes are equal. Similarly, the classes of c_{2k-1} and c_{2k} are equal, so it is enough to prove that the class of c_2 is smaller than the class of c_{2k-1} . For $k = 2$ this follows directly from Lemma 2.3/c. For $k > 2$ the walk $W' = v_1 v_2 \dots v_{2k-1}$ is exactly the type of walk we considered for $k_0 = k - 1$. As it starts in B we have already proved that the class of its first edge is smaller than the class of its last edge. This finishes the proof of the case of a walk starting in A and also the proof of Lemma 2.8. \square

Lemmas 2.7 and 2.8 set the stage to use the thinning procedures recursively to get subgraphs avoiding (k, B) -slow or k -fast walks. In a single application of either thinning procedure the number d of colors in the original coloring is replaced by $t - 1 = \lceil \log d/2 \rceil$ colors in the type coloring. Here $4(t - 1) > \log(4d)$, so after k recursive calls we still have more than $\log^{(k)}(4d)/4$ colors, where $\log^{(k)}$ stands for the k times iterated log functions. (Of course, this only makes sense if $\log^{(k)}(4d) > 2$. Otherwise we can get stuck, as neither thinning procedure is defined in the pathetic case of $d = 1$ colors.) Making optimal random choices we may assume that each thinning procedure yields at least the expected number of edges. Thus, the ratio of the number of edges and the number of colors decreases by at most a factor of 240 in each iteration. Clearly, the only interesting case is when the original average degree was $\Theta(d)$ in which case the average degree after k iterations remains $\Theta(\log^{(k)} d)$. The constant of proportionality depends on k .

Theorem 2. *Let $G = (A, B, E)$ be a bipartite graph with a proper edge*

d -coloring and let $k \geq 2$. There exists a subgraph $G' = (A, B, E')$ of G without a (k', B) -slow walk for any $2 \leq k' \leq k$ and with $|E'| > \frac{\log^{(k-1)} d}{4 \cdot 240^{k-1} d} |E|$. Similarly, there exists a subgraph $G'' = (A, B, E'')$ of G without a k' -fast walk for any $2 \leq k' \leq k$ and with $|E''| > \frac{\log^{(k-1)} d}{4 \cdot 240^{k-1} d} |E|$.

Proof: We apply the thinning procedures recursively. First we use lexicographic and reversed thinning to obtain subgraphs G_1 and G_2 of G , respectively. We make sure these graphs have at least as many edges as the expected number given in Lemma 2.6. If $k = 2$ we are done, $G' = G_1$ and $G'' = G_2$ satisfy the conditions of the theorem. Otherwise we consider G_1 and G_2 with the type edge coloring. We find recursively their subgraphs G' and G'' , respectively, avoiding (k', A) -slow walks for $2 \leq k' \leq k - 1$. This can be done because the sides A and B play symmetric roles. Finally, we apply Lemmas 2.7 and Lemma 2.8 to see that the subgraphs G' and G'' , if considered with the original edge coloring of G , avoid all walks required in the theorem. The number of edges guaranteed in the subgraphs is calculated in the paragraph preceding the theorem and is at least the stated bound. \square

2.4 k -flat graphs

In this subsection we establish that removing a linear number of edges from a k -fast walk free graph the resulting graph has special structural properties. We note here that the recursive thinning construction that we used to arrive at k -fast walk free graphs results in a graph that itself is k -flat as defined below. We chose however to keep the inductive part of the proof simple and concentrated only on (k, B) -slow and k -fast walks. We derive the more complicated properties from these simpler ones. Note that in this subsection we do not use that our graphs are bipartite.

Let G be a graph and χ a proper edge coloring of G . We define the *shaving* of the graph G to be the subgraph obtained from G by deleting the edge with the largest color incident to every (non-isolated) vertex. Clearly, we delete at most n edges, where n is the number of vertices in G . We define the k -*shaving* of G to be the subgraph obtained from G by repeating the shaving operation k times. Clearly, we delete at most kn edges for a k -shaving.

Let W be a walk of length m in a properly edge colored graph G , and assume its coloring is (c_1, \dots, c_m) . We define the *height function* h_W from

$\{1, \dots, m\}$ to the integers recursively letting $h_W(1) = 0$ and

$$h_W(i+1) = \begin{cases} h_W(i) + 1 & \text{if } c_{i+1} > c_i \\ h_W(i) - 1 & \text{if } c_{i+1} < c_i \end{cases}$$

for $1 \leq i < m$. Note $h_W(i) + i$ is always odd. This function considers how the colors of the edges in the walk change, in particular, how many times the next color is larger and how many times it is smaller than the previous color.

We call a graph G with proper edge coloring k -flat if the following is true for every walk W in G . Let $m \geq 2$ be the length of W , let (c_1, \dots, c_m) be the coloring of W and assume that the height function satisfies $h_W(i) < 0$ for $2 \leq i \leq m$. If $m \leq 2k + 1$ or $h_W(i) \geq -k$ for all i then $c_1 > c_m$.

Lemma 2.9. *Let G be properly edge colored graph. Let $k \geq 1$ and assume G has no k' -fast walk for $2 \leq k' \leq k$. Then the $(k-1)$ -shaving G' of G is k -flat.*

Proof: We prove the following slightly stronger statement by induction on m . Let $W = v_0 \dots v_m$ be a walk of length m in G with coloring (c_1, \dots, c_m) . Let $1 \leq j \leq m$ be the largest index such that $h_W(j) = 1 - j$. Assume the walk $v_j v_{j+1} \dots v_m$ is in the $(k-1)$ -shaving G' of G . Also assume that $h_W(i) < 0$ for $2 \leq i \leq m$. If $m \leq 2k + 1$ or $h_W(i) \geq -k$ for all i then we claim $c_1 > c_m$. This statement is stronger than Lemma 2.9 since it allows for the initial decreasing segment of W be outside G' .

If $j = m$ the statement of the claim is obvious from the definition of the height function. This covers the $m = 2$ and $m = 3$ base cases. Let $m \geq 4$ and assume the statement is true for walks of length $m-1$ and $m-2$.

If $j > k + 1$ we have $h_W(j) = 1 - j < -k$ so we must have $m \leq 2k + 1$. Consider the walk $W' = v_1 \dots v_m$ of length $m-1$. We have $h_{W'}(i) = 1 - i < 0$ for $2 \leq i < j$ and $h_{W'}(i) \leq h_{W'}(j-1) + (i - (j-1)) = 3 + i - 2j < 0$ for $j \leq i \leq m-1$. Thus the inductive hypothesis is applicable and we get $c_1 > c_2 > c_m$.

Finally consider the $j \leq k + 1$ case. As the trivial $j = m$ case was already dealt with we also assume $j < m$. Clearly, $h_W(2) < 0$ implies $j \geq 2$. We chose a $w_0 \dots w_{j-2}$ walk in G ending at $w_{j-2} = v_j$ and with coloring (c'_1, \dots, c'_{j-2}) satisfying $c'_1 > c'_2 > \dots > c'_{j-2} > c_{j+1}$. This is possible since the edge $v_j v_{j+1}$ is in the $(k-1)$ -shaving G' of G , so we can find the edge $w_{j-3} w_{j-2}$ in the $(k-2)$ -shaving of G , $w_{j-4} w_{j-3}$ in the $(k-3)$ -shaving, and so on. We must have $c_1 > c'_1$ as otherwise $w_0 w_1 \dots w_{j-3} v_j v_{j-1} \dots v_0$ is a $(j-1)$ -fast walk and no such walk exists in G . Now consider the

walk $W' = w_0 w_1 \dots w_{j-3} v_j v_{j+1} \dots v_m$. This is a walk of length $m - 2$ and satisfies $h_{W'}(i) = 1 - i$ for $1 \leq i \leq j - 1$ and $h_{W'}(i) = h_W(i + 2)$ for $j - 1 \leq i \leq m - 2$. All requirements of the inductive hypothesis are satisfied, so we have $c'_1 > c_m$. Thus $c_1 > c'_1 > c_m$ as claimed. \square

Corollary 2.1. *Let $G = (A, B, E)$ be a bipartite graph with a proper edge d -coloring and let $k \geq 2$. There exists a k -flat subgraph $G' = (A, B, E')$ of G with $|E'| > \frac{\log^{(k-1)} d}{4 \cdot 240^{k-1}} |E| - (k - 1)(|A| + |B|)$.*

Proof: Combine Theorem 2 with Lemma 2.9 and the fact that $(k - 1)$ -shaving keeps all but at most $(k - 1)(|A| + |B|)$ edges of G . \square

The final lemma in this section is a simple but useful observation on k -flat graphs. It can also be stated for longer walks with height function bounded from below, but for simplicity we restrict attention to short walks.

Lemma 2.10. *Let $k \geq 1$ and let G be a properly edge colored k -flat graph. Let $W = v_0 \dots v_m$ be a walk in G of length $m \leq 2k + 1$ with coloring (c_1, \dots, c_m) . If $c_1 \geq c_i$ for all $1 \leq i \leq m$ then $h_W(i) \leq 0$ for all $1 \leq i \leq m$.*

Proof: We prove the contrapositive statement. Assume $h_W(i) > 0$ for some $1 \leq i \leq m$ and let i_0 be the smallest such index. Clearly, $i_0 \geq 2$, $h_W(i_0) = 1$ and for the walk $W' = v_{i_0} v_{i_0-1} \dots v_0$ we have $h_{W'}(i) = h_W(i_0 - i + 1) - 1 < 0$ for $1 < i \leq i_0$. So by the definition of k -flatness we have $c_{i_0} > c_1$. \square

3 Locally plane graphs

Locally plane graphs were introduced in the paper [7] (though the name appears first in this paper). That paper gives a simple construction for 3-locally plane graphs. We recall (a simplified version of) the construction as it is our starting point.

3.1 Construction of 3-locally plane graphs in [7]

Let $d \geq 1$ and consider the orthogonal projection of (the edge graph of) the d dimensional hypercube into the plane. A suitable projection of the “middle layer” of the hypercube provides the 3-locally plane graph. Here is the construction in detail:

Let $d \geq 1$ be fixed and set $b = \lfloor d/2 \rfloor$. The bit at position i in $x \in \{0, 1\}^d$ (the i th coordinate) is denoted by x_i for $1 \leq i \leq d$. We let $A = \{x \in \{0, 1\}^d \mid \sum_{i=1}^d x_i = b\}$ and $B = \{x \in \{0, 1\}^d \mid \sum_{i=1}^d x_i = b + 1\}$. The abstract graph

underlying the geometric graph to be constructed is $G_d = (A, B, E)$ with $(x, y) \in E$ if $x \in A$, $y \in B$ and x differs from y in a single position. This is the middle layer of the d dimensional hypercube. We define the edge coloring $\chi : E \rightarrow \{1, \dots, d\}$ that colors an edge $e = (x, y) \in E$ by the unique position $\chi(e) = i$ with $x_i \neq y_i$. Notice that this is a proper edge coloring. The number of vertices is $n = |A| + |B| = \binom{d}{b} + \binom{d}{b+1} \leq 2^d$, the number of edges is $|E| = \binom{d}{b}(d-b) > nd/4$. The average degree is greater than $d/2 \geq \log n/2$.

To make the abstract graph G_d into a geometric graph we project the hypercube into the plane. We give two possible projections here. The first is more intuitive and it is closer to the actual construction in [7]. We let $a_i = (10^i, i \cdot 10^i)$ for $1 \leq i \leq d$ and use a_i as the projection of the edge of the hypercube parallel to the coordinate axis i . The important property of these vectors is the following: higher indexed vectors have higher slope and much greater length. We identify the vertex $x \in A \cup B$ with the point $P_x = \sum_{i=1}^d x_i a_i$. The edges are represented by the straight line segment connecting their endpoints.

We give the second construction to obtain a graph where all edges are very close in length and direction. Let $0 < \epsilon < 10^{-d}$ arbitrary and consider the vectors $b_i = (\epsilon^{d+1-i}, 10^i \epsilon)$ and identify a point $x \in A \cup B$ with $Q_x = \sum_{i=1}^d x_i b_i$.

It is easy to verify that we get a geometric graph in both cases (i.e., the vertices are mapped to distinct points and no edge passes through a vertex that is not its endpoint). Note that edges of color i are all translates of the same vector a_i or b_i . We do not introduce separate notations for the two geometric graphs constructed this way as they will only be treated separately in the proof of Lemma 3.1, where we refer to them as the first and the second realization of G_d .

3.2 Self-intersecting paths in G_d

In [7] a graph very similar to G_d was shown to be 3-locally plane. Here we do more, we analyze all self-intersecting paths of G_d as follows.

Lemma 3.1. *Let W be a walk in G_d with coloring (c_1, \dots, c_m) and assume $c_1 \geq c_m$ and W and all its non-empty subwalks have a unique edge of maximal color. The first and last edges of W cross in either geometric realization if and only if m is even and there is an odd index $1 < j < m$ satisfying $c_1 > c_m > c_j \geq c_i$ for all $1 < i < m$.*

Proof: Let $W = v_0 \dots v_m$. Note that the first and the last edges cross if

and only if v_0 and v_1 are on different sides of the line ℓ through v_{m-1} and v_m and similarly v_{m-1} and v_m are on different sides of the line ℓ' through v_0 and v_1 . To analyze such separations consider the projection π_i to the y axis parallel to edges of color i .

Let us consider the first realization of G_d with the vectors a_i . We have $\pi_i(x, y) = y - ix$ and the projection of the vector a_j is of length $|i - j|10^j$. Thus higher colored edges map to longer intervals (except color i itself). Under the projection π_{c_1} the direction of the highest colored edge in the walks $v_1 \dots v_{m-1}$ (respectively, $v_1 \dots v_m$) determines which side v_{m-1} (respectively v_m) lies of the line ℓ' . Indeed this highest color cannot be c_1 , so the projections of the other edges will be much shorter and by the unique maximal color property we see that the walk contains at most 2^{k-1} edges having the k th largest color, so these shorter projections cannot add up to be more than the projection of the largest edge.

We can only have v_{m-1} and v_m lying on different sides of ℓ' if these edges of maximal color are distinct, thus we must have $c_m > c_i$ for all $1 < i < m$. From $c_1 \geq c_m$ and the unique maximal property we have $c_1 > c_m$. Taking c_j to maximize c_i for $1 < i < m$ (this is unique again) we have $c_1 > c_m > c_j \geq c_i$ for all $1 < i < m$.

It is left to prove that the first and last edges of W cross if and only if m is even and j is odd.

To prove this claim one has to use that G_d is bipartite with vertex sets A and B , and every edge of color c is a translate of the vector a_c with its head in B and tail in A . Thus, the vector $v_{i-1}v_i$ is either a_{c_i} or $-a_{c_i}$ depending on the parity of i . Which sides of ℓ' v_{m-1} and v_m lie is determined by the projections of the j th and last edge, so they are on opposite sides if j and m have different parities. Similarly, the sides of ℓ on which v_0 and v_1 lies is determined by the π_{c_m} projection of the first and j th edges, but as we have $c_1 > c_m > c_j$ they are on different sides if 1 and j has the same parity. See Figure 1 for a rough depiction of all four cases. This ends the proof of the claim and the part of the lemma regarding the first realization of G_d .

The proof for the second realization of G_d as a geometric graph (involving the vectors b_i) is only slightly more complicated. We have $\pi_i(x, y) = y - \epsilon^{d+1-i}x$ and $\pi_i(b_j) = \epsilon^{d+1-j}(1 + 10^j\epsilon) - 10^j\epsilon^{d+2-i} - \epsilon^{d+1-i}$. The ϵ^{d+1-i} terms alternate in sign in the projection on the edges and cancel completely for an even length walk. For odd length walks a single such term remains but it is dominated by the other terms if the walk has an edge with color above i . If however no such edge exists, the remaining uncanceled ϵ^{d+1-i} term dominates the other terms in the projection. Thus if the projection of the unique largest colored edge of a walk has color $c > i$ or if the walk has even



Figure 1: Rough picture of W in the four cases according to the parity of m and j . The scale is set to match the edge $v_{j-1}v_j$ of color c_j , shorter edges are approximated by zero, and only initial segments of the two longer edges are depicted

length the sign of the π_i image of the largest colored edge determines the sign of the projection of the walk. But in case $c < i$ and the walk has odd length, then the non-canceling ϵ^{d+1-i} term determines the sign.

Let the edge $v_{j-1}v_j$ be the one with the unique largest color in the walk $v_1 \dots v_{m-1}$. A case analysis of the parities of j , m and whether $c_j > c_1$ or $c_j > c_m$ hold show that the first and last edges of W cross if and only if $c_m > c_j$, j is odd and m is even – as claimed in the lemma. \square

We call $v_m \dots v_0$ the *reverse* of the walk $W = v_0 \dots v_m$.

Lemma 3.2. *Let $k \geq 1$ and let G' be a k -flat subgraph of G_d . If the length of a walk W in G' does not exceed $2k + 1$, then W has unique edge of largest color.*

Proof: Let W be a walk of length $3 \leq m \leq 2k + 1$ in G' with coloring (c_1, \dots, c_m) and assume the largest color is not unique. We may assume $c_1 = c_m > c_i$ holds for all $1 < i < m$, otherwise one can take a suitable subwalk of W . By Lemma 2.10 we have $h_W(m) \leq 0$. Consider W' the reverse of the path W . Clearly, $h_{W'}(m) = -h_W(m)$, but also by Lemma 2.10 we have $h_{W'}(m) \leq 0$. So we must have $h_W(m) = 0$ and m must be odd.

The contradiction that proves the statement of the lemma comes from the simple observation that between two consecutive appearances of a color

in any walk of G_d there always are an even number of edges, so m must be even. To see this recall that G_d is bipartite with sides A and B where A consists of the 0-1 sequences of length d with $\lfloor d/2 \rfloor$ ones, while the 0-1 sequences in B contain one more ones. The A end of an edge of color c has 0 at position c , while the B end of this edge has 1 there. Along edges of other colors bit c does not change. Thus if a walk traverses an edge of color c from A to B say, then along the walk bit c remains 1 until the next time the walk traverses an edge of color c and this has to be from B to A . \square

We note that Lemma 3.2 immediately implies that the girth of a k -flat subgraph of G_d is at least $2k+2$. This estimate can be improved by observing that any cycle in G_d has an even number of edges of any color (you need to flip a bit even times to get back to the original state). In particular any cycle has at least two occurrences of the largest color. These edges break the cycle into two paths sharing two edges. Both path has to be of length at least $2k+2$, the length of the cycle is at least $4k+2$.

As every properly edge colored graph is 1-flat the $k=1$ case of the next lemma establishes that G_d is 3-locally plane.

Lemma 3.3. *For any $k \geq 1$, any k -flat subgraph of G_d is $(2k+1)$ -locally plane.*

Proof: Let G' be a k -flat subgraph of G_d . We need to show that no walk (or path) W of length $m \leq 2k+1$ is self-intersecting. It is clearly enough to show that the first and the last edges of W do not cross and we may assume that the color of the first edge is not smaller than that of the last edge (otherwise simply consider the same walk reversed). By Lemma 3.2 W and all its subwalks have a unique edge of maximal color, so Lemma 3.1 applies. It is enough to show that the coloring (c_1, \dots, c_m) of W does not satisfy the conditions of Lemma 3.1. Assume the contrary. So m is even, and there is an odd index j such that $c_1 > c_m > c_j \geq c_i$ for all $1 < i < m$.

Consider the walk $W_1 = v_m v_{m-1} \dots v_{j-1}$ of length $m-j+1$. Its coloring is (c_m, \dots, c_j) and c_m is its largest color. By Lemma 2.10 $h_{W_1}(m-j+1) \leq 0$. In fact, as $m-j+1$ is even $h_{W_1}(m-j+1)$ is odd, so we have $h_{W_1}(m-j+1) \leq -1$.

Consider the walk $W_2 = v_j v_{j-1} \dots v_1$ of length $j-1$ and its coloring (c_j, \dots, c_1) . The largest color in W_2 is c_j , so we have $h_{W_2}(j-1) \leq 0$ by Lemma 2.10. And again by parity considerations $h_{W_2}(j-1) \leq -1$.

It is easy to see that $h_W(m) = -1 - h_{W_2}(j-1) - h_{W_1}(m-j+1)$. So we have $h_W(m) \geq -1 + 1 + 1 = 1$ contradicting Lemma 2.10. The contradiction proves Lemma 3.3. \square

Theorem 3. *For any fixed $k > 0$ and large enough n there exists a $(2k+1)$ -locally plane graphs on n vertices having at least $(\frac{\log^{(k)} n}{240^k} - k)n$ edges. Given two arbitrary disks in the plane, one can further assume that all edges of these graph connect a vertex inside the first disk to one inside the other disk.*

Proof: Simply combine the results of Corollary 2.1 and Lemma 3.3. If n is not the size of the vertex set of G_d for any d , add isolated vertices to the largest G_d with fewer than n vertices. Use the second realization of G_d as a geometric graph with a small enough $\epsilon > 0$ to obtain a geometric graph with all edges connecting two small disks and apply a homothety and a rotation to get to the desired disks. \square

4 Discussion on optimality of thinning

The maximum number of edges of a 3-locally plane graph on n vertices is $\Theta(n \log n)$ as proved in [7]. The lower bound is reproduced here by the $k = 1$ case of Theorem 3, which is therefore tight. The upper bound of [7] extends to *x-monotone topological graphs*, i.e., when the edges are represented by curves with the property that every line parallel to the y axis intersects an edge at most once. Without this artificial assumption on *x-monotonicity* only much weaker upper bounds are known. For higher values of k we do not have tight results even if the edges are straight line segments as considered in this paper. While the number of edges in a k -locally plane graph constructed here deteriorates very rapidly with the increase of k , the upper bound hardly changes. In fact the only general upper bound better than $O(n \log n)$ is for 5-locally plane graphs: they have $O(n \log n / \log \log n)$ edges as shown in [7]. Another result of that paper deals with geometric (or *x-monotone topological*) graphs with the additional condition that a vertical line intersects every edge. It claims that if such graphs are $(2k)$ -locally plane for $k \geq 2$, then they have $O(n \log^{1/k} n)$ edges. The first realization of G_d does not satisfy this condition, but the second one does. Still, the lower and upper bounds for this restricted problem are far apart: for 4-locally plane graphs with cutting line the upper bound on the edges is $O(n\sqrt{\log n})$ while the construction gives $\Omega(n \log \log n)$.

Although we cannot establish that the locally plane graphs constructed are optimal we can prove that the thinning procedure we use is optimal within a constant factor. It follows that any 4-locally plane subgraph of G_d has $O(n \log \log n)$ edges. This optimality result below refers to a single

step of the thinning procedure. It would be interesting to establish a strong upper bound on the number of edges of a k -flat graph for $k \geq 3$.

Let us mention here that the thinning procedures described in this paper found application in the extremal theory of 0-1 matrices, see [12], and there the result is shown to be optimal within a constant factor. Consider an n by n 0-1 matrix that has no 2 by 3 submatrix of either of the following two forms:

$$\begin{pmatrix} 1 & 1 & * \\ 1 & * & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 1 \\ * & 1 & 1 \end{pmatrix},$$

where the $*$ can represent any entry. The maximal number of 1 entries in such a matrix is $\Theta(n \log \log n)$ as proved in [12]. The construction proving the lower bound is based on lexicographic thinning.

The following lemma shows that the number of edges in the subgraphs claimed in Lemma 2.1 and Theorem 1 are optimal in a very strong sense: no properly edge colored graph with significantly more edges than the ones guaranteed by the above results can avoid heavy paths (slow or fast walks, respectively).

Lemma 4.1. *Let $G = (A, B, E)$ be a bipartite graph with proper edge coloring given by $\chi : E \rightarrow \{1, \dots, d\}$.*

- a) *If G does not have a heavy path then $|E| \leq 2\sqrt{d|A||B|} \leq (|A| + |B|)\sqrt{d}$.*
- b) *If G does not have a slow walk then $|E| \leq (|A| + |B|)(\log d + 2)$.*
- c) *If G does not have a fast walk then $|E| = 2(|A| + |B|)(\log d + 2)$.*

Proof: For any vertex $z \in A \cup B$ denote by $m(z) = \max(\chi(e))$, where the maximum is for edges e incident to z . For an edge $e = (x, y) \in E$ we define its *weight* to be $w(e) = m(x) - \chi(e)$, while its *B-weight* is $w_B(e) = m(y) - \chi(e)$. Clearly, both $w(e)$ and $w_B(e)$ are integers in $[0, d - 1]$.

To prove part a) of the lemma assume G does not contain a heavy path. We set a threshold parameter $t = \lfloor \sqrt{d|A||B|} \rfloor$ and call an edge e *B-light* if $w_B(e) < t$, otherwise e is *B-heavy*.

All the edges incident to a vertex $y \in B$ have different colors, thus they also have different B -weights, so at most t of them can be B -light. The total number of B -light edges is at most $|B|t \leq \sqrt{d|A||B|}$.

Now consider two edges $e_1 = (x, y_1)$ and $e_2 = (x, y_2)$ incident to vertex $x \in A$. Assume $\chi(e_2) \leq \chi(e_1)$. If $w_B(e_2) > 0$ we can extend the path formed by these two edges with the edge e_3 incident to y_2 having maximum color $\chi(e_3) = m(y_2)$. Clearly, $\chi(e_3) = \chi(e_2) + w_B(e_2)$ and the resulting path

is heavy unless $\chi(e_2) > \chi(e_1) + w(e_2)$. Therefore, the number of B -heavy edges incident to x is at most $d/(t+1)$. The total number of B -heavy edges is at most $|A|d/(t+1) \leq \sqrt{d|A||B|}$.

For the total number of edges we have $|E| \leq 2\sqrt{d|A||B|} \leq (|A|+|B|)\sqrt{d}$.

For parts b) and c) of the lemma consider an edge $e = (x, y) \in E$. If $w(e) = 0$ we call the edge e *maximal*. Clearly, there are at most $|A|$ maximal edges. If e is not maximal we define $n(e)$ to be the “next larger colored edge at x ”, i.e., $n(e)$ is the edge in E having minimal color $\chi(n(e))$ among edges incident to x and satisfying $\chi(n(e)) > \chi(e)$. We define the *gap* of e to be $g(e) = \chi(n(e)) - \chi(e)$. Clearly, $0 < g(e) \leq w(e)$. We call the edge e *heavy* if $w(e) > 2g(e)$, otherwise e is *light*. Recall, that for maximal edges $n(e)$ and $g(e)$ are not defined and maximal edges are neither light nor heavy.

Let e_1 and e_2 be distinct edges in E incident to a vertex $x \in A$. If $\chi(e_1) < \chi(e_2)$ then $w(e_1) \geq w(e_2) + g(e_1)$. If e_1 is light, $w(e_1) \geq 2w(e_2)$ follows, therefore at most $\lceil \log d \rceil$ light edges are incident to $x \in A$. Thus the total number of light edges in E is at most $\lceil \log d \rceil |A|$.

For part b) of the lemma assume G does not contain a slow walk.

Let $e_2 = (x_2, y)$ and $e_3 = (x_3, y)$ be distinct non-maximal edges in E and assume $\chi(e_2) < \chi(e_3)$. Let $e_1 = n(e_2)$ and let e_4 be the maximal edge incident to x_3 in G . We have $\chi(e_1) = \chi(e_2) + g(e_2)$ and $\chi(e_4) = \chi(e_3) + w(e_3)$. The edges e_1, e_2, e_3 , and e_4 cannot form a slow walk. As $\chi(e_1) > \chi(e_2) < \chi(e_3) < \chi(e_4)$ we must have $\chi(e_4) < \chi(e_1)$. This implies $g(e_2) > w(e_3)$, and if e_2 is heavy $w(e_2) > 2w(e_3)$. Therefore, at most $\lceil \log d \rceil$ heavy edges can be incident to $y \in B$. The total number of heavy edges in G is at most $(\lceil \log d \rceil |B|)$.

For the total number of edges we add the bound obtained for light, heavy, and maximal edges and get $|E| \leq (|A| + |B|)\lceil \log d \rceil + |A|$.

Finally for part c) of the lemma we assume G does not contain a fast walk. Let E_1 consist of the edges $(x, y) \in E$ for which $m(x) \geq m(y)$. Assume without loss of generality that $|E_1| \geq |E|/2$. If this is not the case consider the same graph with its sides switched. We use here that the definition of a fast walk and the claimed bound on the number of edges are both symmetric in the color classes.

As in the previous case consider two non-maximal edges $e_2 = (x_2, y)$ and $e_3 = (x_3, y)$ in E_1 with $\chi(e_2) < \chi(e_3)$. Let e_1 be the maximal edge incident to x_2 and let $e_4 = n(e_3)$. As $\chi(e_2) < \chi(e_3) < \chi(e_4)$ but G does not contain a fast walk we must have $\chi(e_1) < \chi(e_4)$. As $e_2 \in E_1$ we must also have

$\chi(e_1) = m(x_2) \geq m(y) \geq \chi(e_3)$. If e_3 is heavy we also have

$$\begin{aligned} \chi(e_3) + w(e_3) - m(y) &> \chi(e_3) + 2g(e_3) - m(y) \\ &\geq 2(\chi(e_3) + g(e_3) - m(y)) \\ &= 2(\chi(e_4) - m(y)) \\ &> 2(\chi(e_1) - m(y)) \\ &= 2(\chi(e_2) + w(e_2) - m(y)). \end{aligned}$$

For all the heavy edges $e \in E_1$ incident to $y \in B$ the values $\chi(e) + w(e) - m(y)$ increase strictly more than by a factor of 2. As these values are integers from $[0, d - 1]$, there are at most $\lceil \log d \rceil$ heavy edges in E_1 incident to y . The total number of heavy edges in E_1 is at most $\lceil \log d \rceil |B|$.

For the total number of edges in E_1 we sum our bound on heavy edges in E_1 and the bounds on the light and maximal edges in E . We obtain $|E_1| \leq (|A| + |B|)\lceil \log d \rceil + |A|$. Finally we get $|E| \leq 2(|A| + |B|)\lceil \log d \rceil + 2|A|$. \square

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