On the Maximum Number of Edges in Quasi-Planar Graphs

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Abstract

A topological graph is *quasi-planar*, if it does not contain three pairwise crossing edges. Agarwal et al. [2] proved that these graphs have a linear number of edges. We give a simple proof for this fact that yields the better upper bound of $8n$ edges for n vertices. Our best construction with $7n-O(1)$ edges comes very close to this bound. Moreover, we show matching upper and lower bounds for several relaxations and restrictions of this problem. In particular, we show that the maximum number of edges of a *simple* quasi-planar topological graph (i.e., every pair of edges have at most one point in common) is $6.5n-O(1)$, thereby exhibiting that non-simple quasi-planar graphs may have many more edges than simple ones.

1 Introduction

A *topological graph* G is a graph drawn in the plane, that is, its vertex set, $V(G)$, is a set of distinct points, and its edge set, $E(G)$, is a set of Jordan arcs, each connecting two vertices and containing no other vertex. Crossings are allowed, but we assume that each pair of edges has a finite number of common internal points and they properly cross at each (these usual restrictions are not important for the present paper). A *geometric graph* is a topological graph in which the arcs are straight line segments. A topological graph has a simple graph for its underlying abstract graph, i.e., it does not have self-loops or parallel edges. In Section 3 we will also consider generalized topological graphs with several edges connecting the same pair of vertices.

A topological graph is *k-quasi-planar* if no k of its edges are pairwise crossing. We refer to 3-quasi-planar graphs simply as quasi-planar graphs.

It is conjectured [6, Problem 3.3] that for every fixed k, the maximum number of edges in a k-quasi-planar graph on n vertices is $O(n)$. For $k=2$ this conjecture is trivial, as 2-quasi-planar graphs are in fact planar graphs, so for $n>2$ vertices they have at most $3n-6$ edges by Euler's Polyhedral Formula. Agarwal et al. [2] were the first to prove that the number of edges in a (3-)quasi-planar graph on n vertices is $O(n)$. Later, Pach et al. [9] simplified their proof and obtained an upper bound of $65n$ on the number of edges. Their proof relies on an earlier result of Capoyleas and Pach [5], who showed that for every fixed k the number of edges in a k-quasi-planar geometric graph on n points in a *convex* position is at most $2(k-1)n-\binom{2k-1}{2}$. Recently [1], the conjecture was also settled for $k=4$, however, for $k>4$ the best upper bound is $O(n\log^{4k-16} n)$ [1]. In [14] Valtr obtained an upper bound of $O(n\log n)$ for any fixed k for topological graphs with x-monotone edges. Many other Turán-type problems on topological graphs avoiding certain geometric patterns were considered in the literature (see, e.g., [8, 10, 11, 12, 13], and [7] for a survey).

In the following section we prove our main result:

Theorem 1. *The maximum number of edges in a quasi-planar graph on $n \geq 3$ vertices is at most $8n-20$.*

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Agarwal et al. [2] argued that for every integer n there is a quasi-planar graph on n vertices with roughly $6n$ edges. The construction we give establishes the following stronger bound:

Theorem 2. *For every positive integer n, there is a quasi-planar graph on n vertices with $7n - O(1)$ edges.*

At present we cannot close the gap between the bounds in Theorems 1 and 2, but in Section 3 we discuss several natural relaxations and restrictions of this problem where we can give matching lower and upper bounds. The most general problem we consider is about generalized topological graphs, where several edges may connect the same pair of vertices as long as the 2-gon formed by two such edges always contains at least one vertex. We require a weaker property than quasi-planarity, namely that no three pairwise crossing edges determine an *empty* triangle. We prove that the upper bound on the number of edges stated in Theorem 1 is still valid in this more general case. We also give a construction proving that this bound is tight expect for the additive constant.

As a natural restricted case we consider simple topological graphs. A topological graph is *simple* if every pair of its edges meet at most once (either at a vertex or at an intersection). We prove that the maximal number of edges of a simple quasi-planar graph on n vertices is $6.5n - O(1)$. The construction establishing the lower bound uses straight line edges, therefore the bound is tight for geometric graphs too. Contrasting this result with Theorem 2 one sees that restricting attention to simple topological graphs severely reduces how many edges a quasi-planar graph may have. We are not aware of any other example where such a clear distinction is proven between extremal properties of topological graphs and simple topological graphs.

2 Proof of the main results

We use the *discharging method* for proving Theorem 1. In this technique one usually assigns "charges" (or weights) to elements of the input (typically vertices or faces of a planar graph). The total charge is then computed twice: once after assigning the charges, and once more after charges have been redistributed (*discharging* phase). The most famous example of using the discharging method is in the proof of *The Four Color Theorem* [3]. In the current setting we consider the plane graph G' obtained by adding the crossing points of edges of the quasi-planar graph G as new vertices. First, we assign charges to the faces of G', such that the total sum of charges is $O(|V(G)|)$. Then, we redistribute the charges such that the charge of every face is proportional to the number of vertices of G along its boundary. Thus, the overall charge is $\Omega(|E(G)|)$ and, hence, $|E(G)| = O(|V(G)|)$. We now provide the full details of the proof.

Proof of Theorem 1. Consider a quasi-planar topological graph G on n vertices. We assume without loss of generality that among the quasi-planar graphs with the same underlying abstract graph G is chosen to have the minimum overall number of intersections. We can also assume that G is connected as otherwise the statement easily follows by induction. Let $X(G)$ denote the set of points where the edges of G intersect and let G' be the graph obtained by adding the vertices of $X(G)$ to the graph G and subdividing the edges of G accordingly. Note that G' is a crossing-free topological graph, i.e., a plane drawing of a planar graph. We refer to the vertices in $V(G)$ as *original* vertices, while the vertices in $X(G)$ are the *new* vertices. As no three edges of G cross in a single vertex, all new vertices are degree 4 vertices of G'. Denote by $F(G')$ the set of faces of G', and let $|f|$ be the *size* of a face $f \in F(G')$, that is, the number of edges of G' along the boundary of f. (Note that it is possible for an edge of G' to appear twice along the boundary of a face, and in this case it is counted with multiplicity.) Given a face f, we denote by $v(f)$ the number of original vertices that appear along the boundary of f (note that a vertex can appear more than once along the boundary of a face, and again, in this case it is counted with multiplicity). We will use the terms *triangles*, *quadrilaterals*, and *pentagons* to refer to faces of size 3, 4, and 5, respectively. An integer before the name of a face denotes the number of original vertices $v(f)$ along the boundary of the face f. For example, a 2-pentagon is a face of size 5 that has 2 original vertices along its boundary.

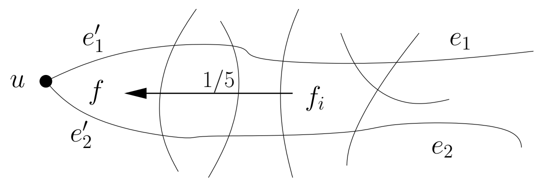


Figure 1: The first round of charges redistribution

We proceed by assigning charges to the faces of G' such that the charge of a face f, $ch(f)$, is $|f| + v(f) - 4$. Summing the total charges over all the faces of G' we have:

$$\sum_{f \in F(G')} ch(f) = \sum_{f \in F(G')} \Big(|f| + v(f) - 4\Big) = 2|E(G')| + \Big(\sum_{f \in F(G')} v(f)\Big) - 4|F(G')| = 4n - 8, \qquad (1)$$

where the last equality follows from Euler's formula and from the next equalities:

$$\sum_{f \in F(G')} v(f) = \sum_{u \in V(G)} d(u) = \sum_{u \in V(G')} d(u) - \sum_{u \in X(G)} d(u) = 2|E(G')| - 4\Big(|V(G')| - |V(G)|\Big),$$

where $d(u)$ denotes the degree of the vertex u.

Since G is drawn with the least possible number of crossings, G' does not contain faces of size one or two (otherwise, such faces could be "opened" without introducing new crossings, thus reducing the overall number of crossings). Faces that are 0-triangles are also impossible, as they consist of three pairwise crossing edges. Thus, every face in G' has a non-negative charge.

Next, we redistribute the charges without affecting the total charge found in (1). We make sure that the new charge $ch'(f)$ of a face f satisfies $ch'(f) \geq v(f)/5$.

Clearly, the only faces that do not satisfy this bound with the original charge ch are 1-triangles. Let f be a 1-triangle, and let u be the original vertex of f. Let e_1' and e_2' be the two sides of f incident to u. Let e_1 and e_2 be the edges of G of which e_1' and e_2' are segments. Let us examine the faces of G' touching e_1 from the same side as f. Along e_1 we find f followed by one or more additional faces $f_1, f_2, \ldots$ Let f_i be the first of these faces that is not a 0-quadrilateral (see Figure 1). Clearly, such a face exists as the last f_j has at least one original vertex on its boundary. Moreover, one can easily see that we obtain the same faces $f_1, \ldots, f_i$ if we switch the roles of e_1 and e_2; therefore, f_i is well defined despite the ambiguity about e_1 and e_2. Note that f_i is not a triangle, as we do not have 0-triangles and e_1 and e_2 are not connecting the same pair of vertices. Let us shift $1/5$ unit of charge from f_i to f. We say that f_i loses this charge *through* the edge it is sharing with f_{i-1} (or f if $i = 1$). Let ch' be the charge obtained from ch after doing this shift for all 1-triangles f.

Notice that $ch'(f) = 1/5 = v(f)/5$ for every 1-triangle f and $ch'(f) = 0 = v(f)/5$ for every 0-quadrilateral f. Let f be a face of G' that is neither a 0-quadrilateral, nor a 1-triangle. We have $ch(f) = |f| + v(f) - 4 \geq 1$ and $ch'(f) = ch(f) - x_f$, where x_f is the total charge f lost in the redistribution. As f lost at most $1/5$ unit of charge through any one of its edges, and only if both endpoints of the edge are new we have $x_f \leq (|f| - v(f))/5$. Thus, we have $ch'(f) \geq (2/5)v(f) + (4/5)(ch(f) - 1) \geq 2v(f)/5$ in this case.

We, therefore, have $ch'(f) \geq v(f)/5$ for all faces f of G'. As a second round of redistribution we collect all the extra charge at faces incident to an original vertex, and place this charge on the *vertex*. Henceforth, both faces of G' and vertices of G will have some charge. For every face f with $v(f) > 0$ we compute the extra charge $ch'(f) - v(f)/5$ and distribute it evenly among the $v(f)$ original vertices along the boundary of f. Each such original vertex receives $(ch'(f) - v(f)/5)/v(f)$ units of charge from f (as always, if the same vertex appears several times along the boundary it receives this charge several times). We call $ch''(f)$ the remaining charge at a face f after this step, and $ch''(u)$ the total charge accumulated at an original vertex u. By the construction of ch'' we have

$$ch''(f) \geq v(f)/5, \tag{2}$$

for every face f, with equality unless $v(f) = 0$. From equation (1) we have

$$\sum_{f \in F(G')} ch''(f) + \sum_{u \in V(G)} ch''(u) = 4n - 8. \tag{3}$$

It remains to prove a lower bound on $ch''(u)$ for original vertices u. We claim

$$ch''(u) \geq 4/5 \tag{4}$$

For the proof of this claim let us consider the graph G_u obtained from G by removing the vertex u and all its incident edges. Let G'_u be the corresponding plane drawing, i.e., we introduce all crossing points as new vertices and subdivide the edges accordingly. Let f_u be the face of G'_u containing u. The size of f_u is at least 2. Here we assume G_u is not empty (if G is a star then the statement of Theorem 1 clearly holds) and no edge is self-crossing (otherwise G does not have the minimal number of overall crossings). Let w be a vertex of f_u. There is at least one face f of G' that touches both u and w. If $|f| \geq 5$, then f alone contributes at least $\frac{4}{5}$ unit of charge to u, so the claim holds. The same is true if f is a 4-quadrilateral, a 3-quadrilateral, or a 2-quadrilateral with the two original vertices not being neighbors. As f cannot be a 1-triangle, the remaining cases are a 2-triangle contributing $3/10$, a 3-triangle contributing $7/15$, a 1-quadrilateral contributing at least $2/5$, and a 2-quadrilateral (with neighboring original vertices) contributing at least $7/10$. One can finish the proof by a case analysis noting that (a) except for the 1-quadrilateral case we find an edge of G incident to u that is not involved in any crossing, and therefore, the faces on both sides of this edge contribute charge to u; and (b) there is at least one other vertex w' of f_u besides w. The minimal value of $ch''(u) = 4/5$ is only possible if f_u is subdivided into two 1-quadrilaterals and a few 1-triangles in G'.

Combining (2), (3), and (4) we have:

$$4n - 8 \geq \sum_{f \in F(G')} v(f)/5 + \sum_{u \in V(G)} \frac{4}{5} \geq \frac{2}{5}|E(G)| + \frac{4}{5}n.$$

Therefore, $|E(G)| \leq 8n - 20$. □

Remarks Note that one can stop after the first round of charge redistribution and use Equation (1) together with $ch'(f) \geq v(f)/5$ to obtain $|E(G)| \leq 10n - 20$. An even simpler way to prove a linear bound on the number of edges in a quasi-planar graph is the following: Let G be a quasi-planar graph on n vertices. We claim $|E(G)| \leq 19n$. For the proof one can assume G is connected and $d(u) \geq 20$ for every $u \in V(G)$, for otherwise we conclude by induction. Now set charges as follows: for every $u \in V(G')$ set $ch(u) := d(u) - 4$, and for every $f \in F(G')$ set $ch(f) := |f| - 4$. Then by Euler's formula the total charge is -8. However, we can distribute the charges such that every element has a non-negative charge as follows. First, every original vertex contributes $\frac{4}{5}$ units of charge to each of the faces incident to it. The only elements with a negative charge after this step are 1-triangles. Each 1-triangle now obtains another $\frac{1}{5}$ units of charge in the way described in the proof of Theorem 1 and ends up with a zero charge.

Let m_0 be the largest value such that the complete graph K_{m_0} can be drawn as a quasi-planar graph. Our Theorem 1 implies $m_0 \leq 14$ but the correct value is probably lower. Figure 2 shows a drawing of K_9 as a quasi-planar *geometric* graph, thus we have $m_0 \geq 9$. Aichholzer and Krasser [4] showed that K_{10} cannot be drawn as a quasi-planar geometric graph, by exploring all the different order-types of 10 points in the plane. It might be possible to draw K_{10} as a *simple topological* quasi-planar graph, but for K_{11} Theorem 5 in Section 3 implies there is no such drawing.

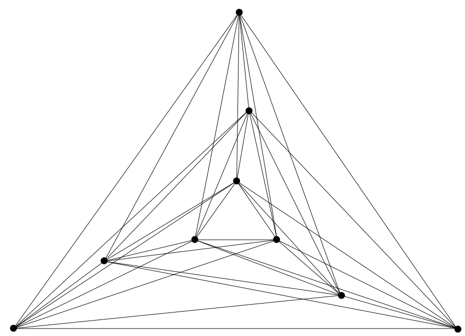
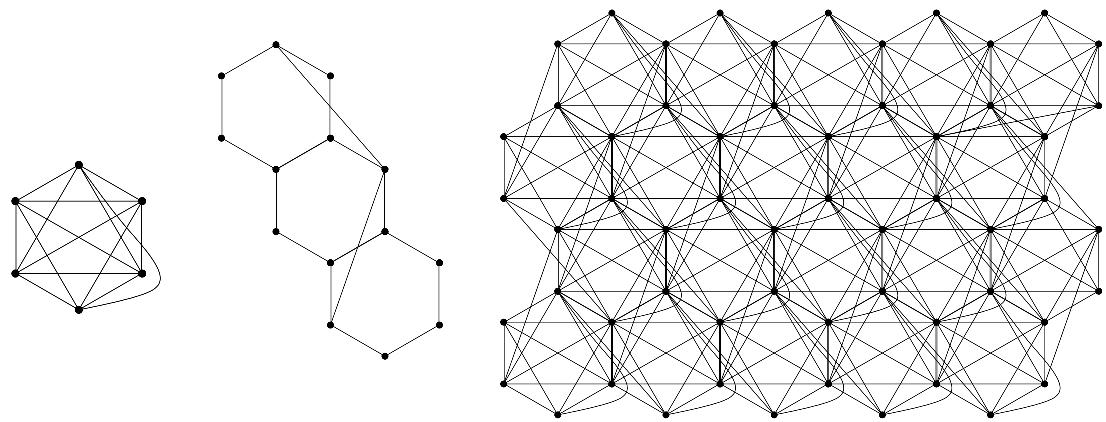


Figure 2: K_9 drawn as a quasi-planar geometric graph.



(a) (b) (c)

Figure 3: A construction for a quasi-planar graph with n vertices and $7n - O(1)$ edges.

We finish this section by providing the example proving Theorem 2. This is an improvement over the result of Agarwal et al. [2], who argued that for sufficiently large n one can always construct a quasi-planar graph with n vertices and roughly $6n$ edges, simply by considering two (almost) edge-disjoint triangulations of the same set of n points.

Proof of Theorem 2. Figure 3 shows a construction that yields n-vertex quasi-planar graphs with $7n - O(1)$ edges: First, consider a hexagonal grid such that for each hexagon we draw all the diagonals but one as straight line edges. Then we add a curved edge for the missing diagonal as in Figure 3(a). Finally we add longer edges, two from every vertex as in Figure 3(b). One can verify by inspection the quasi-planarity of the obtained graph. Note that the degree of vertices that are far enough from the boundary is 14, and that there are $O(\sqrt{n})$ vertices which are close to the boundary and, therefore, have a degree smaller than 14. Thus, this quasi-planar graph already has $7n - O(\sqrt{n})$ edges. To improve on the error term we wrap the graph around a cylinder so that we have three hexagons around the cylinder and draw five more edges on each of the top and bottom faces. Considering m layers of hexagons we have $n = 6m + 6$ vertices and $7n - 29$ edges. For m not divisible by 6 the constant term gets slightly worse. $\square$

3 Generalizations and restrictions

Observe that the proof of Theorem 1 did not use our assumptions defining quasi-planar graphs in their full generality. We assumed that they—as topological graphs—do not contain several edges connecting the same pairs of vertices. This assumption may seem to be crucial as one can have an arbitrary number of parallel edges even in a planar graph. Nevertheless, Euler's formula proves the $3n-6$ bound on the number of edges in any plane drawing as long as there is no face bounded by two edges. In other words, we can allow for parallel edges as long as they are drawn with at least one vertex in any 2-gon they determine. This generalization of the bound on the number edges of a planar graph is often useful. We can generalize Theorem 1 the same way and the proof presented in the previous section will still apply. For the generalization of topological graphs where parallel edges are allowed, we use the term *generalized topological graph*.

We have not used the quasi-planarity assumption in full generality either. All we used is the assumption that the drawing does not yield 0-triangles, i.e., three pairwise crossing edges determining an empty triangle with no vertex inside. Here we consider three edges crossing each other at a single point an "infinitesimal" 0-triangle and so we do not allow it either.

We, therefore, have the following generalization of Theorem 1:

Theorem 3. *Consider a generalized topological graph G on $n \geq 3$ vertices. Assume that*

- *G has no self-loops;*
- *any 2-gon formed by two (possibly intersecting) parallel edges contains a vertex inside; and*
- *any triangle formed by three pairwise intersecting edges has a vertex inside.*

Then G has at most $8n-20$ edges.

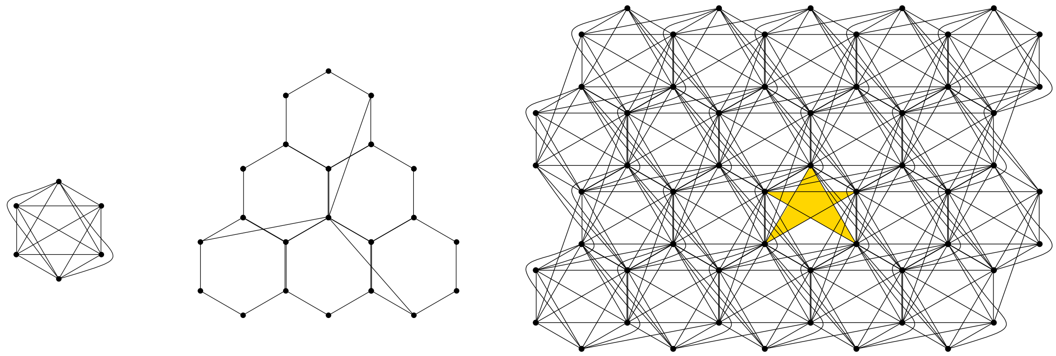
This generalization of Theorem 1 is surprisingly tight.

Theorem 4. *There exist generalized topological graphs satisfying the conditions of Theorem 3 with n vertices and $8n - O(1)$ edges.*

Proof. We start with the construction proving Theorem 2 (before the wrapping up) and add more edges. For each of the curved edges we add a parallel edge that is drawn as the central reflection of the original one (see Figure 4(a)). Each pair of parallel edges yields a 2-gon with two vertices inside. Then we further add the (straight) edges as shown in Figure 4(b). Each of these new edges will create two sets of three pairwise crossing edges with exactly one vertex in either triangle. See Figure 4(c) for the resulting graph. To obtain $8n - O(1)$ edges (rather than $8n - O(\sqrt{n})$) we wrap this graph around a cylinder, as is done in the proof of Theorem 2. $\square$

Remarks It is interesting to see how the charges are distributed in the graph constructed in the proof of Theorem 4. All the faces f end up with a charge of $v(f)/5$, so in particular all 0-faces are 0-quadrilaterals or 0-pentagons. Moreover, the original edges involved in any 0-pentagon form a 5-star with the 0-pentagon in the middle and each arm consisting of a few 0-quadrilaterals and a 1-triangle (see Figure 4(c)). All original vertices u (except for a few close to the boundary) are surrounded by a 2-gon split up into many 1-triangles and two 1-quadrilaterals, so they end up with the minimal charge of $4/5$.

We may allow for only one of (a) parallel edges with at least one vertex in the 2-gon or (b) three pairwise crossing edges if the resulting triangle has a vertex inside. For both resulting generalizations of the quasi-planar graphs we still have that $8n-20$ is an upper bound on the number of edges but the obvious modification of the construction in the proof of Theorem 4 gives only examples with $7.5n-O(1)$ edges.



(a) (b) (c)

Figure 4: A generalized topological graph satisfying the conditions of Theorem 3.

An alternate approach is to consider natural restrictions of the original problem. Recall that a topological graph is *simple* if every pair of its edges meets at most once (either at an endpoint or at a crossing point).

Theorem 5. *The maximum number of edges in a simple quasi-planar graph on $n \geq 4$ vertices is at most $6.5n - 20$, and this bound is tight up to some additive constant.*

Proof. The proof is very similar to the proof of Theorem 1, so we omit most of the details. In this case we can derive a stronger bound since it can be shown that for every vertex $u \in V(G)$ we have $ch''(u) \geq \frac{7}{5}$. The main observation is that the face f_u of G'_u containing u cannot be a 2-gon because G is a simple topological graph, and it can be a triangle, but only if one of the vertices of the triangle is an original vertex (otherwise we would have three pairwise crossing edges). In the extremal configuration giving charge $ch''(u) = \frac{7}{5}$, u is surrounded by two 2-triangles, two 1-quadrilaterals and any number of 1-triangles.

The matching lower bound is obtained from the same construction shown in Figure 3, without adding the curved edges. □

Note that this bound is tight also for geometric graphs. To see this, one has to make the construction consist of straight line edges. It naturally does consist of straight line edges before we wrap it around a cylinder. The wrapping can also be managed with straight line edges if we use five or more hexagons in every layer instead of three.

As mentioned in the Introduction, the pair of Theorems 2 and 5 seems to give the first clear distinction in the extremal function of a forbidden configuration among topological graphs and simple topological graphs.

As a combination of generalization and relaxation we state the following result:

Theorem 6. *Let G be a simple topological graph on $n \geq 4$ vertices and with no three pairwise crossing edges that form an empty triangular face. Then the number of edges in G is at most $7n - 20$, and this bound is tight up to some additive constant.*

Proof. Once again the upper bound is proven following the proof of Theorem 1, and noticing that for every vertex $u \in V(G)$, after re-distributing the charges for the second time, we have $ch''(u) \geq \frac{6}{5}$. Now the minimal contribution to an original vertex comes from three 1-quadrilateral faces.

For the lower bound we use the construction described in Figure 4, this time using no curved edges. □

Finally, we mention that the construction for Theorem 5 has many edges that are not crossed by any other edge. This is no coincidence, since the minimal vertex-charge of 7/5 can only be achieved with such edges. If one only considers simple quasi-planar graphs with *all edges crossed*, then one can prove an upper bound of $6n - O(1)$ on the number of edges and this is tight as witnessed by the construction presented in the proof of Theorem 5 with the non-crossed edges removed.

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