Crossing numbers of imbalanced graphs

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Abstract

The *crossing number*, $\mathrm{cr}(G)$, of a graph G is the least number of crossing points in any drawing of G in the plane. According to the Crossing Lemma of Ajtai, Chvátal, Newborn, Szemerédi [ACNS82] and Leighton [L83], the crossing number of any graph with n vertices and $e > 4n$ edges is at least constant times e^3/n^2. Apart from the value of the constant, this bound cannot be improved. We establish some stronger lower bounds under the assumption that the distribution of the degrees of the vertices is irregular. In particular, we show that if the degrees of the vertices are $d_1 \geq d_2 \geq \ldots \geq d_n$, then the crossing number satisfies $\mathrm{cr}(G) \geq \frac{c_1}{n} \sum_{i=1}^{n} i d_i^3 - c_2 n^2$, and that this bound is tight apart from the values of the constants $c_1, c_2 > 0$. Some applications are also presented.

1 Introduction

Let G be a simple undirected graph with $n = n(G)$ vertices and $e = e(G)$ edges. A *drawing* of G in the *plane* is a mapping f that assigns to each vertex of G a distinct point in the plane and to each edge uv a continuous arc connecting $f(u)$ and $f(v)$, not passing through the image of any other vertex. For simplicity, the arc assigned to uv is also called an *edge*, and if this leads to no confusion, it is also denoted by uv. Assume that no three edges share an interior point. A common interior point of two edges is called a *crossing point*. The *crossing number*, $\mathrm{cr}(G)$, of G is the minimum number of crossing points in any drawing of G.

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The determination of $\mathrm{cr}(G)$ is an NP-complete problem [GJ83]. It was discovered by Leighton [L83] that the crossing number can be used to estimate the chip area required for the VLSI circuit layout of a graph. He proved the general lower bound $\mathrm{cr}(G) > ce^3/n^2 - O(n)$, for some $c > 0$, which was discovered independently by Ajtai, Chvátal, Newborn, and Szemerédi [ACNS82]. The largest known value of the constant, $c = 1024/31827 > 0.032$, was found in [PRTT04].

Székely [Sz95] observed that this result can be elegantly used to deduce the Szemerédi-Trotter theorem on the maximum number of incidences between n points and m lines in the plane and the best known upper bound for the number times the same distance can occur among n points in the plane. Székely's method was further developed to improve the existing lower bounds on the number of distinct distances determined by n points in the plane [ST01], [KT04] and upper bound for the number of different ways how a line can split a set of $2n$ points into two equal parts [D98]. For some other interesting corollaries, consult [PS98], [PT02], [STT02], [MSSW06], [BCSV07].

It is easy to see that the above bound is tight, apart from the value of the constant. However, as was shown in [PST00], it can be strengthened for some special classes of graphs, e.g., for graphs not containing some fixed, so-called forbidden subgraph. In particular, if G contains no cycle of length four, its crossing number is at least $c'e^4/n^3 - O(n)$, for a suitable constant $c' > 0$.

The aim of this note is to establish that the order of magnitude of the bounds of Leighton and Ajtai et al. can be tight only for "nearly regular" graphs and find improved bounds for graphs with irregular degree distributions. Our techniques could also be used to achieve similar improvement on the estimates in [PST00].

Theorem 1. *For any simple graph G on n vertices with vertex degrees $d_1 \geq d_2 \geq \ldots \geq d_n$ we have*

$$\mathrm{cr}(G) \geq \frac{1}{36000n} \sum_{i=1}^{n} i d_i^3 - 4.01n^2.$$

Another result of this kind was found independently by Fox and Cs. Tóth [FT06]. They proved that if $\sum_{i=1}^{n} d_i \geq 16n^{7/5} \log^{2/5} n$, then $\mathrm{cr}(G) \geq \frac{1}{24} \sum_{i=1}^{n} d_i^2$. In this range, our bound in Theorem 1 is stronger. Fox and Tóth also established the following "truncated" inequality for sparse graphs: $\mathrm{cr}(G) \geq \frac{1}{64} \sum_{i=k}^{n} d_i^2$, where k is the smallest index that $\sum_{i=k}^{n} d_i \leq \frac{2}{3} \sum_{i=1}^{n} d_i$. (The degree sequence is decreasing.) For the precise details and for an interesting application of the last inequality, consult [FT06].

As we will see, the main term in the estimate of Theorem 1 cannot be substantially improved if we restrict our attention to bounds that depend monotonically on the degrees of the vertices. In this form of the estimate, the quadratic error term is also unavoidable as shown for example by n vertex star that is planar, but $\frac{1}{n} \sum_{i=1}^{n} d_i^3$ evaluates to $(n-1)^3/n$. Note however that the original version of the crossing number lemma (see Lemma 2.2 (ii)) has a linear error term. It would be desirable to change main term of our estimate in such a way that allows for a linear error term and therefore would allow meaningful estimates for the crossing number, even if it smaller than quadratic.

First, we prove the tightness of Theorem 1 in a bipartite setting.

Theorem 2. *For any sequence of integers $n \geq d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$, there exists a bipartite graph G with n vertices in either vertex class such that the degree sequence in one class is exactly $d_1, \ldots, d_n$ and*

$$\mathrm{cr}(G) \leq \frac{8}{n}\sum_{i=1}^{n} i d_i^3.$$

Theorems 1 and 2 are almost complementary. Building on this we show in Section 4 that the minimum crossing number of a graph with degrees *at least* $d_1 \geq \ldots \geq d_n$ is estimated up to a constant factor by the expression $\frac{1}{n}\sum_{i=1}^{n} i d_i^3$, provided that this value exceeds a constant multiple of n^2. This is not true if we consider graphs with degree sequence *exactly* $d_1 \geq \ldots \geq d_n$ as shown by the observation that while the addition of isolated vertices does not change the crossing number of a graph, adding extra zeroes to the degree sequence decreases $\frac{1}{n}\sum_{i=1}^{n} i d_i^3$ as n increases.

We mention some alternative forms of this estimate. As we will see in Section 3, $\sum_{i=1}^{n} i d_i^3$ is within a constant factor of $\sum_{i=1}^{n} s_i d_i^2$, where $s_i = \sum_{j=1}^{i} d_i$ and $d_1 \geq \ldots \geq d_n \geq 0$. In some situations, it is more convenient to use the latter variant.

The expression $(\sum_{i=1}^{n} d_i^{3/2})^2$ is also closely related to the sum $\sum_{i=1}^{n} i d_i^3$. It is more attractive, in the sense that it does not depend on the order of the elements $d_1, \ldots, d_n$. It is easy to prove that for any sequence of nonnegative reals $d_1 \geq \ldots \geq d_n$, we have

$$\sum_{i=1}^{n} i d_i^3 \leq \left(\sum_{i=1}^{n} d_i^{3/2}\right)^2 \leq (\ln n + 1)\sum_{i=1}^{n} i d_i^3.$$

For the first inequality use that $d_i^{3/2} d_j^{3/2} \geq d_j^3$ for $i \leq j$, for the second inequality use Cauchy-Schwarz with $i^{1/2} d_i^{3/2}$ and $i^{-1/2}$. The logarithmic factor on the right-hand side cannot be eliminated, as is shown, for example, by the sequence $d_i = i^{-2/3}$. In order to obtain an integer sequence (suitable for degree sequence), one can scale this up to $d_i = \lfloor n/i^{2/3} \rfloor$.

Theorems 1 and 2 are proved in Sections 2 and 3. Section 4 contains some direct applications of our results and concluding remarks.

2 Imbalanced bipartite Crossing Lemma

Our computations will be based on the simple observation that in an *imbalanced* bipartite graph the number of crossings is always larger than the bound guaranteed by the Crossing Lemma.

Lemma 2.1. *Let $G(A,B)$ be a bipartite graph with vertex classes A and B, and suppose that its number of edges satisfies $e \geq 6\max(|A|,|B|)$. Then we have*

$$\mathrm{cr}(G(A,B)) \geq \frac{1}{108}\frac{e^3}{|A||B|}.$$

Proof. If $G(A,B)$ is planar, then it follows from Euler's Polyhedral Formula that $e \leq 2(|A|+|B|)-4$, provided that $|A|+|B| \geq 3$. This yields, by induction on e, that for not necessarily planar bipartite graphs on at least 3 vertices

$$\mathrm{cr}(G(A,B)) \geq e - 2(|A|+|B|) + 4 \tag{1}$$

holds.

Select each vertex of A independently with probability p_1, and let A' denote the set of selected vertices. Analogously, let B' be a randomly chosen subset of B, whose elements are selected from B independently with probability p_2. Letting $G(A',B')$ denote the subgraph of $G(A,B)$ induced by $A' \cup B'$, (1) implies that

$$\mathrm{cr}(G(A',B')) > e' - 2(|A'|+|B'|),$$

where e' stands for the number of edges of $G(A',B')$. Taking expectations of both sides, we obtain

$$\begin{aligned} p_1^2 p_2^2 \mathrm{cr}(G(A,B)) &\geq E[\mathrm{cr}(G(A',B'))] \\ &> E[e'] - 2E[|A'|+|B'|] \\ &= p_1 p_2 e - 2(p_1|A| + p_2|B|) \\ \mathrm{cr}(G(A,B)) &> \frac{1}{p_1 p_2}\left(e - 2\left(\frac{|A|}{p_2} + \frac{|B|}{p_1}\right)\right). \end{aligned}$$

Setting $p_1 := \frac{6|B|}{e}$ and $p_2 := \frac{6|A|}{e}$, the result follows. □

We can get rid of the assumption $e \geq 6\max(|A|,|B|)$ in Lemma 2.1 by introducing an error term. For comparison and later reference, we also state the original version of the Crossing Lemma (with the better constant obtained in [PRTT04]).

Lemma 2.2. (i) *Let G be a bipartite graph with vertex classes of size k and ℓ with $k \leq \ell$ and e edges. We have*

$$\mathrm{cr}(G) \geq \frac{1}{108}\frac{e^3}{k\ell} - 2\frac{\ell^2}{k}.$$

(ii) *For an arbitrary simple graph G with n vertices and e edges, we have*

$$\mathrm{cr}(G) \geq \frac{1}{32}\frac{e^3}{n^2} - 2n.$$

Proof of (i). Adjusting the constant in the error term, we can achieve that the bound becomes negative and therefore trivially holds when the assumption of Lemma 2.1 is not satisfied. □

Proof of Theorem 1. Fix a graph G with vertex set $V(G) = \{v_1, \dots, v_n\}$ with $d(v_i) = d_i$ for $i = 1, \dots, n$.

Let $l_0 = \lfloor \log_2 n \rfloor$. For $1 \leq l \leq l_0$, consider the pairwise disjoint sets $V_l = \{v_i | 2^{l-1} \leq i < 2^l\} \subseteq V(G)$, and let H_l denote the subgraph of G induced by V_l. Let $H'_l \subseteq G$ be the bipartite subgraph, consisting of all edges of G running between V_l and its complement $V(G) \setminus V_l$. Finally, let f_l and f'_l denote the number of edges in H_l and H'_l.

Set $t_l = \sum_{v_i \in V_l} d_i$. Clearly, we have $t_l = 2f_l + f'_l$ for every l, so that $\max(f_l, f'_l) \geq t_l/3$. Applying parts (ii) and (i) of Lemma 2.2 to H_l and H'_l, respectively, we obtain that

$$\mathrm{cr}(H_l) \geq \frac{f_l^3}{2^{2l+3}} - 2^l \text{ and } \mathrm{cr}(H'_l) \geq \frac{f_l'^3}{54 \cdot 2^l n} - \frac{4n^2}{2^l}.$$

This yields

$$\max(\mathrm{cr}(H_l), \mathrm{cr}(H'_l)) \geq \frac{t_l^3}{1500 \cdot 2^l n} - \frac{4n^2}{2^l}.$$

The graphs H_l and H'_l $(1 \leq l \leq l_0)$ have the property that every edge belongs to at most two of them. Thus, we have

$$\mathrm{cr}(G) \geq \sum_{l=1}^{l_0} \frac{\mathrm{cr}(H_l) + \mathrm{cr}(H'_l)}{2} \geq \frac{1}{3000n} \sum_{l=1}^{l_0} \frac{t_l^3}{2^l} - 4n^2.$$

In order to prove Theorem 1, it is enough to show that the above bound exceeds the one stated in the theorem. It follows from the fact that the sequence $d_1, d_2, \dots$ is monotone decreasing that

$$t_l = \sum_{i=2^{l-1}}^{2^l-1} d_i \geq 2^{l-1} d_{2^l},$$

for $1 \leq l \leq l_0$.

Consider the partial sum

$$D_l = \sum_{i=2^l}^{\max(2^{l+1}-1, n)} i d_i^3.$$

Obviously, we have

$$\sum_{l=1}^{l_0} D_l = \sum_{i=2}^{n} i d_i^3 = \sum_{i=1}^{n} i d_i^3 - d_1^3.$$

Using again the monotonicity of the degree sequence, we conclude that

$$D_l \leq 3 \cdot 2^{2l-1} d_{2^l}^3 \leq 12 \frac{t_l^3}{2^l},$$

so that

$$\begin{aligned}
\mathrm{cr}(G) &\geq \frac{1}{3000n}\sum_{l=1}^{l_0}\frac{t_l^3}{2^l} - 4n^2 \\
&\geq \frac{1}{36000n}\sum_{l=1}^{l_0} D_l - 4n^2 \\
&\geq \frac{1}{36000n}\sum_{i=1}^{n} i d_i^3 - 4.01n^2. \quad \Box
\end{aligned}$$

3 A bipartite construction

For the proof of Theorem 2, we need the following technical lemma.

Lemma 3.1 *Let $d_1, \ldots, d_n$ be a sequence of non-negative reals. For $1 \leq i \leq n$, let $s_i = \sum_{j=1}^{i} d_j$. We have*

$$\sum_{i=1}^{n} s_i d_i^2 \leq 4 \sum_{i=1}^{n} i d_i^3.$$

Proof. First, notice that

$$2\sum_{j=1}^{i} s_j d_j = 2\sum_{j=1}^{i}\sum_{k=1}^{j} d_j d_k = s_i^2 + \sum_{j=1}^{i} d_j^2.$$

Therefore, we have

$$s_i \leq \frac{2}{s_i}\sum_{j=1}^{i} s_j d_j,$$

for all i.

Introducing the notation $A = \sum_{i=1}^{n} s_i d_i^2$ and $B = \sum_{i=1}^{n} i d_i^3$, in view of the last inequality, we have

$$A \leq 2\sum_{i=1}^{n}\frac{d_i^2}{s_i}\sum_{j=1}^{i} s_j d_j,$$

$$\frac{1}{2}A - B \leq \sum_{i=1}^{n} d_i^2 \sum_{j=1}^{i}\left(\frac{s_j}{s_i}d_j - d_i\right).$$

Using the estimate $d_i(x - d_i) \leq x^2/4$ for $x = (s_j/s_i)d_j$, and switching the order of the summations, we obtain

$$\frac{1}{2}A - B \leq \sum_{i=1}^{n}\frac{d_i}{4}\sum_{j=1}^{i}\left(\frac{s_j}{s_i}d_j\right)^2 = \frac{1}{4}\sum_{j=1}^{n} s_j^2 d_j^2 \sum_{i=j}^{n}\frac{d_i}{s_i^2}$$

Notice that

$$\frac{d_i}{s_i^2} \le \frac{1}{s_{i-1}} - \frac{1}{s_i},$$

so we have

$$\sum_{i=j}^{n} \frac{d_i}{s_i^2} \le \frac{d_j}{s_j^2} + \frac{1}{s_j}.$$

This yields

$$\frac{1}{2}A - B \le \frac{1}{4}\sum_{j=1}^{n}(s_j d_j^2 + d_j^3) = \frac{1}{4}(A + \sum_{j=1}^{n} d_j^3),$$

and, by rearranging the terms, $A \le 4B + \sum_{i=1}^{n} d_i^3$.

To get rid of the error term of $C = \sum_{i=1}^{n} d_i^3$, we simply apply the last inequality to the sequence $(d_i')_{i=1}^{2n}$ obtained from $(d_i)_{i=1}^{n}$ by repeating each term twice. The corresponding sums for this sequence are $A' = 4A - C$, $B' = 4B - C$, and $C' = 2C$. We obtain $A' \le 4B' + C'$, which implies $A \le 4B$, as claimed. □

We suspect that Lemma 3.1 remains true with the constant 4 replaced by 3. However, as is shown by the sequence $d_i = i^{-2/3}$, the claim is certainly false with any constant smaller than 3. Note that the reverse inequality $\sum_{i=1}^{n} i d_i^3 \le \sum_{i=1}^{n} s_i d_i^2$ trivially holds if d_i is non-increasing sequence of non-negative reals, since in this case $s_i \ge i d_i$.

Proof of Theorem 2. We construct G together with a straight-line drawing that will demonstrate the upper bound on the crossing number of G. Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ be the two vertex classes of G. Pick a line ℓ, and place the points $w_1, \ldots, w_n$ on ℓ in this order, from left to right. The positions of the vertices $v_1, \ldots, v_n$ will be determined one by one, so that no v_i lies on ℓ or on any previously drawn edge, and each v_i is connected by a segment to the prescribed number d_i of points w_j.

Place v_1 at any point that does not belong to ℓ, and connect it to the vertices $w_1, \ldots, w_{d_1}$. Now let $i > 1$, and assume that the position of all vertices $v_1, \ldots, v_{i-1}$ has already been fixed. If $d_i = 0$, then v_i is an isolated vertex and it can be placed anywhere outside of ℓ and the previously drawn edges. If $d_i > 0$, then let w_j denote the last vertex (that is, the one with the highest index) incident to v_{i-1}.

If $n - j \ge d_i$, connect v_i to $w_{j+1}, w_{j+2}, \ldots, w_{j+d_i}$. If $n - j < d_i$, connect v_i to $w_1, \ldots, w_{d_i}$. Place v_i at a point (not belonging to ℓ or to any previously drawn edge) which lies so close to one of its neighbors that any edge incident to v_i can cross only those edges that are incident to a neighbor of v_i. As the neighbors of v_i are consecutive points on ℓ, this can indeed be achieved.

Obviously, the resulting geometric graph is bipartite and the degrees of its vertices satisfy $d(v_i) = d_i$. It remains to estimate the number of crossings. Fix a vertex v_i and consider its neighbors w_j. Let s_i denote the maximum degree of a neighbor of v_i in the graph induced by the vertices $w_1, \ldots, w_n$ and $v_1, \ldots, v_{i-1}$. In other words, s_i is the number of times our drawing algorithms had to "start over" at w_1 before processing v_i. The numbers d_i were listed in decreasing order, therefore in each round at least $n/2$ edges were drawn. Thus, the total number of edges drawn before processing v_i is $\sum_{j=1}^{i-1} d_j \geq s_i n/2$, so that

$$s_i \leq \frac{2}{n} \sum_{j=1}^{i-1} d_j.$$

According to the drawing rules, the introduction of each v_i creates at most $s_i d_i^2$ new crossings. Thus, we have $\mathrm{cr}(G) \leq \sum_{i=1}^{n} s_i d_i^2$. Applying Lemma 3.1, we obtain

$$\mathrm{cr}(G) \leq \frac{8}{n} \sum_{i=1}^{n} i d_i^3,$$

as required. □

4 Applications and concluding remarks

Given a set P of $2n$ points in general position in the plane, two elements of P form a *halving pair* if the line connecting them divides P into two parts of equal cardinality [L71]. The best known upper bound for the number of halving pairs, $O(n^{4/3})$, was established by Dey [D98]. His result was strengthened by Andrzejak, Aronov, Har-Peled, Seidel, and Welzl [AAHSW98, AW03], as follows. Define the *halving-edge graph* $H(P)$ of P, as a geometric graph on the vertex set P, where two vertices are connected by a straight-line segment (edge) if and only if they form a halving pair. For the degree sequence $d_1, d_2, \ldots, d_{2n}$ of the vertices of $H(P)$, Andrzejak et al. found the beautiful formula

$$\kappa(H(P)) + \sum_{i=1}^{2n} \binom{(d_i+1)/2}{2} = \binom{n}{2},$$

where $\kappa(H(P))$ denotes the number of crossing pairs of edges of $H(P)$. It follows that $\kappa(H(P)) < n^2/2$, and combining this bound with the Crossing Lemma, we immediately obtain Dey's result. Note that here the contribution of the sum of the squares of the degrees is negligible, therefore we gain no information on the degree distribution. By Theorem 1, we have

Corollary 3. *For the degree sequence $d_1 \geq d_2 \geq \ldots \geq d_{2n}$ of the halving-edge graph $H(P)$ of a $2n$-element point set P in general position in the plane, we have*

$$\sum_{i=1}^{2n} i d_i^3 \leq C n^3,$$

where C is a positive constant.

This inequality is a strengthening of Dey's bound. It implies that if there exists a point set with $\Omega(n^{4/3})$ halving pairs, then its halving-edge graph must have a fairly even degree distribution. Analogously, we can generalize other applications of the Crossing Lemma, and conclude that the resulting estimates cannot be asymptotically tight, unless the degrees of the vertices in the corresponding graph are roughly the same.

Finally, we return to the claim we made in the Introduction that the minimum crossing number of a simple graph on n vertices having degrees at least $n-1 \geq d_1 \geq \ldots \geq d_n$ is approximated within a constant factor by $\frac{1}{n}\sum_{i=1}^{n} i d_i^3$, provided this value exceeds a certain constant multiple of n^2. The lower bound directly follows from Theorem 1, as increasing the degrees can only increase the bound claimed there. For the upper bound, we slightly modify the construction in the proof of Theorem 2 to obtain a graph G with vertices $v_1, \ldots, v_n$ satisfying $d(v_i) \geq d_i$ and a drawing of G in the plane with straight-line edges and with $O(\sum_{i=1}^{n} i d_i^3)$ edge crossings. For simplicity, we assume n is even and set $m = n/2$. If $d_m > m$, we have $\frac{1}{n}\sum_{i=1}^{n} i d_i^3 = \Omega(n^4)$, so any straight-line drawing of $G = K_n$ will do. Otherwise, we apply the construction of Theorem 2 to the degree sequence $d'_1, \ldots, d'_m$, where $d'_i = \max(d_i, m)$. We obtain a bipartite graph G_0 between the vertices $v_1, \ldots, v_m$ and $w_1 = v_{m+1}, \ldots, w_m = v_{2m}$, and a straight-line drawing of G_0 such that $d(v_i) = d'_i$ for $i \leq m$ and the number of edge crossings is at most $\frac{8}{m}\sum_{i=1}^{m} i {d'_i}^3$. If v_{m+1} is connected to v_{i+1} and v_i is connected to v_j, we connect v_i to every vertex v_k with $k > j$. In other words, we finish each "pass" of the construction of G_0 by connecting the last vertex in the pass with the remaining vertices v_j, $j > m$ "not used in that pass". In the resulting graph G_1, we still have $d(v_i) \geq d'_i$ for $i \leq m$, and now we have $d(v_i) \geq d'_m = d_m \geq d_i$ for $i > m$. The number of edge crossings in G_1 is still at most $\frac{16}{m}\sum_{i=1}^{m} i {d'_i}^3$. Finally, we complete the construction of G by connecting every vertex v_i of G_1 with $d_i > m$ to all the other vertices of G_1. The resulting graph satisfies $d(v_i) \geq d_i$ for all i and the number of edge crossings is still $O(\frac{1}{m}\sum_{i=1}^{m} i {d'_i}^3) = O(\frac{1}{n}\sum_{i=1}^{n} i d_i^3)$.

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