

# The Visible Perimeter of an Arrangement of Disks\*

Gabriel Nivasch<sup>†</sup>  
gabrieln@ariel.ac.il

János Pach<sup>‡</sup>  
pach@cims.nyu.edu

Gábor Tardos<sup>§</sup>  
tardos@renyi.hu

May 20, 2013

## Abstract

Given a collection of  $n$  opaque unit disks in the plane, we want to find a stacking order for them that maximizes their *visible perimeter*, the total length of all pieces of their boundaries visible from above. We prove that if the centers of the disks form a *dense* point set, *i.e.*, the ratio of their maximum to their minimum distance is  $O(n^{1/2})$ , then there is a stacking order for which the visible perimeter is  $\Omega(n^{2/3})$ . We also show that this bound cannot be improved in the case of a sufficiently small  $n^{1/2} \times n^{1/2}$  uniform grid. On the other hand, if the set of centers is dense and the maximum distance between them is small, then the visible perimeter is  $O(n^{3/4})$  with respect to any stacking order. This latter bound cannot be improved either.

Finally, we address the case where no more than  $c$  disks can have a point in common.

These results partially answer some questions of Cabello, Haverkort, van Kreveld, and Speckmann.

Keywords: Visible perimeter, disk, unit disk, dense set.

## 1 Introduction

In cartography and data visualization, one often has to place similar copies of a symbol, typically an opaque disk, on a map or a figure at given locations [De99], [Gr90]. The size of the symbol is sometimes proportional to the quantitative data associated with the location. On a cluttered map, it is difficult to identify the symbols. Therefore, it has been investigated in several studies how to minimize the amount of overlap [GrC78], [SIM03].

In the present note, we follow the approach of Cabello, Haverkort, van Kreveld, and Speckmann [CaH10]. We assume that the symbols used are opaque circular disks of the same size. Given a collection  $\mathcal{D}$  of  $n$  distinct *unit* disks in the  $(x, y)$ -plane, a *stacking order* is a one-to-one assignment  $f : \mathcal{D} \rightarrow \{1, 2, \dots, n\}$ . We consider the integer  $f(D)$  to be the  $z$ -coordinate of the disk  $D \in \mathcal{D}$ . The *map* corresponding to this stacking order is the 2-dimensional view of this arrangement from the point at negative infinity of the  $z$ -axis (for notational convenience,

---

\*A preliminary version of this paper appeared in *Graph Drawing 2012* (LNCS 7704, pp. 364–375, 2013).

<sup>†</sup>Ariel University, Ariel, Israel. Work done when the author was at EPFL, Lausanne, Switzerland.

<sup>‡</sup>EPFL, Lausanne, Switzerland and Rényi Institute, Budapest, Hungary. Work supported by Hungarian Science Foundation EuroGIGA Grant OTKA NN 102029, by Swiss National Science Foundation Grants 200020-144531 and 20021-137574, and by NSF Grant CCF-08-30272.

<sup>§</sup>Rényi Institute, Budapest, Hungary. Work supported by an NSERC grant and by OTKA grants T-046234, AT048826 and NK-62321.

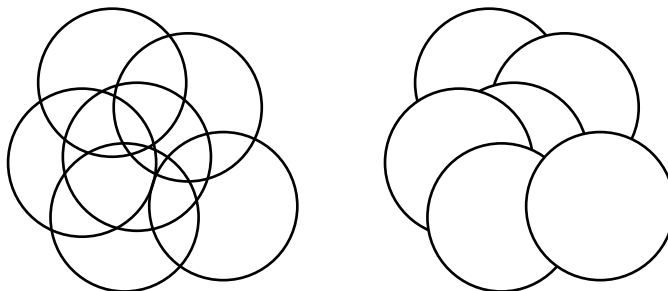


Figure 1: Left: A collection of unit disks in the plane. Right: A stacking order for them.

we look at the arrangement from below rather than from above.) In particular, for the lowest disk  $D$ , we have  $f(D) = 1$ , and this disk, including its full perimeter, is visible from below. The total length of the boundary pieces of the disks visible from below is the *visible perimeter* of  $\mathcal{D}$  with respect to the stacking order  $f$ , denoted by  $\text{visible}(\mathcal{D}, f)$ . We are interested in finding a stacking order for which the visible perimeter of  $\mathcal{D}$  is as large as possible. See Figure 1.

There are other situations in which this setting is relevant. Sometimes the vertices of a graph are not represented as points but as circles of a given radius. It may happen that some vertices overlap in the visualization (especially if they have further constraints on their geometric position), and then it becomes important to choose a convenient stacking order that maximizes the visible perimeter.

Given an integer  $n$ , we define

$$v(n) = \inf_{|\mathcal{D}|=n} \max_f \text{visible}(\mathcal{D}, f), \quad (1)$$

where the maximum is taken over all stacking orders  $f$ . We would like to describe the asymptotic behavior of  $v(n)$ , as  $n$  tends to infinity.

Cabello *et al.* have already noted that  $v(n) = \Omega(n^{1/2})$ ; in other words, every set  $\mathcal{D}$  of  $n$  disks of unit radii admits a stacking order with respect to which its visible perimeter is  $\Omega(n^{1/2})$ . Indeed, by a well-known result of Erdős and Szekeres [ErSz35], we can select a sequence of  $\lceil n^{1/2} \rceil$  disks  $D_i \in \mathcal{D}$  ( $1 \leq i \leq \lceil n^{1/2} \rceil$ ) such that their centers form a monotone sequence. More precisely, letting  $x_i$  and  $y_i$  denote the coordinates of the center of  $D_i$ , we have  $x_1 \leq x_2 \leq x_3 \leq \dots$  and either  $y_1 \leq y_2 \leq y_3 \leq \dots$  or  $y_1 \geq y_2 \geq y_3 \geq \dots$ . Then, in any stacking order  $f$  such that  $f(D_i) = i$  for every  $i$ ,  $1 \leq i \leq \lceil n^{1/2} \rceil$ , a full quarter of the perimeter of each  $D_i$  ( $1 \leq i \leq \lceil n^{1/2} \rceil$ ) is visible from below. Therefore, the visible perimeter of  $\mathcal{D}$  with respect to  $f$  satisfies

$$\text{visible}(\mathcal{D}, f) \geq \frac{\pi}{2} \lceil n^{1/2} \rceil.$$

At the problem session of *EuroCG'11* (Morschach, Switzerland), Cabello, Haverkort, van Kreveld, and Speckmann asked whether  $v(n) = \Omega(n)$ ; in other words, does there exist a positive constant  $c$  such that every set of  $n$  unit disks in the plane admits a stacking order, with respect to which its visible perimeter is at least  $cn$ ? We answer this question in the negative; *cf.* Theorems 2 and 5 below.

Given a set of points  $P$  in the plane, let  $\mathcal{D}(P)$  denote the collection of disks of radius 1 centered at the elements of  $P$ . For any positive real  $\varepsilon$ , let  $\varepsilon P$  stand for a similar copy of  $P$ , scaled by a factor of  $\varepsilon$ . For a stacking order  $f$  of  $\mathcal{D}(P)$  we will study the quantity

visible( $\mathcal{D}(\varepsilon P), f$ ). (Note the slight abuse of notation: We denote the stacking order of  $\mathcal{D}(P)$  and the corresponding stacking order of  $\mathcal{D}(\varepsilon P)$  by the same symbol  $f$ . The two orders are also identified in Lemmas 1 and 7 and in Theorems 2, 3, and 5.) It is not hard to verify that, as  $\varepsilon$  gets smaller, the function visible( $\mathcal{D}(\varepsilon P), f$ ) decreases. To see this, it is enough to observe, as was also done by Cabello *et al.* (unpublished), that as we contract the set of centers, the part of the boundary of each unit disk visible from below shrinks. As we will see in Lemma 7, the limit in the following lemma has a simple alternative geometric interpretation.

**Lemma 1.** *For every point set  $P$  in the plane and for every stacking order  $f$  of the collection of disks  $\mathcal{D}(P)$ , we have*

$$\text{visible}(\mathcal{D}(P), f) \geq \lim_{\varepsilon \rightarrow 0} \text{visible}(\mathcal{D}(\varepsilon P), f).$$

As in [AlKP89], [Va92], and [Va96], we consider  $C$ -dense  $n$ -element point sets  $P$ , *i.e.*, point sets in which the ratio of the maximum distance between two points to the minimum distance satisfies

$$\frac{\max(|pq| : p, q \in P)}{\min(|pq| : p, q \in P, p \neq q)} \leq Cn^{1/2}.$$

(The above ratio is sometimes called the *spread* of  $P$  [Er03]; thus, we consider point sets with spread at most  $Cn^{1/2}$ .)

**Theorem 2.** *For any  $C$ -dense  $n$ -element point set  $P$  in the plane and for any stacking order  $f$ , we have*

$$\lim_{\varepsilon \rightarrow 0} \text{visible}(\mathcal{D}(\varepsilon P), f) \leq C'n^{3/4},$$

where  $C'$  is a constant depending only on  $C$ .

The order of magnitude of the upper bound in Theorem 2 cannot be improved:

**Theorem 3.** *For every positive integer  $n$ , there exists a 4-dense  $n$ -element point set  $P_n$  in the plane and a stacking order  $f$  such that*

$$\lim_{\varepsilon \rightarrow 0} \text{visible}(\mathcal{D}(\varepsilon P_n), f) \geq n^{3/4}.$$

In the general case, where  $P$  is an arbitrary  $n$ -element point set in the plane, we have been unable to improve on the easy lower bound

$$\max_f \text{visible}(\mathcal{D}(P), f) = \Omega(n^{1/2}),$$

sketched above. However, under special assumptions on  $P$ , we can do better.

**Theorem 4.** *Every  $C$ -dense  $n$ -element point set  $P$  in the plane admits a stacking order  $f$  with*

$$\text{visible}(\mathcal{D}(P), f) \geq C''n^{2/3},$$

where  $C'' > 0$  depends only on  $C$ .

In particular, Theorem 4 provides an  $\Omega(n^{2/3})$  lower bound for the visible perimeter of a collection of  $n$  unit disks centered at the points of an  $n^{1/2} \times n^{1/2}$  uniform grid, under a suitable stacking order. If the side length of the grid is very small, this is better than the line-by-line “lexicographic” stacking order, for which the visible perimeter is only  $\Theta(n^{1/2} \log n)$ . It turns out that in this case there is no stacking order for which the order of the magnitude of the visible perimeter would exceed  $n^{2/3}$ .

**Theorem 5.** *Let  $n$  be a perfect square and let  $G_n$  denote an  $n^{1/2}$  by  $n^{1/2}$  uniform grid in the plane. For any stacking order  $f$ , we have*

$$\lim_{\varepsilon \rightarrow 0} \text{visible}(\mathcal{D}(\varepsilon G_n), f) = O(n^{2/3}).$$

Consequently, we have  $v(n) = O(n^{2/3})$ .

Lemma 1 implies that the worst collections of disks are those whose centers are very close to each other, so all disks have a point in common. This is, of course, not a realistic assumption in the labeling problem in cartography that has motivated our investigations. In practical applications, only a bounded number of unit disks share a point. For such a case, we have the following result:

**Theorem 6.** *Let  $\mathcal{D}$  be a collection of  $n$  unit disks in which at most  $c$  disks have a point in common. Then there exists a stacking order  $f$  for which*

$$\text{visible}(\mathcal{D}, f) = \Omega(v(c)n/c),$$

where  $v(c)$  is given in (1). This bound is worst-case asymptotically tight.

In Section 2, we establish Theorems 2 and 3. The proof of Theorem 4 is presented in Section 3. In Section 4, we consider the square grid and present a much simpler proof of this special case of Theorem 4 based on Jarnik’s theorem [Ja25]; we then prove Theorem 5, which states that the bound of Theorem 4 is tight in this case. In Section 5, we prove Theorem 6. The last section contains concluding remarks and open problems.

## 2 Dense Sets with Largest Visible Perimeter

In this section, we prove Theorems 2 and 3.

First, we express the limit of visible perimeters in a simpler form. Given a set of points  $P$  in the plane, let  $\text{conv } P$  stand for its convex hull. Let  $D(p)$  denote the unit disk centered at  $p$  and let  $\mathcal{D}(P)$  stand for the set  $\{D(p) : p \in P\}$ .

Fix an orthogonal system of coordinates in the plane. For any point  $p = (x, y)$  and for any  $\varepsilon > 0$ , let  $\varepsilon p$  denote the point with coordinates  $(\varepsilon x, \varepsilon y)$ .

**Lemma 7.** *Let  $P = \{p_1, p_2, \dots, p_n\}$  be a set of points in the plane, let  $\varepsilon > 0$ , and let  $f$  be the stacking order of  $\mathcal{D}(\varepsilon P)$  given by  $f(D(\varepsilon p_i)) = i$  for  $i = 1, 2, \dots, n$ .*

*We have*

$$\lim_{\varepsilon \rightarrow 0} \text{visible}(\mathcal{D}(\varepsilon P), f) = \sum_{i=1}^n \tau_i,$$

where  $\tau_1 = 2\pi$ , and for all other indices,  $\tau_i = 0$  if  $p_i$  belongs to  $\text{conv}\{p_1, p_2, \dots, p_{i-1}\}$ , and  $\tau_i$  is equal to the external angle of the convex polygon  $\text{conv}\{p_1, p_2, \dots, p_i\}$  at vertex  $p_i$ , otherwise.

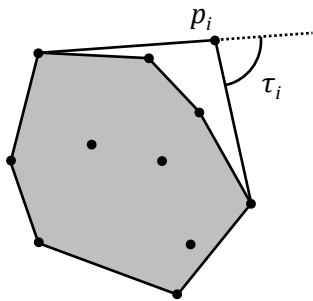


Figure 2: If  $p_i$  lies outside the convex hull of the preceding points, then  $\tau_i$  is defined as the external angle of the polygon  $\text{conv}\{p_1, \dots, p_i\}$  at vertex  $p_i$ .

See Figure 2.

*Proof.* We prove that the contribution of  $\mathcal{D}(\varepsilon p_i)$  to the visible perimeter tends to  $\tau_i$  as  $\varepsilon \rightarrow 0$  for each  $1 \leq i \leq n$ .

Since  $D(\varepsilon p_1)$  is the lowest disk in  $\mathcal{D}(\varepsilon P)$ , its whole boundary is visible from below. Therefore, its contribution is  $2\pi$ . Let  $i > 1$ . If  $p_i$  belongs to the interior of  $\text{conv}\{p_1, p_2, \dots, p_{i-1}\}$ , then there is a threshold  $\varepsilon_0 > 0$  such that

$$D(\varepsilon p_i) \subset \bigcup_{j=1}^{i-1} D(\varepsilon p_j),$$

for every  $\varepsilon < \varepsilon_0$ . In this case, no portion of the boundary of  $D(\varepsilon p_i)$  is visible from below, provided that  $\varepsilon$  is sufficiently small. If  $p_i$  lies on the boundary of  $\text{conv}\{p_1, p_2, \dots, p_i\}$ , then it is in between some points  $p_j$  and  $p_k$  with  $1 \leq j < k < i$  and although  $\mathcal{D}(\varepsilon p_i)$  will not be entirely covered by earlier disks for any  $\varepsilon > 0$ , the part of its boundary outside  $\mathcal{D}(\varepsilon p_j) \cup \mathcal{D}(\varepsilon p_k)$  tends to zero as  $\varepsilon \rightarrow 0$ .

Finally, if  $p_i$  lies outside  $\text{conv}\{p_1, \dots, p_{i-1}\}$ , then it is a vertex of  $\text{conv}\{p_1, \dots, p_i\}$ . Consider the external unit normal vectors to the two sides of  $\text{conv}\{p_1, \dots, p_i\}$  that meet at  $\varepsilon p_i$  (or in case the convex hull is a single segment, the two unit normal vectors for this segment). Drawing these vectors from  $\varepsilon p_i$ , the arc on the boundary of  $\mathcal{D}(\varepsilon p_i)$  between them is of length  $\tau_i$  and it is not covered by  $\bigcup_{j=1}^{i-1} D(\varepsilon p_j)$ . Thus, it is visible from below, and, as  $\varepsilon \rightarrow 0$ , the total contribution of the remaining part of the boundary of  $D(\varepsilon p_i)$  to the visible perimeter tends to 0, concluding the proof.  $\square$

**Theorem 2.** For any  $C$ -dense  $n$ -element point set  $P$  in the plane and for any stacking order  $f$ , we have

$$\lim_{\varepsilon \rightarrow 0} \text{visible}(\mathcal{D}(\varepsilon P), f) \leq C' n^{3/4},$$

where  $C'$  is a constant depending only on  $C$ .

*Proof.* Consider a  $C$ -dense point set  $P$  in the plane and let  $f$  be a stacking order for  $\mathcal{D}(P)$ . Using Lemma 7, it is enough to prove  $\sum_{i=1}^n \tau_i \leq C' n^{3/4}$  for the angles  $\tau_i$  defined in the lemma. As  $\tau_i = 0$  whenever  $p_i$  is contained in  $\text{conv}\{p_1, \dots, p_{i-1}\}$ , we can assume this is never the case.

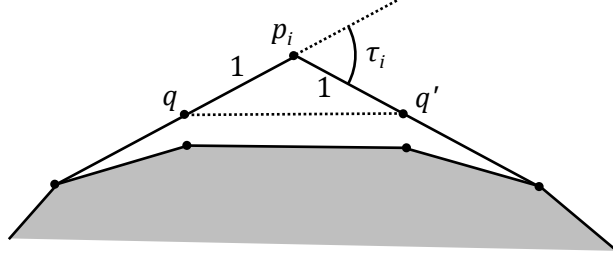


Figure 3: The triangle  $p_i q q'$  lies entirely outside the convex hull of  $p_1, \dots, p_{i-1}$ .

Since the quantity  $\sum \tau_i$  is independent of scale, we can assume without loss of generality that the minimum distance between points is 1; thus, the maximum distance (diameter) is at most  $Cn^{1/2}$ . We write  $P = \{p_1, p_2, \dots, p_n\}$  with  $f(D(p_i)) = i$ .

For every  $i$  ( $1 \leq i \leq n$ ), let  $\text{per}(i)$  denote the perimeter of  $\text{conv}\{p_1, p_2, \dots, p_i\}$ . We define the perimeter of a segment to be twice its length and the perimeter of a point to be 0. Let  $2 \leq i \leq n$ , consider the two sides of the polygon  $\text{conv}\{p_1, p_2, \dots, p_i\}$  meeting at  $p_i$ , and denote by  $q$  and  $q'$  the points on these sides at unit distance from  $p_i$ . Since no point of  $P$  is closer to  $p_i$  than 1, the triangle  $p_i q q'$  does not contain any element of  $\{p_1, p_2, \dots, p_{i-1}\}$ . Hence,  $\text{conv}\{p_1, p_2, \dots, p_{i-1}\}$  is contained in the convex region obtained from  $\text{conv}\{p_1, p_2, \dots, p_i\}$  by cutting off the triangle  $p_i q q'$ . (In the degenerate case when  $\text{conv}\{p_1, \dots, p_i\}$  is a segment, we have  $q = q'$ , and the empty “triangle” becomes just a unit segment.) This observation implies that the perimeter of  $\text{conv}\{p_1, p_2, \dots, p_{i-1}\}$  satisfies

$$\text{per}(i-1) \leq \text{per}(i) - |p_i q| - |p_i q'| + |qq'| = \text{per}(i) - 2 + 2 \cos \frac{\tau_i}{2} \leq \text{per}(i) - \frac{\tau_i^2}{5}.$$

Here we used that the external angle of the triangle  $p_i q q'$  at vertex  $p_i$  is  $\tau_i$ .

Thus, we have

$$\text{per}(i) - \text{per}(i-1) \geq \frac{\tau_i^2}{5},$$

for all  $i > 1$ . Adding up these inequalities, we obtain

$$\text{per}(n) \geq \sum_{i=2}^n \frac{\tau_i^2}{5}.$$

Since  $\text{per}(n)$  is at most  $\pi$  times the diameter of  $P$ , that is,  $\text{per}(n) \leq \pi Cn^{1/2}$ , we have

$$\sum_{i=2}^n \tau_i^2 \leq 5\pi Cn^{1/2}.$$

Applying the relationship between the arithmetic and quadratic means, we can conclude that

$$\sum_{i=2}^n \tau_i \leq (n-1)^{1/2} \left( \sum_{i=2}^n \tau_i^2 \right)^{1/2} < (5\pi C)^{1/2} n^{3/4}.$$

Taking into account that  $\tau_1 = 2\pi$ , the theorem follows by Lemma 7.  $\square$

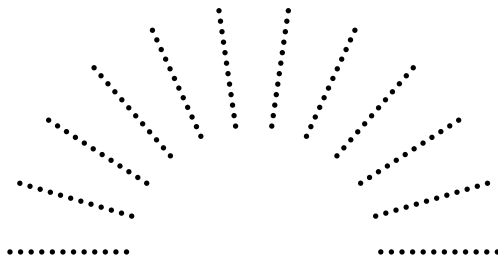


Figure 4: A dense point set that has a good stacking order.

**Theorem 3.** *For every positive integer  $n$ , there exists a 4-dense  $n$ -element point set  $P_n$  in the plane and a stacking order  $f$  such that*

$$\lim_{\varepsilon \rightarrow 0} \text{visible}(\mathcal{D}(\varepsilon P_n), f) \geq n^{3/4}.$$

*Proof.* Suppose for simplicity that  $n = k^2$  for some integer  $k \geq 3$ . Our point set  $P_n$  consists of the points having polar coordinates  $(r, \theta) = (i, j\pi/(k-1))$  for  $i \in \{k, k+1, \dots, 2k-1\}$  and  $j \in \{0, 1, \dots, k-1\}$ . See Figure 4. The smallest distance between two points in  $P_n$  is 1, and the largest distance is less than  $4k$ ; thus,  $P_n$  is 4-dense, as required.

Our stacking order  $f$  takes the points by increasing  $r$ , and for each  $r$  by increasing  $\theta$ .

We apply Lemma 7 and calculate the sum of the external angles determined by  $f$ . Denote by  $C_i$  the circle of radius  $i$  centered at the origin. Consider a point  $p \in P_n$  on  $C_i$ . Let  $\ell$  be the ray leaving  $p$  towards the right tangent to  $C_i$ , and let  $\ell'$  be the ray leaving  $p$  towards the left tangent to  $C_{i-1}$ . Let  $q$  be the point of tangency between  $\ell'$  and  $C_{i-1}$ . Then all the points of  $P_n$  that precede  $p$  in the order  $f$  lie below  $\ell$  and  $\ell'$ . Thus, the external angle  $\tau$  contributed by  $p$  is at least the supplement  $\alpha$  of the angle between  $\ell$  and  $\ell'$ . We have  $\alpha = \angle p0q \geq \sin \alpha = \sqrt{2i-1}/i \geq n^{-1/4}$ . The theorem follows.  $\square$

### 3 All Dense Sets Have Good Stacking Orders

We now turn to Theorem 4.

**Theorem 4.** *Every  $C$ -dense  $n$ -element point set  $P$  in the plane admits a stacking order  $f$  with*

$$\text{visible}(\mathcal{D}(P), f) \geq C'' n^{2/3},$$

where  $C'' > 0$  depends only on  $C$ .

Throughout this section, let  $P$  be a  $C$ -dense  $n$ -point set in the plane. We will define a stacking order  $f$  for  $\mathcal{D}(P)$  for which the external angles  $\tau_i$  defined in Lemma 7 satisfy  $\sum_{i=1}^n \tau_i \geq C'' n^{2/3}$ , for some constant  $C'' > 0$  depending only on  $C$ . Then the theorem follows from Lemma 7.

Assume without loss of generality that the minimum distance in  $P$  is 1. Then, since  $P$  is  $C$ -dense, there exists a disk of radius  $Cn^{1/2}$  that contains all of  $P$ . Let  $D$  be such a disk, and let  $K$  be a circle of radius  $2Cn^{1/2}$  concentric with  $D$ .

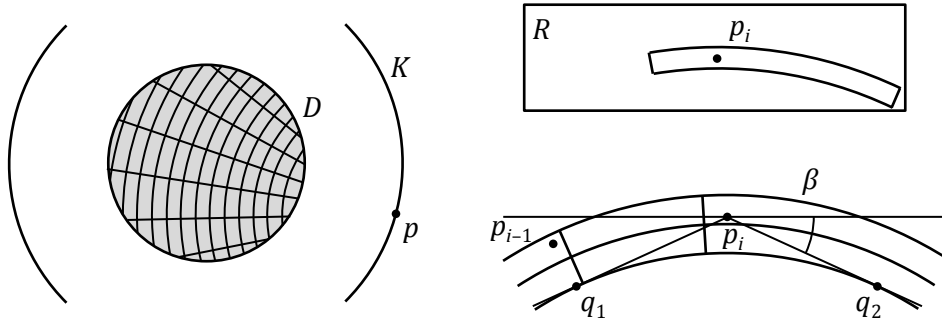


Figure 5: Left: Partition of  $D$  into annular sectors centered at a point  $p \in K$ . Top right: The sector containing  $p_i$  is contained in the rectangle  $R$  centered at  $p_i$ . Bottom right: Point  $p_i$  contributes external angle at least  $\beta$ .

Given a point  $p \in K$ , we define a family  $F = F(p)$  of annular sectors that disjointly cover the plane, as follows: For each positive integer  $i$ , let  $K_i = K_i(p)$  be a circle centered at  $p$  with radius  $in^{-1/6}$ ; then divide each annulus between two consecutive circles into sectors of angular length  $\alpha = C^*n^{-1/3}$  for a large enough constant  $C^*$  (as will be specified below). See Figure 5 (left).

Note that each annular sector that intersects  $D$  has area  $\Theta(1)$  (since the radius of such a sector is  $\Theta(n^{1/2})$ ). The number of annular sectors that intersect  $D$  is  $\Theta(n^{1/2}n^{1/6}n^{1/3}) = \Theta(n)$ . Call a sector *occupied* if it contains at least one point of  $P$ .

**Lemma 8.** *There exists a point  $p \in K$  for which  $\Omega(n)$  sectors of  $F(p)$  are occupied.*

*Proof.* Choose  $p$  uniformly at random on  $K$  and construct the sectors using  $p$  and dividing the annuli into the correct-length sectors in an arbitrary way. For each point  $p_i \in P$ , define the random variable  $n(p_i)$  to be the number of points of  $P$  contained in the sector of  $F(p)$  that contains  $p_i$ . We claim that the expected value  $E[n(p_i)]$  of  $n(p_i)$  satisfies

$$E[n(p_i)] \leq k$$

for some constant  $k$ .

Indeed, let  $R = R_{p_i}(p)$  be a rectangle centered at  $p_i$ , with dimensions  $(k'n^{1/6}) \times (k'n^{-1/6})$ , and with short sides parallel to the line  $pp_i$ , for an appropriate constant  $k'$ . If  $k'$  is large enough (but constant with respect to  $n$ ), then  $R$  completely contains the sector of  $F(p)$  that contains  $p_i$ . See Figure 5 (top right). Thus, it suffices to bound the expected number of points of  $P$  in  $R$ . Note that, as  $p$  rotates around  $K$ ,  $R$  rotates around its center together with  $p$ .

Partition the plane into annuli centered at  $p_i$  by tracing circles around  $p_i$  of radii  $1, 2, 4, 8, \dots$ . The annulus with inner radius  $r$  and outer radius  $2r$  contains at most  $k_2r^2$  points of  $P$ , for some constant  $k_2$ . Each such point has probability at most  $k_3n^{-1/6}r^{-1}$  of falling in  $R$  (over the choice of  $p$ ), for another constant  $k_3$ ; therefore, the expected contribution of this annulus to  $n(p_i)$  is at most  $k_2k_3rn^{-1/6}$ . Summing up for all annuli with inner radius  $r \leq k'n^{1/6}$ , we obtain that  $E[n(p_i)] \leq k$  for some constant  $k$ , as claimed.

Now, call point  $p_i$  *isolated* if  $n(p_i) \leq 2k$ . By Markov's inequality, each point  $p_i$  has probability at least  $1/2$  of being isolated. Therefore, the expected number of isolated points is at least  $n/2$ . There must exist a  $p$  that achieves this expectation, and for it we obtain at least  $n/(4k)$  occupied sectors, proving the lemma.  $\square$



*Proof of Theorem 4.* Fix a point  $p$  for which  $F(p)$  has  $\Omega(n)$  occupied sectors. Color the sectors with four colors, using colors 1 and 2 alternatingly on the odd-numbered annuli and colors 3 and 4 alternatingly on the even-numbered annuli.

There must be a color for which  $\Omega(n)$  sectors are occupied. Consider only the occupied sectors with this color. Let these sectors be  $S_1, S_2, \dots, S_m$ , listed by increasing distance from  $p$ , and for each fixed distance, in clockwise order around  $p$ . Select one point  $p_i \in P \cap S_i$  from each of these sectors. Let the stacking order  $f$  start with these points, that is,  $f(\mathcal{D}(p_i)) = i$  for  $i = 1, \dots, m$ . The order of the remaining points in  $P$  is arbitrary.

We claim that each selected point  $p_i$  contributes an external angle of  $\tau_i = \Omega(n^{-1/3})$ , which implies that  $\sum \tau_i = \Omega(n^{2/3})$ , as desired.

Indeed, consider the  $i$ -th selected point  $p_i$ . Suppose without loss of generality that  $p$  lies directly below  $p_i$ . Let  $K_k$  and  $K_{k+1}$  be the inner and outer circles bounding the annulus that contains  $p_i$ . Trace rays  $z_1$  and  $z_2$  from  $p_i$  tangent to  $K_{k-1}$ , touching  $K_{k-1}$  at points  $q_1$  and  $q_2$ . See Figure 5 (bottom right).

Every point  $p_j$ ,  $j < i$ , that is *not* contained in the same annulus as  $p_i$  lies below these rays. Moreover, the angle  $\beta$  that these rays make with the horizontal is  $\Theta(n^{-1/3})$ : Consider, for example, the ray  $z_1$ . The triangle  $pp_iq_1$  is right-angled, with angle  $\angle p_i p q_1 = \beta$ . We have  $pq_1 = \Theta(n^{1/2})$  and  $pp_i = pq_1 + \Theta(n^{-1/6})$ . It follows that  $p_i q_1 = \Theta(n^{1/6})$ , and so  $\beta \approx \tan \beta = p_i q_1 / pq_1 = \Theta(n^{-1/3})$ .

Now suppose that  $p_{i-1}$  lies in the same annulus as  $p_i$ . If the constant  $C^*$  in the definition of  $\alpha$  is chosen large enough, then  $p_{i-1}$  must have a smaller  $y$ -coordinate than  $p_i$ . (In the worst case,  $p_i$  lies near the bottom-left corner of its sector and  $p_{i-1}$  lies near the top-right corner of its sector.)

Thus,  $p_i$  contributes external angle  $\tau_i \geq \beta = \Omega(n^{-1/3})$ , as claimed. □

## 4 The “Worst” Dense Set: the Grid

In this section, we assume that  $n$  is a square number and  $G_n$  denotes an  $n^{1/2}$  by  $n^{1/2}$  integer grid. Note that  $G_n$  is a  $\sqrt{2}$ -dense set consisting of  $n$  points.

As we mentioned in the Introduction, in the special case where  $P = \varepsilon G_n$ , Theorem 4 has a simple proof. For  $\mathcal{D}(\varepsilon G_n)$ , one can produce a stacking order with large visible perimeter using the following greedy algorithm (which can also be applied to any other point set  $P$ ): Set  $P_n = G_n$ , and select a vertex of  $\text{conv}(P_n)$  whose external angle is maximum. Let this vertex be  $p_n$ , the last element in the desired order  $f_{\text{greedy}}$ . Repeat the same step for the set  $P_{n-1} = P_n \setminus \{p_n\}$ , and continue in this fashion until the first element  $p_1$  gets defined.

By Jarnik’s theorem [Ja25], every convex polygon has  $O(n^{1/3})$  vertices in  $G_n$ . Therefore, at each step, the greedy algorithm selects a point  $p_i$  that makes an external angle  $\tau_i = \Omega(n^{-1/3})$ . Hence,  $\sum \tau_i = \Omega(n^{2/3})$  for the order  $f_{\text{greedy}}$ . Lemma 7 completes the proof.

Now we turn to Theorem 5.

**Theorem 5.** *Let  $n$  be a perfect square and let  $G_n$  denote an  $n^{1/2}$  by  $n^{1/2}$  uniform grid in the plane. For any stacking order  $f$ , we have*

$$\lim_{\varepsilon \rightarrow 0} \text{visible}(\mathcal{D}(\varepsilon G_n), f) = O(n^{2/3}).$$

Our proof is an improved version of the proof of Theorem 2. There we were concerned with how the *perimeter* of the convex hull grows as we add the points of our set one by one

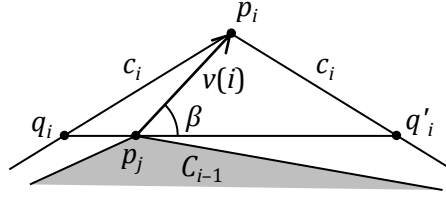


Figure 6: The triangle  $p_i q_i q'_i$  is the largest isosceles triangle at point  $p_i$  that does not intersect the interior of  $C_{i-1}$ .

as prescribed by the stacking order. As is well known, the perimeter of a convex set in the plane is the integral of its width in all directions (this is known as Cauchy's theorem; see *e.g.* [PaA95], Theorem 16.15). The proof of Theorem 5 is very similar, but we deal with the widths in different directions in a non-uniform way. The width in a direction close to the direction of a short grid vector is more important in the analysis than widths in other directions.

*Proof of Theorem 5.* Let  $G_n = \{p_1, \dots, p_n\}$  be an enumeration of the points of  $G_n$  according to a given stacking order, and let  $\tau_i$  denote the corresponding external angles, as defined in Lemma 7. According to the lemma, we need to prove that  $\sum_{i=1}^n \tau_i = O(n^{2/3})$ . Let us partition this sum into several parts, and bound the contribution of each part separately.

Let  $[n] = \{1, \dots, n\}$ . We start with the small angles. Let

$$I_0 = \{i \in [n] \mid \tau_i < n^{-1/3}\}.$$

Clearly, we have

$$\sum_{i \in I_0} \tau_i < n \cdot n^{-1/3} = n^{2/3}.$$

As in the proof of Theorem 2, let  $C_i = \text{conv}\{p_1, \dots, p_i\}$  and denote the perimeter of  $C_i$  by  $\text{per}(i)$ . Since  $G_n$  is an  $n^{1/2} \times n^{1/2}$  integer grid, we have  $\text{per}(n) = 4(n^{1/2} - 1)$ . Consider only those indices  $i > 1$  that do not belong to  $I_0$ . For these indices, we have  $\tau_i > 0$ , so that  $p_i$  must be a vertex of  $C_i$ . For each such point  $p_i$ , let  $c_i$  denote the smallest number satisfying the following condition: the segment connecting the points  $q_i$  and  $q'_i$  that lie on the boundary of  $C_i$  at distance  $c_i$  from  $p_i$ , intersects  $C_{i-1}$ . (In the case where  $C_i$  is a segment, we have  $q_i = q'_i \in C_{i-1}$ .) Note that the segment  $q_i q'_i$  contains a point  $p_j$  with  $1 \leq j < i$ . See Figure 6.

In the proof of Theorem 2, we argued that  $\text{per}(i) - \text{per}(i-1) \geq \tau_i^2/5$ . Now the same argument gives that  $\text{per}(i) - \text{per}(i-1) > c_i \tau_i^2/5$ . Let

$$I_1 = \{i \in [n] \setminus (I_0 \cup \{1\}) \mid c_i \tau_i > n^{-1/6}\}.$$

For  $i \in I_1$ , we have  $\text{per}(i) - \text{per}(i-1) \geq \tau_i n^{-1/6}/5$ . Since  $\text{per}(i)$  is monotone in  $i$ , we conclude that

$$\sum_{i \in I_1} \tau_i \leq 5n^{1/6}(\text{per}(n) - \text{per}(1)) < 20n^{2/3}.$$

Let

$$I_2 = [n] \setminus (\{1\} \cup I_0 \cup I_1).$$

To bound the angles  $\tau_i$  for indices  $i \in I_2$ , we need a charging scheme and we need to consider the growth of the width of  $C_i$  in some specific directions. The *width* of a planar set in a given direction is the diameter of the orthogonal projection of the set to a line in this direction. Let us associate the directions in the plane with the points of the unit circle  $K$ . We identify opposite points of this circle as the widths of the same set in opposite directions are the same. This makes the total length of  $K$  become  $\pi$ . We define a set of arcs along  $K$  as follows. For any non-zero grid vector  $v$  from the integer grid and for any integer  $\ell \geq 0$ , let  $V_{v,\ell}$  denote the arc of length  $2^{-\ell}$  symmetric around the direction of  $v$ . For any direction  $\alpha \in K$ , let  $\rho_i(\alpha)$  denote the width of  $C_i$  in the direction *orthogonal* to  $\alpha$  (*i.e.*, where the corresponding projection is parallel to  $\alpha$ ).

The perimeter  $\text{per}(i)$  is equal to the integral of  $\rho_i(\alpha)$  along the circle  $K$  (note that after the identification of opposite points the length of  $K$  became  $\pi$ ). We have  $\rho_i(\alpha) = \rho_{i-1}(\alpha)$ , unless the direction  $\alpha$  is tangent to  $C_i$  at the vertex  $p_i$ . Let  $U_i$  denote the arc of directions where such a tangency occurs. Clearly, the length of  $U_i$  is  $\tau_i$ , and for any arc  $V$  that contains  $U_i$ , we have

$$\int_V (\rho_i(\alpha) - \rho_{i-1}(\alpha)) d\alpha = \text{per}(i) - \text{per}(i-1) \geq c_i \tau_i^2 / 5.$$

For each index  $i \in I_2$ , choose a grid point  $p_j$  on the segment  $q_i q'_i$ . (Recall that the points  $q_i$  and  $q'_i$  are at distance  $c_i$  from  $p_i$ , and that there is always a grid point between them.) We *charge* the index  $i$  to the pair  $(v(i), \ell(i))$ , where  $v(i)$  is the grid vector pointing from  $p_j$  to  $p_i$  and  $\ell(i)$  is the largest integer such that  $V_{v(i), \ell(i)}$  contains  $U_i$ . Notice that  $|v(i)| \leq c_i$ . Denote by  $I_2(v, \ell)$  the set of indices  $i \in I_2$  that are charged to the pair  $(v, \ell)$ .

Note that  $U_i$  is symmetric around the direction of the segment  $q_i q'_i$ . For the angle  $\beta$  between this direction and the direction of  $v(i)$  we have  $|v(i)| \sin \beta = c_i \sin(\tau_i/2)$  (refer again to Figure 6). This implies  $\beta < c_i \tau_i / |v(i)|$ , and hence  $2^{-\ell(i)} < 4\beta + 2\tau_i < 6c_i \tau_i / |v(i)|$ . Finally, we also have

$$\int_{V_{v(i), \ell(i)}} (\rho_i(\alpha) - \rho_{i-1}(\alpha)) d\alpha \geq c_i \tau_i^2 / 5 > 2^{-\ell(i)} |v(i)| \tau_i / 30.$$

Let  $s(v, \ell) = \sum_{i \in I_2(v, \ell)} \tau_i$ . The integral  $\int_{V_{v, \ell}} \rho_i(\alpha) d\alpha$  is monotone in  $i$  and grows by at least  $2^{-\ell} |v| \tau_i / 30$  at every  $i \in I_2(v, \ell)$ . We have  $\rho_1(\alpha) = 0$  and  $\rho_n(\alpha) < (2n)^{1/2}$ , so that the final integral satisfies  $\int_{V_{v, \ell}} \rho_n(\alpha) d\alpha \leq 2^{-\ell} (2n)^{1/2}$ . Therefore,  $\sum_{i \in I_2(v, \ell)} (2^{-\ell} |v| \tau_i / 30) \leq 2^{-\ell} (2n)^{1/2}$ , which implies that  $s(v, \ell) \leq 30\sqrt{2}n^{1/2} / |v|$ .

Consider the set of all pairs  $(v, \ell)$  such that there is an index  $i \in I_2$  charged to them. We have  $c_i \tau_i \leq n^{-1/6}$ ,  $\tau_i \geq n^{-1/3}$  and  $|v| \leq c_i$ , which implies that  $|v| \leq n^{1/6}$ . We proved that  $2^{-\ell} < 6c_i \tau_i / |v| \leq 6n^{-1/6} / |v|$ . On the other hand, we also have  $2^{-\ell} \geq 2\tau_i \geq 2n^{-1/3}$ . Thus, for any given grid vector  $v$ , there are at most  $\log(6n^{1/6} / |v|)$  possible values of  $\ell$ , where  $\log$  denotes the binary logarithm.

Hence,

$$\sum_{i \in I_2} \tau_i = \sum_{v, \ell} s(v, \ell) \leq \sum_{|v| \leq n^{1/6}} \frac{30\sqrt{2}n^{1/2}}{|v|} \log \frac{6n^{1/6}}{|v|}.$$

To evaluate this sum, we note that the number of grid vectors  $v$  satisfying  $2^k \leq |v| < 2^{k+1}$  is

$\Theta(2^{2k})$ . Thus,

$$\sum_{i \in I_2} \tau_i = O\left(n^{1/2} \sum_{k=0}^{\log n^{1/6}} (\log n^{1/6} - k) 2^k\right) = O(n^{1/2} n^{1/6}) = O(n^{2/3}).$$

In conclusion, we have

$$\sum_{i=1}^n \tau_i = \tau_1 + \sum_{i \in I_0} \tau_i + \sum_{i \in I_1} \tau_i + \sum_{i \in I_2} \tau_i = 2\pi + O(n^{2/3}) + O(n^{2/3}) + O(n^{2/3}) = O(n^{2/3}),$$

completing the proof of the theorem.  $\square$

## 5 Collections of disks with bounded overlap

In this section, we prove Theorem 6.

**Theorem 6.** *Let  $\mathcal{D}$  be a collection of  $n$  unit disks in which at most  $c$  disks have a point in common. Then there exists a stacking order  $f$  for which*

$$\text{visible}(\mathcal{D}, f) = \Omega(v(c)n/c),$$

where  $v(c)$  is given in (1). This bound is worst-case asymptotically tight.

Note that Lemma 7 is not relevant in this case, since we cannot contract the set of centers of  $\mathcal{D}$ .

*Proof.* Partition the plane into an infinite grid of axis-parallel square cells of side-length 4, where the position of the grid is chosen uniformly at random. For each unit disk, the probability that it belongs entirely to a single cell is  $1/4$ . Thus, we can fix the grid in such a way that at least  $n/4$  disks lie entirely in a cell. Let  $k_i$  be the number of disks entirely contained in cell  $i$ . By area considerations, we have  $k_i \leq (16/\pi)c$ .

For each cell  $i$ , we independently select a stacking order that achieves visible perimeter at least  $v(k_i)$ ; then we place all the remaining disks behind them. Thus, our stacking order achieves visible perimeter at least  $\sum_i v(k_i)$ .

For any  $n$ -element point set  $\mathcal{D}$ , we can take an  $rn$ -element point set  $\mathcal{D}'$  as the union of  $r$  pairwise disjoint translates of  $\mathcal{D}$ . We clearly have  $\max_f \text{visible}(\mathcal{D}', f) \leq r \max_f \text{visible}(\mathcal{D}, f)$ . This implies that  $v(rn) \leq rv(n)$ . Let  $r_i = \lceil c/k_i \rceil \leq (16/\pi)c/k_i$ , and we have  $v(c) \leq v(r_i k_i) \leq r_i v(k_i)$ , thus

$$v(k_i) \geq v(c)/r_i \geq \frac{\pi v(c)}{16c} k_i.$$

Since  $\sum k_i \geq n/4$ , the claimed bound follows.

To show that this bound is worst-case asymptotically tight, take the union of  $\lceil n/c \rceil$  worst case sets of  $c$  disks far from each other.  $\square$

## 6 Concluding remarks

**A.** The greedy algorithm described at the beginning of Section 4 was first considered by Cabello *et al.* (unpublished) in the context of maximizing the *minimum* visible perimeter of a single disk. They showed that the order  $f_{\text{greedy}}$  is always optimal for this purpose. Unfortunately, this stacking order is *not* always optimal with respect to the total visible perimeter. Indeed, let  $n$  be a perfect square and consider the set of points  $\{p_i \mid 1 \leq i \leq n\}$ , where the polar coordinates of  $p_i$  are  $(r_i, \theta_i) = (e^{bi}, 2\pi i/n^{1/2})$  with  $b > 0$  sufficiently small. This point set is obtained as the intersection of  $n^{1/2}$  equally-spaced rays emanating from the origin, and  $n^{1/2}$  “rounds” of a very tight logarithmic spiral centered at the origin. The greedy algorithm produces the stacking order indicated by the indices, so it takes the points of  $P$  outwards along the spiral. The contribution  $\tau_i$  is equal for every point  $p_i$  with  $i \geq n^{1/2}$  and tends to  $2\pi/n^{1/2}$  as  $b$  goes to zero, making  $\sum \tau_i = \Theta(n^{1/2})$  if  $b$  is small enough. However, taking the points ray by ray in a cyclic order, going outwards along each ray, the contribution  $\tau_i$  is a constant for the first half of the points, making  $\sum \tau_i = \Theta(n)$ .

**B.** Theorem 4 can be generalized to point sets satisfying weaker density conditions. Indeed, let  $P$  be a set of  $n$  points in the plane with diameter  $D$  and minimum distance  $d$ . A randomized construction, similar to the one used in the proof Theorem 4, guarantees the existence of a stacking order  $f$  such that  $\text{visible}(D(P), f) = \Omega(n/(D/d)^{2/3})$ . This beats the  $\Omega(n^{1/2})$  bound mentioned in the Introduction as long as  $D/d = o(n^{3/4})$ .

### Acknowledgements

The authors express their gratitude to Radoslav Fulek and Andres Ruiz Vargas (EPFL), for many insightful discussions on the subject, as well as to the anonymous referees for their useful comments.

### References

- [AIKP89] N. Alon, M. Katchalski, and W. R. Pulleyblank: The maximum size of a convex polygon in a restricted set of points in the plane, *Discrete Comput. Geom.* **4** (1989), 245–251.
- [CaH10] S. Cabello, H. Haverkort, M. van Kreveld, and B. Speckmann: Algorithmic aspects of proportional symbol maps, *Algorithmica* **58** (2010), 543–565.
- [De99] B. Dent: *Cartography. Thematic Map Design, 5th edn*, McGraw-Hill, New York, 1999.
- [ErSz35] P. Erdős and G. Szekeres: A combinatorial problem in geometry, *Compositio Math.* **2** (1935), 463–470.
- [Er03] J. Erickson: Nice point sets can have nasty Delaunay triangulations, *Discrete Comput. Geom.* **30** (2003), 109–132.
- [Gr90] T. Griffin: The importance of visual contrast for graduated circles, *Cartography* **19** (1990), 21–30.
- [GrC78] R. E. Groop and D. Cole: Overlapping graduated circles: Magnitude estimation and method of portrayal, *Can. Cartogr.* **15** (1978), 114–122.

- [Ja25] V. Jarník: Über die Gitterpunkte auf konvexen Kurven, *Mathematische Zeitschrift* **24** (1926), 500–518.
- [PaA95] J. Pach and P. K. Agarwal: *Combinatorial Geometry*, Wiley, New York, 1995.
- [SLM03] T. A. Slocum, R. B. McMaster, F. C. Kessler, and H. H. Howard: *Thematic Cartography and Geographic Visualization, 2nd edn*, Prentice Hall, New York, 2003.
- [Va92] P. Valtr: Convex independent sets and 7-holes in restricted planar point sets, *Discrete Comput. Geom.* **7** (1992), 135–152.
- [Va96] P. Valtr: Lines, line-point incidences and crossing families in dense sets, *Combinatorica* **16** (1996), 269–294.