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An algorithm for finding many disjoint monochromatic edges in a complete 2-colored geometric graph

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We present an $O(n^{\log\log n+2})$-time algorithm for finding n disjoint monochromatic edges in a complete geometric graph of $3n-1$ vertices, where the edges are colored by two colors.

A *geometric graph* is a graph drawn in the plane so that every vertex corresponds to a point, and every edge is a closed straight-line segment connecting two vertices but not passing through a third. The $\binom{n}{2}$ segments determined by n points in the plane, no three of which are collinear, form a *complete* geometric graph with n vertices (see [PA95]).

Does a given geometric graph contain a non-crossing subgraph of a certain type? It is conjectured that essentially all such problems are NP–hard, regardless of the complexity of their non-geometric analogues. This conjecture is supported by several striking examples. For instance, finding a largest tree or a maximum matching in a graph are easily solvable problems in polynomial time. Yet it is known to be NP–hard to decide whether a geometric graph has a non-crossing spanning tree, or to find the maximum number of disjoint edges [JW93],[KLN91],[KN90].

In this note we would like to show that the complexity of these problems may radically decrease if we are allowed to choose where to find a non-

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crossing spanning tree (or k disjoint edges): in the graph G or in its complement $\overline{G}$. In fact, in this case classical Ramsey–theory works to our advantage by ensuring that either G or $\overline{G}$ contains a large clique and many other large subgraphs [GRS90]. Some of these results can be extended to geometric graphs to yield the existence of certain non-crossing subgraphs and subgraphs satisfying some other geometric conditions.

In [KPT96], we settled a problem of Bialostocki and Dierker [BD94] by proving the following theorem. For any 2-coloring of the edges of a complete geometric graph with $3n-1$ vertices, there exist n pairwise disjoint edges of the same color. Here we present an $O(n^{\log\log n+2})$-time algorithm for finding these edges, based on a different proof. The analogous theorem for abstract graphs (that is, when the geometric constraints are ignored) was found by Gerencsér and Gyárfás [GG67]. It is easy to see that the above results are false for graphs with fewer than $3n-1$ vertices.

We can also design an $O(n^{\log\log n+2})$-time algorithm for finding a non-crossing monochromatic spanning tree in a complete geometric graph of n vertices, whose edges are colored by two colors ([KPT96]). Since the second algorithm is based on the same idea, but is somewhat simpler, we present only the first one, called Disjoint-Edges(n).

Before formally describing the algorithm, we briefly sketch the ideas it is based on.

The algorithm first checks the edges of the convex hull and if they are not monochromatic, then one simple recursive call to Disjoint-Edges$(n-1)$ is sufficient (Steps 1, 2).

If the convex hull is monochromatic (let us call this color red and the other color blue), then the algorithm picks a vertex p of the convex hull and sorts the other points in order of visibility from p (Step 3).

For each $0<j<n$, both the first and the last $3j-1$ vertices span j monochromatic disjoint edges. Let c_j resp. c'_j denote their colors. Let c_0 and c'_0 be red.

If both $c_{\lceil n/2\rceil}$ and $c'_{\lfloor n/2\rfloor}$ are red, the union of the corresponding two sets of disjoint red edges provides a solution to our problem (see Step 5).

Otherwise, one of them, say $c_{\lceil n/2\rceil}$, is blue, so there exists a q ($0\leq q\leq\lfloor n/2\rfloor$) such that c_q is red and c_{q+1} is blue. Given such a q (and the corresponding two sets of monochromatic edges), two more recursive calls to Disjoint-Edges suffice (see Steps 7, 8, 9).

Note that we make only one recursive call to Disjoint-Edges(m) with $m>\lceil n/2\rceil$ (namely the last one). As we cannot limit the parameter m of this last recursive call (it may be as big as $n-1$), we make sure instead that in case m is close to n we do little additional work before the call to DISJOINT-EDGES(m). Thus, instead of a straightforward binary search, first we find the smallest power of 2 above q (Step 4) and use binary search (Step 6) only afterward.

For the correctness of the algorithm one only has to observe that the output given in steps 0, 2, 5, 8, and 9 is always a monochromatic set of n disjoint edges.

Next, we give a more precise description of the algorithm.

The input of DISJOINT-EDGES(n) is a complete geometric graph K_{3n-1} all of whose edges are colored by red or blue. The output of DISJOINT-EDGES(n) is a set of n pairwise disjoint monochromatic edges of K_{3n-1}.
DISJOINT-EDGES(n)

STEP 0. If $n = 1$ return the only edge of G. Stop.

STEP 1. Construct the convex hull of the points in $O(n \log n)$ steps (see [E87]). Check the colors of all the edges along the convex hull. If all of them are of the same color, say red, then go to Step 3.

STEP 2. Choose three consecutive vertices of the convex hull, p_1, p_2, and p_3, so that p_1p_2 and p_2p_3 are of different color. Remove them from K_{3n-1} and call DISJOINT-EDGES($n-1$) for the geometric graph spanned by the remaining $3n-4$ points and complete the resulting set of $n-1$ pairwise disjoint monochromatic edges by p_1p_2 or p_2p_3. Stop.

STEP 3. Take a point p on the convex hull. Sort the other $3n-2$ points in the clockwise order of visibility from p and denote them by $p_1, p_2, \ldots p_{3n-2}$.
For every j, $1 \le j \le n-1$, let K^l_j and K^r_j denote the subgraphs of K_n induced by the points $\{p_1, \ldots, p_{3j-1}\}$ and $\{p_{3j}, \ldots, p_{3n-2}\}$, respectively. Set $i = 0$.

STEP 4. If $2^i \ge n/2$, then go to Step 5. Otherwise call DISJOINT-EDGES(2^i) for $K^l_{2^i}$ and for $K^r_{n-2^i}$. If any of the two resulting set of 2^i pairwise disjoint monochromatic edges is of color blue, say, in $K^l_{2^i}$ DISJOINT-EDGES(2^i) has found 2^i blue edges, then go to Step 6. If both sets of 2^i edges are red, then let $i = i+1$ and repeat Step 4.

STEP 5. Call DISJOINT-EDGES($\lceil n/2 \rceil$) for $K^l_{\lceil n/2 \rceil}$ and DISJOINT-EDGES($\lfloor n/2 \rfloor$) for $K^r_{\lceil n/2 \rceil}$. If the two set of edges are of the same color, take their union and stop. If they are of different colors, then one of them is blue (say, in $K^l_{\lceil n/2\rceil}$ DISJOINT-EDGES($\lceil n/2\rceil$) has found $\lceil n/2\rceil$ blue edges).

STEP 6. In case $i=0$ set $q=0$ and go to Step 8. Otherwise call DISJOINT-EDGES recursively (at most $i-1$ times) to find a q by binary search on the interval $2^{i-1}\le q<\min\{2^i,\lceil n/2\rceil\}$ so that we have q pairwise disjoint red edges of K^l_q and $q+1$ pairwise disjoint blue edges of K^l_{q+1}.

STEP 7. Let $\bar{K}^l$ be the subgraph of K_n induced by the points $\{p_2,\dots,p_{3q}\}$. Call DISJOINT-EDGES(q) to get q pairwise disjoint monochromatic edges of $\bar{K}^l$. If their color is blue then go to Step 9.

STEP 8. Call DISJOINT-EDGES($n-q-1$) to get $n-q-1$ pairwise disjoint monochromatic edges of K^r_{q+1}.

If they are blue, take their union with the $q+1$ pairwise disjoint blue edges of K^l_{q+1}. Stop.

If they are red, take their union with the red edge p_1p and in case $q>0$ then also with the q pairwise disjoint red edges of $\bar{K}^l$. Stop.

STEP 9. Let $\bar{K}^r$ be the subgraph of K_n induced by the points $\{p,p_{3q+1},\dots,p_{3n-2}\}$. Call DISJOINT-EDGES($n-q$) to get $n-q$ pairwise disjoint monochromatic edges of $\bar{K}^r$.

If they are blue, take their union with the q pairwise disjoint blue edges of $\bar{K}^l$. Stop.

If they are red, take their union with the q pairwise disjoint red edges of K^l_q. Stop.

Let $g(n)$ denote the running time of DISJOINT-EDGES(n). Using the monotonicity of g, we obtain the following recurrence relation:

$$g(n)\le \max_{1\le i<\log n}\Big\{\underbrace{2(g(1)+\dots+g(2^{i-1})+g(\min\{2^i,\lceil n/2\rceil\}))}_{\textsc{Steps } 4,\ 5}+$$
$$+\underbrace{ig(\min\{2^i,\lceil n/2\rceil\})}_{\textsc{Steps } 6,\ 7}+\underbrace{g(n-2^{i-1})}_{\textbf{Steps 2, 8, 9}}\Big\}+\underbrace{cn\log n}_{\textsc{Steps } 1,\ 3},$$

where c is a suitable constant.

Claim

If $g(n)$ satisfies the above recurrence relation then $g(n)=O(n^{\log\log n+2})$, where the logarithm is of base 2.

Proof. Choose $C \geq \max\{c, g(1)\}$ such that $Cn^{\log\log n+2} > g(n)$ for $1 < n < 256$. Let $f(1) = C$, $f(n) = Cn^{\log\log n+2}$ for $n > 1$ and let $n \geq 256$ be fixed.

We prove that

$$\max_{1\leq i<\log n}\{2(f(1)+\cdots+f(2^{i-1}))+(i+2)f(\min\{2^i,\lceil n/2\rceil\}+f(n-2^{i-1})\} + cn\log n < f(n)$$

holds.

$$\max_{1\leq i<\log n}\{2(f(1)+\cdots+f(2^{i-1}))+(i+2)f(\min\{2^i,\lceil n/2\rceil\}+f(n-2^{i-1})\} + cn\log n$$

$$< \max_{2\leq x\leq\lceil n/2\rceil}\{f(n-x/2)+(\log x+7)f(x)\}+Cn\log n$$

$$= \max_{2\leq x\leq\lceil n/2\rceil}\{C(n-x/2)^{\log\log(n-x/2)+2}+C(\log x+7)x^{\log\log x+2}\}+Cn\log n$$

$$< \max_{2\leq x\leq\lceil n/2\rceil}\{C(n-x/2)^{\log\log n+2}+C(\log x+7)x^{\log\log x+2}\}+Cn\log n.$$

We used that $\sum_{j=0}^{i-2} f(2^j) < f(2^{i-1})$. So it is enough to check that for all $2 \leq x \leq \lceil n/2\rceil$,

$$F(x) = n^{\log\log n+2} - (n-x/2)^{\log\log n+2} - (\log x+7)x^{\log\log x+2} - n\log n > 0.$$

Routine calculation shows that $F(2) > 0$, $F(\lceil n/2\rceil) > 0$, and $F(x)$ is concave on $[2, \lceil n/2\rceil]$. So $F(x) > 0$ in this interval. ∎

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