Where have all the grasshoppers gone?

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Abstract

Let P be an N-element point set in the plane. Consider N (pointlike) grasshoppers sitting at different points of P. In a "legal" move, any one of them can jump over another, and land on its other side at exactly the same distance. After a finite number of legal moves, can the grasshoppers end up at a point set, similar to, but larger than P? We present a linear algebraic approach to answer this question. In particular, we solve a problem of Brunck by showing that the answer is yes if P is the vertex set of a regular N-gon and $N \neq 3, 4, 6$. Some generalizations are also considered.

1 Introduction

In 2001, at the Moscow Oral Team Mathematics Olympiad [3], the following question was asked: *Four grasshoppers are sitting at the vertices of a square. Every minute, one of them jumps over another and lands at the point symmetric to it. Prove that the grasshoppers can never end up sitting at the vertices of a larger square.*

The solution is simple. Suppose, for contradiction, that the grasshoppers can end up at the vertices of a *larger* square. Since the moves are reversible, starting with the final position and reversing the sequence of moves, we can get from a square to a *smaller* square. However, this is impossible. Indeed, we can assume by scaling that the initial positions of the grasshoppers are $(0, 0), (0, 1), (1, 0)$, and $(1, 1)$. Then they will keep jumping around on the points of the integer grid $\mathbb{Z}^2$, which has no four points that form a square of sidelength smaller than 1.

The same argument works when we have *six* grasshoppers sitting at the vertices of a regular hexagon T. The only difference is that now the possible positions of the grasshoppers belong to a triangular lattice induced by two adjacent side vectors of T. As T is the smallest regular hexagon in this lattice, the grasshoppers can never reach a regular hexagonal position of smaller—and,

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therefore, by reversibility—of larger side length than T. Exactly the same argument works if we start with a regular triangle (or with an arbitrary triangle). However, in this case, we have an even simpler argument: a legal move never changes the *area* of the triangle determined by the three pieces. Therefore, after any number of steps, the positions of the pieces must form a triangle whose area is the same as that of the initial triangle.

As far as we know, it was Florestan Brunck [1] who first asked: what happens if there are 5 grasshoppers, and their starting position is the vertex set of a regular pentagon? Can they be taken into the vertices of a larger regular pentagon?

Somewhat surprisingly, the answer to this question is affirmative.

Theorem 1.1. *Let $N \geq 3$, and let C_N denote a configuration consisting of N pieces sitting at the vertices of a regular N-gon. In a legal move, any piece can jump over any other piece and land at the point centrally symmetric about it.*

Unless $N = 3, 4,$ or 6, there is a finite number of moves that take the pieces of C_N into a strictly larger similar configuration.

One can raise the following more general question. Given any two N-element point sets in the plane or in higher dimensions, is it possible to transform the first one to the second by a sequence of legal moves? In Section 3, we develop a linear algebraic approach to answer this question.

Throughout this note, we fix an orthogonal system of coordinates in $\mathbf{R}^d$. Let $P \subset \mathbf{R}^d$ be a configuration of $N = n+1$ points. The position of each piece can be described by a *column vector* of length d. Assume without loss of generality that initially one of the pieces is at the origin 0. We distinguish this piece and call it *stationary* or *special*. All other pieces are said to be *ordinary*. The movement of the stationary piece is artificially restricted: any ordinary piece is allowed to jump over it, but this piece is not allowed to jump. Note that it is not forbidden for two or more pieces to sit at the same point at the same time.

Enumerate the elements of P by the integers $0, 1, 2, \ldots, n$, where 0 denotes the stationary piece. We identify P with the $d \times n$ real matrix, whose ith column gives the position of the ordinary piece i for $1 \leq i \leq n$. Without any danger of confusion, we denote both the configuration and the corresponding matrix by P. Part 1 of Theorem 3.1, stated in the next section, characterizes all configurations P that can be reached from the initial position under the restriction that piece 0 is stationary. In part 2, we get rid of this unnatural condition.

In the special case where we want to transform P into a larger configuration similar to it, we obtain the following result.

Theorem 1.2. *Let P be a configuration of $N = n+1$ points in $\mathbf{R}^d$, one of which initially sits at the origin but it is still allowed to jump. Let P also denote the $d \times n$ matrix associated with it.*

Then P can be transformed into a similar but larger configuration, using legal moves, if and only if there is an $n \times n$ integer matrix A with $|\det A| = 1$ such the configuration associated with PA is similar to and larger than the initial configuration P.

In Sections 4 and 5, we apply this result to prove Theorem 1.1 and the following very similar statement, respectively.

Theorem 1.3. *Let $n \geq 5$, and let C_n^* be a configuration consisting of $n+1$ pieces sitting at the vertices and the center of a regular n-gon.*

Unless $n = 6$, there is a finite number of legal moves that take the pieces of C_n^* into a similar but larger configuration.

Our paper is organized as follows. In Section 2, we prove a simple lemma for matrices. It is instrumental for our general linear algebraic approach to the reconfiguration problem, presented in Section 3. The next two sections contain the proofs of Theorems 1.1 and 1.3, while in the last section we mention some open problems and make a few remarks.

2 Admissible transformations of matrices

We follow the notation introduced above: A configuration P consisting of n ordinary pieces and a stationary piece at the origin in $\mathbf{R}^d$ is identified with a $d \times n$ matrix, also denoted by P, the columns of which correspond to the positions of the ordinary pieces.

The legal move by which the ordinary piece i ($1 \leq i \leq n$) jumps over piece j ($0 \leq j \leq n$, $i \neq j$) brings configuration P to PA_{ij}, where $A_{ij} = (a_{kl}^{(ij)})_{k,l=1}^n$ is the $n \times n$ real matrix defined as follows.

$$a_{kl}^{(ij)} = \begin{cases} 1 & \text{for } k = l \neq i \\ -1 & \text{for } k = l = i \\ 2 & \text{for } k = j, l = i \\ 0 & \text{otherwise.} \end{cases}$$

For example, for $n = 4$, we have

$$A_{2,4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad A_{2,0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the first matrix corresponds to the jump of piece 2 over piece 4, and the second matrix to the jump of 2 over the stationary piece 0. Indeed, denoting the position vectors of 2 and 4 by v_2 and $v_4 \in \mathbf{R}^d$, resp., after the first move, the new position of 2 will be $v_4 + (v_4 - v_2) = -v_2 + 2v_4$. This change of coordinates is reflected by the second column of $A_{2,4}$.

We call the matrices A_{ij} for $1 \leq i \leq n$, $0 \leq j \leq n$, $i \neq j$, *elementary involutions*. (Note that they are indeed involutions, that is, $A_{ij}^2 = I$ holds, reflecting the fact that all legal moves are reversible. Here I stands for the identity matrix.) The positions reachable from an initial position P (with the piece at the origin remaining stationary) are precisely the configurations PA, where A belongs to the *matrix group* generated by the *elementary involutions*.

Our first lemma gives a simple characterization of this matrix group. It will be used in the next section. We write $\det A$ for the determinant of the square matrix A.

Lemma 2.1. *An $n \times n$ integer matrix $A = (a_{ij})$ is generated by elementary involutions if and only if*

(i) $|\det A| = 1$*;*

(ii) a_{ij} *is* even *if* $i \neq j$ *and it is* odd *if* $i = j$.

Proof. All elementary transpositions have determinant -1 and all are equal to the identity matrix I modulo 2. This proves the "only if" part of the lemma.

It is instructive to think of the elementary involutions as column operations: For $1 \leq i \leq n$ multiplying by A_{i0} from the right is the same as multiplying column i by -1. If we further have $1 \leq j \leq n$, $j \neq i$, then multiplying by $A_{ij}A_{i0}$ is the same as *subtracting* twice column j from column i, while multiplying by $A_{i0}A_{ij}$ is the same as *adding* column j twice to column i. We call these column operations *admissible.*

We prove the "if" part of the lemma by showing that any matrix satisfying conditions (i) and (ii) in the lemma can be transformed into the identity matrix I by admissible operations. We do this by induction on n. For $n = 1$, the statement is trivially true. Suppose now that $n > 1$ and that our statement holds for $n - 1$. Let A be an $n \times n$ matrix satisfying conditions (i) and (ii).

Consider the last row of A. As long as we can find two entries a_{ni} and a_{nj} with $|a_{ni}| > |a_{nj}| > 0$, by subtracting or adding twice the jth column to the ith column, we can reduce the value of $|a_{ni}|$. In this step, no other entry of the last row changes. Hence, the sum of the absolute values of the elements of the last row strictly decreases. After a finite number of such steps (admissible operations), we will get stuck. At this point, every entry of the last row is either 0 or $\pm a$ for some integer $a > 0$. During this process, the absolute value of the determinant of the matrix has remained 1. Now it is divisible by a, so we must have $a = 1$. Since the parities of the elements have not changed, all entries of the last row must be 0, with the exception of the last element, which is either $+1$ or -1. Applying a single admissible transformation if necessary, we can achieve that this last element is $+1$.

Let B denote the matrix constructed so far, and let B' be the $(n-1)\times(n-1)$ submatrix of B, obtained by deleting its last column and last row. Expanding the determinant of B along its last row, we get that $|\det B| = |\det B'| = 1$. Clearly, B' also satisfies condition (ii) of the lemma. Thus, we can apply the induction hypothesis to B', and conclude that B' can be transformed to an $(n-1) \times (n-1)$ identity matrix by performing a number of admissible transformations. These transformations can, of course, be extended to the whole matrix B. During this process, the last row of B remains the same: all of its entries, except the last one, remain 0. Its last column also remains unchanged.

The matrix obtained so far coincides with the identity matrix I, apart from the non-last entries of its last column that are arbitrary even integers. We can make them vanish by adding or subtracting each of the first $n-1$ columns by an even number of times. Thus, by a sequence of admissible transformations, A can be transformed into I, as required. $\square$

3 Characterization of all reachable positions

Our next theorem gives a complete linear algebraic description of all configurations that can be obtained from a given initial position of the pieces by a sequence of legal moves.

Theorem 3.1. *Let P be a $d \times n$ matrix associated with the initial configuration of the $n+1$ pieces in $\mathbf{R}^d$, one of them in the origin.*

1. *The configurations that can be reached by a sequence of legal moves that keep the stationary piece at the origin, are exactly the positions corresponding to a matrix PA, where A is a matrix generated by the elementary involutions.*
2. *If we drop the assumption that the stationary piece must stay at the origin and allow it to jump over any of the remaining n pieces, every reachable configuration is a translate of some configuration that can be reached without moving the stationary piece. Moreover, a translate of such a configuration is reachable if and only if the translation vector is of the form $2Pw$ for an integer vector w.*

Proof. Part 1 is immediate from the observation made in the previous section, according to which the effect of a legal jump by an ordinary piece on the matrix associated with the configuration is exactly a multiplication by an elementary involution.

Consider now the case when piece 0 can also jump over other pieces. We cannot call it the stationary piece any more, so we will call it the *special* piece. For a configuration Q we denote the translate of Q where the special piece is at the origin Q_0 and as above we identify Q_0 with a matrix.

First we show that if the configuration Q can be reached then, Q_0 can be reached without ever jumping with the special piece. This will prove the first statement in part 2 of the theorem.

Indeed if configuration Q' is reached from configuration Q by jumping the ordinary piece j over any other piece j, then the same jump also reaches Q'_0 from Q_0. If Q' is reached from Q by jumping the special piece over the ordinary piece i, then special piece is moved by the vector $2v_i$, where v_i is the ith column of Q_0 (i.e., the vector from the position of the special piece to piece i in Q). We move instead all ordinary pieces with the vector $-2v_i$ to achieve a translate Q'' of Q. This we can do by letting the ordinary pieces $j \neq i$ first jump over piece i and then over the special piece and then jumping with piece i over the special piece. As Q'' is a translate of Q' we have $Q''_0 = Q'_0$ and this configuration can be achieved from Q_0 using the same jumps.

Whenever a configuration Q in which the special piece is at v is reachable by legal moves, we have $Q_0 = PA$, for some A generated by elementary involutions. We can, therefore, use part 1 of the theorem to reach any other configuration Q' with $Q'_0 = PB$, where B is generated by the elementary involutions and the special piece is at the same point v. To complete the proof of the theorem, it remains to show that the places where the special piece can go are exactly the points $2Pw$ with some integer vector w.

As all pieces start in the additive subgroup $S = \{Pw \mid w \in \mathbb{Z}^n\}$ of d-space, they must always remain in S. At each jump, every piece moves by twice a vector in S. The special piece starts at the origin, so it must always arrive at a point in $2S$, namely ar $2Pw$ with an integer vector w. (Note that S may or may not be a discrete lattice depending on the initial configuration P.)

Finally, we need to show that the special piece can indeed arrive at any vector of the form $2Pw$ with w integer. For this, it is enough to show that if it can arrive at a position v, then it can also arrive at $v + 2p_i$ and $v - 2p_i$ for any column vector p_i of P. However, we proved that if a configuration with the special piece at v is reachable, then by applying further legal moves which keep the special piece at v, we can achieve a configuration Q with $Q_0 = P$. At this point, a single jump by the special piece will move it to $v + 2p_i$, for any column p_i of P. If we jump piece i over the special piece first, and then jump the special piece over piece i, it will land on $v - 2p_i$. This finishes the proof. $\square$

Theorem 1.2 provides a simpler condition for the special case where the final configuration is a similar but larger copy of the initial one. A configuration is similar to but larger than the one identified with the matrix P if its matrix is λSP, where $|\lambda| > 1$ and S is a $d \times d$ *orthogonal* matrix.

Proof of Theorem 1.2. The proof of this result would be an immediate consequence of Theorem 3.1 and Lemma 2.1 if we had also required that the matrix A satisfies the parity condition that it agrees with the identity matrix I modulo 2. The main thing here is that this requirement can be dropped.

To see this, assume that A satisfies the conditions of Theorem 1.2. We have $|\det A| = 1$, but as the parity condition is dropped, A is not necessarily generated by elementary involutions.

Consider the matrix A modulo 2. Its determinant is 1, so it is an element of the $n \times n$ *matrix group* $GL(n, 2)$. As $GL(n, 2)$ is a *finite* group, A must have a finite order $t \geq 1$, for which A^t agrees with the identity matrix modulo 2. We also have $|\det(A^t)| = |\det A|^t = 1$. Hence, A^t satisfies both conditions in Lemma 2.1 and is, therefore, generated by elementary involutions. Hence, by Theorem 3.1, PA^t is reachable from the original configuration P by legal jumps (leaving the special piece stationary). By our assumption on A, we have $PA = \lambda SP$, for some $|\lambda| > 1$ and some orthogonal matrix S. This implies that $PA^t = \lambda^t S^t P$, which means that the configuration PA^t is similar to but larger than P. This completes the proof. $\square$

4 Regular polygons—Proof of Theorem 1.1

Let $N = n + 1$ and let $p_0, p_1, p_2, \ldots, p_n \in \mathbf{R}^2$ denote the vertices of a regular N-gon in the plane, enumerated in positive (counter-clockwise) cyclic order, and suppose that $p_0 = 0$. The initial position with the N pieces at these vertices is identified with the $2 \times n$ matrix P, the ith column of which is p_i, for $1 \le i \le n$.

Let M denote the $n \times n$ matrix

$$M = \begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Note that the ith column of PM is $p_{i+1} - p_1$ for $1 \le i < n$ and $-p_1$ for $i = n$ and thus can be obtained as the rotation of p_i with the positive angle $2\pi/N$. Let

$$S = \begin{bmatrix} \cos\frac{2\pi}{n+1} & -\sin\frac{2\pi}{n+1} \\ \sin\frac{2\pi}{n+1} & \cos\frac{2\pi}{n+1} \end{bmatrix}$$

be the orthogonal matrix representing this rotation. Now we have $PM = SP$.

At this point, we are almost done. We have $|\det M| = 1$, and the position PM is similar to P, that is, it is also a regular $(n+1)$-gon. If it was also larger than P, we could apply Theorem 1.2 and finish the proof. Unfortunately, the configuration $PM = SP$ is of exactly the same size as P, because S is a rotation.

We solve this problem by considering polynomials $f(M)$ of M. Obviously, we have $Pf(M) = f(S)P$. Notice that all polynomials $f(S)$ of the matrix S are of the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \lambda T,$$

for some reals a, b, λ and for an orthogonal matrix T. So, the configuration $Pf(M)$ is similar to P, i.e., it is a (possibly degenerate) regular $(n+1)$-gon. We can finish the proof by applying Theorem 1.2 if we find an integer polynomial f such that

(i) $|\det f(M)| = 1$, and

(ii) the regular $(n+1)$-gon identified with $Pf(M)$ is larger than P.

For a positive integer k, consider the matrix

$$B_k = \sum_{j=0}^{k-1} M^j.$$

Note that the characteristic polynomial of M is $\sum_{j=0}^{n} t^j$. Recall that $N = n+1$. Thus, we have $B_N = 0$. We further have

$$B_{kN} = B_N \sum_{j=0}^{k-1} M^{jN} = 0,$$

for all $k > 0$, and

$$B_{kN+1} = I + MB_{kN} = I.$$

Let us fix an integer i with $1 \leq i < N$ which is coprime to N. We can find positive integers l and k such that $il = kN + 1$, and

$$B_i \sum_{j=0}^{l-1} M^{ij} = B_{il} = B_{kN+1} = I.$$

This shows that the integer matrix B_i is invertible and its inverse is also an integer matrix. Therefore, we have $|\det B_i| = 1$. This was condition (i) above that a polynomial of M had to satisfy to finish the proof.

It remains to verify condition (ii) and check that the configuration identified with PB_i (a regular N-gon) is strictly larger than the starting configuration. The side length of the N-gon associated with the configuration PB_i is the distance between $p_1 B_i$, the position of piece 1, and the special piece at the origin. Here, $p_1 M^j = p_{j+1} - p_j$ for $0 \leq j < N - 1$, so that

$$p_1 B_i = \sum_{j=0}^{i-1} p_1 M^j = \sum_{j=0}^{i-1} (p_{j+1} - p_j) = p_i.$$

Therefore, the side length of the regular $(n+1)$-gon corresponding to the configuration PB_i is equal to $|p_i|$. This is the length of a proper diagonal of the original configuration P provided that $1 < i < N - 1$, so strictly larger than original side length of $|p_1|$, .

To complete the proof of Theorem 1.1, it is enough to observe that there always exists an integer i, $1 < i < N - 1$, which is coprime to N, provided that $N \geq 5$ and $N \neq 6$. $\square$

5 Adding the center—Proof of Theorem 1.3

As another application of Theorem 1.2, we consider the case where we have $N+1$ grasshoppers, originally sitting at the vertices and center of a regular N-gon. If $N = 3$, 4 or 6, the grasshoppers jump around on the points of a triangular or square lattice, which implies that they cannot end up at the vertices and center of a smaller regular n-gon. Hence, by the reversibility of the moves, they cannot end up at the vertices and center of a larger regular n-gon either. Theorem 1.3 claims that this can be achieved in all remaining cases. Its proof is similar to the proof of Theorem 1.1 presented in the previous section.

Proof of Theorem 1.3. Denote by $q_1, \dots, q_N$ the vertices of a regular n-gon, listed in positive cyclic order. Assume that the center of the N-gon is the origin. We put a piece in the vertices and the center of this N-gon. The piece at the origin is treated as the *special (stationary)* piece and the ith piece starts at q_i for $i = 1, \dots, n$. As before, the position of the pieces is always identified with a matrix. This time we have $N+1$ pieces, so we identify their configurations with a $2 \times N$ matrix. In particular, let Q denote the matrix identified with the initial configuration having q_i as column i for $1 \le i \le N$. Consider the $N \times N$ matrix

$$M' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The ith column of QM' is q_{i-1} for $1 < i < N$ and the first column of QM is q_N. This represents a rotation with a *negative* angle of $2\pi/N$, so we have $QM' = S^{-1}Q$, where S is the rotation matrix as in the proof of Theorem 1.1. For $0 < i < N$, let $D_i = \sum_{j=0}^{i-1} M'^j$. We have

$$QD_i = \left(\sum_{j=0}^{i-1} S^{-j} \right) Q.$$

Thus, QD_i is similar to the configuration Q. To figure out its side length, note that in the configuration QM'^j, $(0 \le j \le N-2)$, pieces 1 and 2 are at q_{j+1} and q_{j+2}, respectively. Therefore, in QD_i, they are at $r = \sum_{j=1}^{i} q_j$ and $s = \sum_{j=2}^{i+1} q_j$, respectively. Thus, the side length of the configuration QD_i is $|r-s| = |q_{i+1} - q_1|$, which is the length of a proper diagonal and therefore strictly larger then the side length $|q_2 - q_1|$ of the initial configuration Q, provided that $1 < i < N-1$. In these cases, QD_i similar to, but larger than Q.

To complete the proof applying Theorem 1.2, we would need that $|\det D_i| = 1$. Unfortunately, this fails even for many values of i that are coprime to N. We circumvent this difficulty by considering the matrix $D_{i,k} = D_i + kJ$, where J is the $N \times N$ matrix with 1 entries in its last column and zeros everywhere else. We clearly have $QJ = 0$. Hence, $QD_{i,k} = QD$ is similar to the initial configuration Q, and is larger when $1 < i < N-1$. We need to find suitable integers $1 < i < N-1$ and k such that $|\det D_{i,k}| = 1$, and then we can conclude the proof by Theorem 1.2.

Fix i to be coprime to N. If $N \ge 5$ and $N \ne 6$, we can find such an i with $1 < i < N-1$. For now, we keep k as a free variable, and we try to compute the determinant of $D_{i,k}$. First, we subtract the last row of $D_{i,k}$ from all other rows. This does not alter the determinant. The last entry of the last row of the resulting matrix $D'_{i,k}$ is $1+k$, but no other entriy of the matrix depends on k. When we expand $\det D'_{i,k}$ along the last row, no summand depends on k, except the very last one, which is $(k+1)\det E_i$, where E_i is the matrix obtained from $D'_{i,k}$ after deleting its last row and last column. The matrix E_i is independent of k and, by coincidence, it happens to be the transpose of the matrix B_i used in the proof of Theorem 1.1 for the same value of N. To see this, first note that both B_i and E_i are $(N-1)\times(N-1)$ matrices and it is not hard to evaluate and compare their entries. For the entries of $E_i = (e^{(i)}_{lm})$ we have

$$e^{(i)}_{lm} = \begin{cases} 1 & \text{for } \max(i,l) \le m < i+l \\ -1 & \text{for } i+l-N \le m < \min(i,l) \\ 0 & \text{otherwise.} \end{cases}$$

This means that $|\det E_i| = |\det B_i| = 1$, which implies that $\det D_{i,k} = \det D'_{i,k}$ can be made 1 by an appropriate integer choice of k. With that choice, $D_{i,k}$ satisfies the conditions of Theorem 1.2. This finishes the proof. $\square$

6 Remarks and further questions

In the present note, we defined a game with grasshoppers jumping over each other and discussed when a configuration of the grasshoppers *can be reached* from another configuration but we neglected the question of *how many jumps* it takes to reach the desired configuration. Here, we list several problems related to this latter question.

A. The smallest special case of Theorem 1.1 is that of the regular pentagon. One can implement the steps of the proof above to find a concrete sequence of jumps for the grasshoppers starting at the vertices of a regular pentagon that yields a larger regular pentagon, but the sequence so obtained is very long. Here we give short such sequence. We number the grasshoppers 0 through 4 and denote by J_{ij} the jump of grasshopper i over grasshopper j. The following sequence of jumps results in a regular pentagon that is $\sqrt{5}+2$ times larger than the original:

$$J_{42}J_{20}J_{31}J_{21}J_{10}J_{41}J_{32}J_{23}J_{30}J_{13}J_{31}J_{34}J_{14}J_{10}$$

We do not know whether there exists a shorter such sequence.

B. The same question can be asked more generally:

Problem 6.1. Given an integer N which is either 5 or larger than 6, and N pieces sitting at the vertices of a regular N-gon, what is the smallest number of legal moves that takes them into the vertex set of a larger regular N-gon?

How many legal moves are needed to turn a configuration consisting of the vertices and the center of regular N-gon to a similar but larger configuration?

We can turn the proofs of Theorems 1.1 and 1.3 to give explicit bounds but these bounds will be exponential in N for two reasons. First, the application of Lemma 2.1 may itself yield exponentially long sequences; see more on this below. Second, in the proof of Theorem 1.2 we take an integer matrix A and raise it to a power that is the order of A in $GL(n, 2)$. But $GL(n, 2)$ has elements of order $2^n - 1$.

Still, we conjecture that a regular N-gon ($N > 6$) with or without the center can be transformed to a similar but larger configuration in polynomially many moves in N.

C. We can also ask how many legal jumps are needed to bring a given configuration P to another configuration Q whenever this is possible. Here we cannot hope for a bound that depends only on the number N of pieces in the configurations. Theorem 3.1 claims that if P can be transformed to Q at all, then $Q = PA$ for an $(N-1) \times (N-1)$ matrix A generated by the elementary involutions. (Here we assume both P and Q have a stationary piece in the origin and we identify the configurations with matrices as usual.) It would be interesting to bound the number of legal moves needed in terms of N and the maximum absolute value of an entry in A. This is the same as asking how long a product of elementary transformations may be needed to obtain a specific matrix with bounded entries.

As before, our proof of Theorem 3.1 can be turned into a explicit bound, but that bound will be exponential. This is because the recursive nature of the algorithm turning a matrix into the identity matrix using admissible operations that we use to prove Lemma 2.1 may result in the size of entries growing substantially throughout the procedure. We do not know whether better bounds are possible.

D. What if we are given a single configuration of N pieces and we are promised that it can be transformed to a similar but larger configuration by legal jumps? Can we bound the number of jumps needed solely in terms of N?

E. And finally a question that is *not* about the number of jumps needed to arrive to a configuration.

Problem 6.2. For what Platonic solids T can we start this game at the vertices of T and achieve a similar but larger configuration with legal jumps?

There is no way to arrive to a similar but larger configuration in case T is a tetrahedron, a cube or an octahedron, because the vertices of these solids can be on a cubic lattice. If we start with a configuration on the cubic lattice forming the vertex set of the smallest solid similar to T, then we cannot arrive to an even smaller such configuration, and by reversibility, neither can we arrive to a larger similar configuration.

We conjecture that starting from the vertices of either of the remaining two Platonic solids: the dodecahedron or the icosahedron one can arrive to a similar but larger configuration.

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