TURÁN PROBLEMS FOR EDGE-ORDERED GRAPHS

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Abstract. In this paper we initiate a systematic study of the Turán problem for edge-ordered graphs. A simple graph is called *edge-ordered* if its edges are linearly ordered. This notion allows us to study graphs (and in particular their maximum number of edges) when a subgraph is forbidden with a specific edge-order but the same underlying graph may appear with a different edge-order.

We prove an Erdős-Stone-Simonovits-type theorem for edge-ordered graphs—we identify the relevant parameter for the Turán number of an edge-ordered graph and call it the *order chromatic number*. We establish several important properties of this parameter.

We also study Turán numbers of edge-ordered paths, star forests and the cycle of length four. We make strong connections to Davenport-Schinzel theory, the theory of forbidden submatrices, and show an application in discrete geometry.

1. Introduction

The most basic *Turán-type extremal problem* asks the maximum number $\mathrm{ex}(n, H)$ of edges in an n vertex simple graph that does not contain a "forbidden" graph H as a subgraph. For a family $\mathcal{H}$ of forbidden graphs we write $\mathrm{ex}(n, \mathcal{H})$ to denote the maximal number of edges of a simple graph on n vertices that contains no member of $\mathcal{H}$ as a subgraph. This problem has its roots in the works of Mantel, [29] and Turán, [43], where they considered the case where the forbidden graph is a complete graph. For a survey see Füredi and Simonovits, [19]. Several extensions of Turán-type extremal problems for graphs have been studied. For a survey on extremal hypergraph problems see Keevash, [23]. The extremal theory of graphs with a circular or linear order on their *vertex set* has a rich history. For example, see Braß, Károlyi, Valtr, [5] or Tardos, [42], respectively. In this paper we initiate a systematic study of Turán-type problems for *edge-ordered graphs* and establish several fundamental results.

An *edge-ordered graph* is a finite simple graph $G = (V, E)$ with a linear order on its edge set E. We often give this linear order (that we also call *edge-ordering* or *edge-order*, in short) with a *labeling* $L : E \to \mathbb{R}$. In this case we denote the edge-ordered graph obtained by G^L, and we also call it a *labeling* of G. Note that we always assume the function $L : E \to \mathbb{R}$ is injective (so that it defines a linear order on the edges) and we use the labeling only to define this edge-order, so G^L and $G^{L'}$ represent the same edge-ordered graph if for any pair of edges $e, f \in E$, $L(e) < L(f)$ holds if and only if $L'(e) < L'(f)$ holds.

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An isomorphism between edge-ordered graphs must respect the edge-order. A subgraph of an edge-ordered graph is itself an edge-ordered graph with the induced edge-order. We say that the edge-ordered graph G *contains* another edge-ordered graph H if H is isomorphic to a subgraph of G. Otherwise we say that G *avoids* H. We say that G *avoids* a family of edge-ordered graphs if it avoids every member of the family. When speaking of a family of edge-ordered graphs we always assume that all members of the family are *non-empty*, that is they have at least one edge. This is necessary for the definition of the Turán number below to make sense. Note that similar extremal problems for vertex-ordered graphs (where the linear order is on the vertices instead of edges) has been studied before, see for example [26, 31, 32, 42].

The Turán problem for edge-ordered graphs can be formulated as follows.

Definition 1.1. For a positive integer n and a family of edge-ordered graphs $\mathcal{H}$, let the *Turán number of* $\mathcal{H}$ be the maximal number of edges in an edge-ordered graph on n vertices that avoids $\mathcal{H}$, and let this maximum be denoted by $\mathrm{ex}_<(n, \mathcal{H})$. If there is only one forbidden edge-ordered graph H, we simply write $\mathrm{ex}_<(n, H)$ instead of $\mathrm{ex}_<(n, \{H\})$.

Any Turán-type problem from classical extremal graph theory can also be formulated in this language. Indeed, let $\mathcal{H}$ be a family of forbidden simple graphs and define $\mathcal{H}' = \{H^L : H \in \mathcal{H}, L \text{ is a labeling of } H\}$. We clearly have

$$\mathrm{ex}(n, \mathcal{H}) = \mathrm{ex}_<(n, \mathcal{H}').$$

As a consequence, we have the following simple but useful bound for any simple graph H and any labeling L:

$$\mathrm{ex}_<(n, H^L) \geq \mathrm{ex}(n, H).$$

Notation. We will denote the edge-order of short paths and cycles by simply giving labels to the edges along the path or cycle. For example, the edge-ordering of a path P_4 on four vertices, say $abcd$, that gives the edge ab the label 1, the edge bc the label 3, and the edge cd the label 2 is denoted by P_4^{132}. (In other words, this labeling denotes the edge-ordering $ab < cd < bc$.) Similarly, C_4^{1234} denotes the cyclically increasing labeling of the cycle C_4.

1.1. **History.** Only a few special instances of the Turán problem for edge-ordered graphs have been investigated so far. In most of these cases the aim was to find an increasing path or trail, defined as follows: We call a sequence $v_1, \ldots, v_{k+1}$ of vertices in an edge-ordered graph an *increasing trail* of length k if $v_i v_{i+1}$ form a strictly increasing sequence of edges for $1 \leq i \leq k$. If all the vertices v_i are distinct we call it an *increasing path* of length k.

Chvátal and Komlós [9] asked for the length of the longest increasing trail that one can guarantee in any edge-ordering of the complete n-vertex graph K_n. This question was solved by Graham and Kleitman in [21]. In the same paper [9], Chvátal and Komlós also asked the corresponding question for a *path* (rather than a trail). More precisely, they asked: What is the maximum integer k such that every edge-ordering of K_n has an increasing path of length k? It is very natural to ask this question for arbitrary host graphs (rather than just for complete graphs): The *altitude* of a simple graph G is defined as the maximum k such that every edge-ordering of

G has an increasing path of length k. This seemingly simple question turned out to be quite challenging. Let P_{k+1}^{inc} denote the increasing path of length k. The maximal number of edges in a graph on n vertices with a given altitude k can have is precisely $\text{ex}_<(n, P_{k+2}^{\text{inc}})$.

Rödl [40, 44] proved that any graph G with average degree $d \geq k(k+1)$ has altitude at least k. In other words, $\text{ex}_<(n, P_k^{\text{inc}}) < \binom{k}{2}n$. (On the other hand, $\text{ex}_<(n, P_k^{\text{inc}}) \geq \text{ex}(n, P_k) = \frac{k-2}{2}n - O(k^2)$.) For sufficiently dense graphs, Milans [33] proved that any graph G with average degree d has altitude at least $\Omega(d/(n^{1/3}(\log n)^{2/3}))$, where n is the number of vertices in G. Very recently, Bucić, Kwan, Pokrovskiy, Sudakov, Tran, Wagner [7] significantly improved this bound, showing that the altitude is almost as large as d, provided d is not too small. This result is close to being optimal because the longest path in a graph G with average degree d may be as short as d (for example, if G is a disjoint union of cliques of size $d+1$). Inspired by the question of Chvátal and Komlós, several authors studied the altitude of various special classes of graphs including the hypercube [10], the random graph [10, 28]. Closely related problems were also studied with respect to geometric graphs [11].

Concerning the case when the forbidden edge-ordered graph is not a path (or a trail), a preliminary result was shown by Gerbner, Patkós and Vizer, [20], who proved that $\text{ex}_<(n, C_4^{1243}) = O(n^{5/3})$ and applied it to a problem in extremal set theory. Another interesting result is an unpublished result of Leeb (see the paper of Nešetřil and Rödl [35]), stating that for any given $n \in \mathbb{N}$, every large enough edge-ordered complete graph contains a copy of K_n such that the edges of this copy induce one of four special edge-orderings (see Section 2.1 for more details). Ramsey numbers of edge-ordered graphs have been studied recently (motivated by this paper), see [3, 16].

1.2. **Outline of the paper and main results.** In Section 2 we present the analogue of Erdős-Stone-Simonovits theorem that applies to edge-ordered graphs. This theorem ties the Turán number of an edge-ordered graph to its *order chromatic number*, a notion that we will introduce. Order chromatic number is in turn strongly connected to some special edge-orders called *canonical edge-orders*. This connection is discussed in Section 2.1, where we also prove several important properties of order chromatic number (see, for example, Theorem 2.5 and Corollary 2.6). In particular, it turns out that the order chromatic number behaves rather differently compared to the usual chromatic number in several aspects. For example the order chromatic number of a family of edge-ordered graphs can be substantially smaller than that of any single member of the family, and the order chromatic number of a finite edge-ordered graph can be infinite. In Section 2.2 we consider edge-ordered graphs with finite order chromatic number and estimate how large the order chromatic number can be in this case. Among other things, we will show that even when the order chromatic number of an edge-ordered graph G is finite, it can grow exponentially in the number of vertices of G (see Theorem 2.14). Finally, in Section 2.3, we briefly study the smallest and the largest possible order chromatic number of a graph G over all possible edge-orderings of G. For most graphs, this latter number is infinite as shown by Theorem 2.18.

In Section 3, we study Turán numbers of edge-ordered star forests. Recall that a *star* is a simple, connected graph in which all edges share a common vertex and a *star forest* is a non-empty graph whose connected components are all stars. We show a strong connection between this problem and Davenport-Schinzel theory and prove that the Turán number is close to being linear for any given edge-ordered star forest (see Corollary 3.4).

In Section 4 we study Turán numbers of edge-ordered paths. For edge-ordered paths with three edges, we determine the Turán number exactly or up to an additive constant in Section 4.1. And for most edge-ordered paths with four edges, in Section 4.2 we show that the Turán number is either $\Theta(n)$, $\Theta(n \log n)$ or $\Theta(n^2)$ (see the table at the beginning of Section 4.2 for a complete list of results). This section also makes connections to the theory of forbidden submatrices.

In Section 5 we study Turán numbers of edge-ordered 4-cycles. The 4-cycle C_4 has three non-isomorphic edge-orderings. The most interesting one is C_4^{1243}. For this one, using a special weighting argument, we show that the answer is close to $\Theta(n^{3/2})$ (as in the case of the usual Turán problem). It is easy to show that the Turán number of the other two edge-orderings of C_4 is $\binom{n}{2}$.

Lastly, in Section 6 we make some concluding remarks. Turán theory for edge-ordered graphs is very likely to have applications in other areas. As an example, using one of our results, we show that the maximum number of unit distances among n points in convex position in the plane is $O(n \log n)$, reproving a result of Edelsbrunner-Hajnal [12], and Füredi [17] (see Section 6.1). We finish the paper with some open problems in Section 6.3.

Throughout the paper, we use log to denote the binary logarithm.

2. Erdős-Stone-Simonovits theorem for edge-ordered graphs and order chromatic number

The most general result in Turán-type extremal graph theory is the Erdős-Stone-Simonovits theorem, stated below. Note that when using asymptotic notation to estimate the Turán numbers of families of graphs or edge-ordered graphs, we always consider the family to be fixed. In particular, the $o(1)$ term in the following theorem tends to zero as n goes to infinity, for a fixed family $\mathcal{H}$.

Theorem 2.1 (Erdős-Stone-Simonovits theorem, [13, 14])**.** *Any family $\mathcal{H}$ of simple graphs with $r+1 := \min\{\chi(H) : H \in \mathcal{H}\} \geq 2$ satisfies*

$$\mathrm{ex}(n, \mathcal{H}) = \left(1 - \frac{1}{r} + o(1)\right) \frac{n^2}{2}.$$

The lower bound is given by the Turán graph $T(n, r)$, which is the complete r-partite graph with each part having size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Here $\chi(H)$ stands for the *chromatic number* of the graph H. The key to extending this result to edge-ordered graphs is to find the notion that can play the role of chromatic number in the original theorem. We do this as follows.

Definition 2.2. We say that a simple graph G *can avoid* a family $\mathcal{H}$ of edge-ordered graphs if there is labeling G^L of G that avoids $\mathcal{H}$. (In other words, a graph G *cannot avoid* $\mathcal{H}$ if every labeling G^L of G contains a member of $\mathcal{H}$.)

Let $\chi_<(\mathcal{H})$, the *order chromatic number* of $\mathcal{H}$ stand for the smallest chromatic number $\chi(G)$ of a finite graph G that cannot avoid $\mathcal{H}$. In case all finite simple graphs can avoid $\mathcal{H}$ we define $\chi_<(\mathcal{H}) = \infty$. In case the family $\mathcal{H}$ contains a single edge-ordered graph we write $\chi_<(H)$ to denote $\chi_<(\{H\})$.

Remark. Recall that when speaking of a family of edge-ordered graphs we assume no member of the family is empty. This makes the order chromatic number at least 2.

We consider only finite graphs and edge-ordered graphs in this paper, so all members of $\mathcal{H}$ are finite and so is G. But here we remark that the definition of the order chromatic number would not be altered if we allowed for infinite graphs G – this can be shown using compactness.

Theorem 2.3 (Erdős-Stone-Simonovits theorem for edge-ordered graphs)**.** *If* $\chi_<(\mathcal{H}) = \infty$*, then*

$$\mathrm{ex}_<(n, \mathcal{H}) = \binom{n}{2}.$$

If $\chi_<(\mathcal{H}) = r + 1 < \infty$*, then*

$$\mathrm{ex}_<(n, \mathcal{H}) = \left(1 - \frac{1}{r} + o(1)\right)\frac{n^2}{2}.$$

Proof. Clearly, if the simple graph $G = (V, E)$ can avoid $\mathcal{H}$, then $\mathrm{ex}_<(|V|, \mathcal{H}) \geq |E|$. If $\chi_<(\mathcal{H}) = \infty$ (or just larger than n), then the complete graph K_n can avoid $\mathcal{H}$, and this proves the first statement.

The lower bound for the second statement can be proved similarly as the Turán graph $T(n, r)$ with n vertices and r classes has $(1 - \frac{1}{r})\frac{n^2}{2} - O(r^2)$ edges and it can avoid $\mathcal{H}$.

For the upper bound in the second statement, let F be a simple graph with minimum chromatic number $\chi(F) = r + 1$ that cannot avoid $\mathcal{H}$. Clearly, we have $\mathrm{ex}_<(n, \mathcal{H}) \leq \mathrm{ex}(n, F)$. The bound then follows from the Erdős-Stone-Simonovits theorem: $\mathrm{ex}(n, F) = (1 - \frac{1}{r} + o(1))\frac{n^2}{2}$. □

Let us emphasize here that Theorem 2.3 relates the Turán number of a family of edge-ordered graphs to the order chromatic number of the *family*. This is in contrast to the original Erdős-Stone-Simonovits theorem (or the vertex-ordered graph version in [36]), that speaks of chromatic number (interval chromatic number) of a *single graph* (a single vertex-ordered graph, respectively) and relates the Turán number of a family to the least (interval) chromatic number of a member of the family. As we will see, this is a meaningful difference, because the order chromatic number of a family can be substantially smaller than that of any single member in the family, see Proposition 2.10. In the context of extremal hypergraph theory such families are called non-principal. More precisely, a family $\mathcal{H}$ of r-uniform hypergraphs is called *non-principal* if any r-uniform hypergraph avoiding the family contains an asymptotically smaller fraction of the hyperedges of a complete r-uniform hypergraph than hypergraphs avoiding just a single element of the family. Balogh, [4], found non-principal families of 3-uniform hypergraphs of finite size. Later Mubayi and Pikhurko, [34], found a non-principal family of size two.

In light of Theorem 2.3, the asymptotics of the Turán number of an edge-ordered graph is precisely captured by its order chromatic number unless it is 2. In the following subsections we will prove several properties of this parameter. In particular, we will show that the order chromatic number is strongly connected to the notion of canonical edge-orders, studied in the next subsection.

2.1. **Canonical edge-orders.** The Erdős-Stone-Simonovits theorem (Theorem 2.1) connects the classical Turán number to the well-established notion of chromatic number, while the vertex-ordered version of the Erdős-Stone-Simonovits theorem, [36], is connected to *interval chromatic number*, a simple and easy to compute parameter. Theorem 2.3 shows that the order chromatic number is the relevant parameter for the Turán number of edge-ordered graphs, but this notion seems less accessible. In Theorem 2.5 we give criteria to determine the order chromatic number of a family of edge-ordered graphs. To decide whether the order chromatic number is two (that is, whether the Turán number is quadratic in n) is especially simple, see Corollary 2.6.

For the upcoming characterization of the order chromatic number (see Theorem 2.5) we need to introduce the notion of canonical edge-orders. Let us assume that the vertices of a complete graph K_n are linearly ordered, say, they are $v_1, \dots, v_n$. We call an edge-ordering of K_n *canonical* if the order of two edges is always determined by the relative order of their endpoints. Clearly, the canonical edge-order is determined by the order of the six edges spanned by any four vertices. In fact, the order of the three edges in a triangle almost determine the canonical order: four of the six possible orders of the edges in a triangle determine the entire canonical edge-order, while for the other two possible orders the relative order of "overlapping" edges like v_1v_3 and v_2v_4 is not determined, so they yield two canonical edge-orders each. The total of eight canonical edge-orders are isomorphic in pairs; the isomorphism is provided by reversing the order of vertices. Thus, it is enough to give name to the four non-isomorphic edge-orders of K_n:

- *min-labeling* of K_n: For $1 \le i < j \le n$ the label of the edge v_iv_j is $L_1(v_iv_j) = ni + j$.
- *max-labeling* of K_n: For $1 \le i < j \le n$ the label of the edge v_iv_j is $L_2(v_iv_j) = nj + i$.
- *inverse min-labeling* of K_n: For $1 \le i < j \le n$ the label of the edge v_iv_j is $L_3(v_iv_j) = ni - j$.
- *inverse max-labeling* of K_n: For $1 \le i < j \le n$ the label of the edge v_iv_j is $L_4(v_iv_j) = nj - i$.

We need a similar notion of canonical edge-ordering for complete multi-partite graphs as well. Let us denote by $K_{k\times n}$ the complete balanced k-partite graph on kn vertices. We denote the vertices of $K_{k\times n}$ by $v_{i,j}$ with $1 \le i \le k$, $1 \le j \le n$. For $1 \le i \le k$ we call the set $V_i = \{v_{i,j} \mid 1 \le j \le n\}$ a *class* of vertices and two vertices in $K_{k\times n}$ are adjacent if and only if they belong to distinct classes, i.e., $v_{i_1,j_1}v_{i_2,j_2}$ is an edge if and only if $i_1 \ne i_2$.

We call an edge-ordering of $K_{k\times n}$ *canonical* if the order of two edges is determined by the classes of the vertices they connect and in case some of these vertices belong to the same class, then also by the order of those vertices within that class. Thus, in a canonical edge-order the order of the edges $e = v_{i_i,j_1}v_{i_2,j_2}$ and $f = v_{i_3,j_3}v_{i_4,j_4}$ is typically determined by the indices $i_1, i_2, i_3,$

and i_4, but in case some of these indices coincide, like $i_l = i_m$, then the order of j_l and j_m may also influence the order of e and f.

Let us first concentrate on the complete bipartite graphs induced by $V_{i_1} \cup V_{i_2}$ that we call the *parts* of the $K_{k\times n}$. Here $1 \le i_1, i_2 \le k$, $i_1 \ne i_2$. (Note the slightly unusual use of the word 'part', which sometimes means a vertex class of $K_{k\times n}$ but in the rest of this subsection we use it in the sense just introduced.)

By the definition above, the edge-order within a part induced by $V_{i_1} \cup V_{i_2}$ is determined by the relative order of the endpoints of the edges, so it is completely determined by the order of the four edges in any $K_{2\times 2}$ subgraph. A closer inspection shows that for all $n \ge 2$ there are exactly eight possible canonical edge-orders on any part and these can be given by the following labelings:

- $L_1(v_{i_1,j_1}v_{i_2,j_2}) = nj_1 + j_2$
- $L_2(v_{i_1,j_1}v_{i_2,j_2}) = nj_1 - j_2$
- $L_3(v_{i_1,j_1}v_{i_2,j_2}) = -nj_1 + j_2$
- $L_4(v_{i_1,j_1}v_{i_2,j_2}) = -nj_1 - j_2$
- $L_5(v_{i_1,j_1}v_{i_2,j_2}) = nj_2 + j_1$
- $L_6(v_{i_1,j_1}v_{i_2,j_2}) = nj_2 - j_1$
- $L_7(v_{i_1,j_1}v_{i_2,j_2}) = -nj_2 + j_1$
- $L_8(v_{i_1,j_1}v_{i_2,j_2}) = -nj_2 - j_1$

In the first four cases, class V_{i_1} is *dominant*, while in the last four class, V_{i_2} is dominant.

To specify a canonical edge-order of $K_{k\times n}$ for $k > 2$, one has to also say how edges from different parts compare. We say that a part of $G_{k\times n}$ *precedes* another part in an edge-ordering of $G_{k\times n}$ if all edges in the former part come before all edges in the latter part. By our definition, between two vertex disjoint parts in a canonical edge-order one has to precede the other. But this does not have to hold for the distinct parts induced by $V_{i_1} \cup V_{i_2}$ and $V_{i_1} \cup V_{i_4}$. If neither of these parts precedes the other, then we say that these parts *interleave*. In this case the order of the edges $e = v_{i_1,j_1}v_{i_2,j_2}$ and $f = v_{i_1,j_3}v_{i_4,j_4}$ must be determined solely by the order of j_1 and j_3. This leads to the following four possibilities for a fixed pair of interleaving parts:

- $e < f$ if and only if $j_1 < j_3$,
- $e < f$ if and only if $j_1 \le j_3$,
- $e < f$ if and only if $j_1 > j_3$ or
- $e < f$ if and only if $j_1 \ge j_3$.

Clearly, the choice of the canonical edge-orders on the individual parts of $K_{k\times n}$ and the choices for their pairwise behavior determines the relation of every pair of edges. Some combination of these choices do not actually yield a transitive relation, but those that yield a transitive relation, give rise to a canonical edge-order of $K_{k\times n}$. It is easy to see that the same choices yield canonical edge-orders independent of the value of n as long as $n \ge 3$, so the number of canonical edge-orders of $K_{k\times n}$ depends only on k. (The case $n = 2$ is exceptional as in $K_{k\times 2}$ for $k \ge 4$ we may have distinct parts A, B and C such that A and B interleave, so do B and C but A precedes C. Such a configuration is not possible in a canonical edge-order of $K_{k\times n}$ for $n \ge 3$.)

In the simplest case of $K_{2\times n} = K_{n,n}$, we have a single part only, so we have exactly eight canonical edge-orders. All eight of them yield isomorphic edge-ordered complete bipartite graphs. (The isomorphisms are given by reversing the order of the vertices in one or both of the classes and/or switching the two classes.) We denote this canonical edge-ordered complete bipartite graph by $K_{n,n}^{\text{can}}$ (which is therefore unique up to isomorphism). For greater values of k, however, both the number of canonical edge-orders of $K_{k\times n}$ and the number of non-isomorphic edge-ordered graphs obtained is growing fast. It is easier to count canonical edge-orders without interleaving parts: They are determined by an arbitrary order of the $\binom{k}{2}$ parts and also arbitrary canonical edge-orders on each of them. This yields $\binom{k}{2}! \cdot 8^{\binom{k}{2}}$ canonical edge-orders without an interleaving pair of parts and $\binom{k}{2}!8^{\binom{k}{2}}/(k!2^k)$ non-isomorphic edge-ordered graphs. For $k = 3$ this is 3072 canonical edge-orders and 64 non-isomorphic edge-ordered graphs. Still for $k = 3$ there are 768 additional canonical edge-orders with a pair of interleaving parts yielding 16 additional non-isomorphic edge-ordered graphs.

Note that if the part A induced by $V_i \cup V_j$ and the part B induced by $V_i \cup V_{j'}$ interleave in a canonical edge-order of $K_{k\times n}$, then any part not containing V_i must either be preceded by both interleaving parts A and B or it must precede both A and B. This yields the useful observation that in any canonical ordering of $K_{3\times n}$ there must be a part that precedes the other two parts or a part that is preceded by the other two parts.

More generally, given an edge-ordering K of $K_{k\times n}$, the relation "precedes" defines a partial order on the parts. This partial order, plus the ordering within the parts and between incomparable (that is, interleaved) pairs of parts defines the entire edge-order. It is not hard to prove the following characterization of canonical edge-orders.

Proposition 2.4. *An edge-order K of $K_{k\times n}$ ($k \geq 2$, $n \geq 3$) is canonical if and only if "interleaved" is an equivalence relation on the parts with each equivalence class forming a star-like structure with a common class V_{i_0}, and other classes $V_{i_1}, \dots, V_{i_m}$, such that the restriction of K to the bipartite graph between V_{i_0} and $V_{i_1} \cup \dots \cup V_{i_m}$ can be made canonical (with V_{i_0} the dominant side if $m > 1$) by appropriately ordering its vertices such that the vertices of each class V_{i_j} appear consecutively and in monotone (increasing or decreasing) order.*

From the proposition above one can see that such a star can have $m!2^{m+1}$ possible edge-orders for $m > 1$, while singleton equivalence classes have 8 possible edge-orders.

The following theorem connects order chromatic number with the notion of canonical edge-orders. The first part of this theorem is not new and it goes back to an unpublished result of Leeb (see [35]), but we include its simple proof for completeness.

Theorem 2.5.
- *The order chromatic number of a family $\mathcal{H}$ of edge-ordered graphs is infinity if and only if one of the canonical edge-orders of K_n avoids $\mathcal{H}$ for all n.*
- *$\chi_<(\mathcal{H}) > k$ holds for a family $\mathcal{H}$ of edge-ordered graphs and $k \geq 2$ if and only if for all n, one of the canonical edge-orders of $K_{k\times n}$ avoids $\mathcal{H}$.*

Note that if $\mathcal{H}$ is finite, then the "for all n" requirement in both parts of this theorem can be equivalently replaced by setting n to be the largest number of vertices of any member of $\mathcal{H}$.

Proof. We start with the proof of the first claim of the theorem. If a canonical (or any) edge-order of K_n avoids $\mathcal{H}$, then K_n and therefore any of its subgraphs can avoid $\mathcal{H}$. If this holds for all n, then all finite graphs can avoid $\mathcal{H}$, so its order chromatic number is infinity. This proves the "if" part of the claim.

Assume now that the order chromatic number is infinity, therefore any graph can avoid $\mathcal{H}$. Take an edge-ordering K of K_m that avoids $\mathcal{H}$ and color the 4-subsets of its vertices according to the order of the six edges between these vertices. That is, for $1 \leq j_1 < j_2 < j_3 < j_4 \leq m$ we color the set $\{v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}\}$ by the order of the six edges $v_{j_a}v_{j_b}$ in the induced subgraph. This is a 720-coloring of the 4-subsets. Let us choose a monochromatic subset $\{v_{j_l} \mid 1 \leq l \leq n\}$ such that $j_1 < j_2 < \cdots < j_n$. By Ramsey's theorem we can do this for any fixed n if we start with a large enough complete graph K_m. Clearly, the order of two edges is determined by the color of any 4-subset containing their endpoints. This means that the monochromatic subset induces a canonically edge-ordered copy of K_n as long as $n \geq 5$. (The $n = 4$ is exceptional here as any 4-subset is monochromatic, but not all are canonically edge-ordered.) Being a subgraph of K that avoids $\mathcal{H}$, this canonical edge-ordering of K_n also avoids $\mathcal{H}$ proving the "only if" part of the first claim.

For the proof of the second claim assume a canonical (or any) edge-order of $K_{k\times n}$ avoids $\mathcal{H}$, so $K_{k\times n}$ can avoid $\mathcal{H}$. As any finite graph of chromatic number at most k is a subgraph of $K_{k\times n}$ for an appropriate n, we find that it can also avoid $\mathcal{H}$ proving the "if" part of the second claim.

Assume now that $\chi_<(\mathcal{H}) > k$, therefore $K_{k\times m}$ can avoid $\mathcal{H}$ for any m. Let us fix an edge-ordering K of $K_{k\times m}$ avoiding $\mathcal{H}$. We color the 4-subsets $H = \{\{j_1, j_2, j_3, j_4\} : 1 \leq j_1 < j_2 < j_3 < j_4 \leq m\}$ with the order of the edges between the $4k$ vertices in $H^* = \{v_{i,j_l} \mid 1 \leq i \leq k, 1 \leq l \leq 4\}$. There are $16\binom{k}{2}$ such edges, so we have $(16\binom{k}{2})!$ colors. Let us assume that the subset S formed by $j_1 < j_2 < \cdots < j_{kn}$ is monochromatic. By Ramsey's theorem we can find such a set for any n if we start with a large enough value of m. Now we consider the edge-ordered subgraph G of K induced by the vertices v_{i,j_l} for $1 \leq i \leq k$ and $(i-1)n < l \leq in$. Clearly, the underlying simple graph of G is isomorphic to $K_{k\times n}$ with the isomorphism mapping $v_{i,l}$ of $K_{k\times n}$ to $v_{i,j_{(i-1)k+l}}$ in G. This isomorphism induces an edge-order on $K_{k\times n}$ and the fact that S is monochromatic implies that this edge-order is canonical if $nk \geq 5$. Indeed, for any pair of edges in G their order is determined by the color of any set H with H^* containing all four endpoints, and thus by the common color of all 4-subsets of S. In particular, the order between two edges of $K_{k\times n}$ whose endpoints are in four distinct classes is determined by these classes, and the order between two edges whose endpoints are in fewer classes is determined by the classes and the relative order of the endpoints in the common classes. The requirement that $nk \geq 5$ is needed to ensure that if a subset is monochromatic with respect to our coloring of 4-subsets, then the same subset is also monochromatic with respect to a similar coloring of the 3-subsets.

Since G is a subgraph of K, G avoids $\mathcal{H}$, so the canonical edge-order of $K_{k\times n}$ isomorphic to G also avoids $\mathcal{H}$ proving the "only if" part of the second claim and finishing the proof of the theorem. □

Theorem 2.5 implies the following corollary.

Corollary 2.6. *Let $\mathcal{H}$ be a family of edge-ordered graphs.*

1. *The order chromatic number of a subfamily $\mathcal{H}' \subseteq \mathcal{H}$ satisfies $\chi_<(\mathcal{H}') \geq \chi_<(\mathcal{H})$.*
2. *$\chi_<(\mathcal{H}) = 2$ holds if and only if there exists $G \in \mathcal{H}$ with $\chi_<(G) = 2$.*
3. *An edge-ordered graph G on n vertices satisfies $\chi_<(G) = 2$ if and only if $K^{\mathrm{can}}_{n,n}$ contains G.*
4. *If $\chi_<(\mathcal{H})$ is finite, then there exists a subfamily $\mathcal{H}' \subseteq \mathcal{H}$ of size at most four with $\chi_<(\mathcal{H}')$ finite.*
5. *For $k \geq 3$ there exists a number c_k (depending only on k) such that if $\chi_<(\mathcal{H}) = k$, then there exists a subfamily $\mathcal{H}' \subseteq \mathcal{H}$ of size at most c_k with $\chi_<(\mathcal{H}') = k$. One can choose $c_3 = 80$.*

Proof. The monotonicity claimed in part 1 of the corollary follows directly from the definition of the order chromatic number: if a graph can avoid a family $\mathcal{H}$, then it can also avoid all its subfamilies.

For part 4 we use the first claim of Theorem 2.5. As the order chromatic number of $\mathcal{H}$ is finite, none of the four canonical edge-orders of K_n avoid $\mathcal{H}$ for all n. For each one of the four canonical edge-orders, we can find a value of n and an element H of $\mathcal{H}$ such that K_n with that particular canonical edge-order does not avoid H. By the first part of Theorem 2.5 again, the subfamily consisting of these four elements of $\mathcal{H}$ has finite order chromatic number.

For part 5 we argue very similarly. Let c_k be the number of canonical edge-orders of $K_{k\times n}$. By the second part of Theorem 2.5 for each of the canonical edge-orders there is a choice of n such that $K_{k\times n}$ with that edge-order contains a particular element H of $\mathcal{H}$. Let $\mathcal{H}'$ consist of the elements of $\mathcal{H}$ selected for one of those canonical edge-orders. By Theorem 2.5 the order chromatic number of $\mathcal{H}'$ is at most k. But by part 1 above it is at least k, so we have $\chi_<(\mathcal{H}') = k$.

Note that in this argument we could set c_k to be the number of non-isomorphic edge-ordered graphs obtained from canonical edge-orderings of $K_{k\times n}$ as isomorphic edge-ordered graphs avoid the same edge-ordered graphs.

This makes us able to choose $c_3 = 80$ as claimed in part 5 and also proves parts 2 and 3 of the corollary as we know that each canonical edge-order of $K_{2\times n}$ is isomorphic to $K^{\mathrm{can}}_{n,n}$. □

The following two simple observations are related to $K^{\mathrm{can}}_{n,n}$ and part 3 of Corollary 2.6.

Proposition 2.7. *If m is large enough compared to n, then $K_{m,m}$ cannot avoid $K^{\mathrm{can}}_{n,n}$.*

Proof. $K^{\mathrm{can}}_{2n,2n}$ contains $K^{\mathrm{can}}_{n,n}$, so by part 3 of Corollary 2.6 we have $\chi_<(K^{\mathrm{can}}_{n,n}) = 2$. By definition, this implies the existence of a bipartite graph G that cannot avoid $K^{\mathrm{can}}_{n,n}$. Clearly, G is a subgraph of $K_{m,m}$ if m is large enough, so neither can $K_{m,m}$ avoid $K^{\mathrm{can}}_{n,n}$. □

Note that Proposition 2.7 is also the two dimensional special case of a 1993 result by Fishburn and Graham, [15]. Recently Bucić, Sudakov and Tran, [8] proved that the choice $m = 2^{2^{(4+o(1))n^2}}$ is enough for the statement of the proposition to hold. Note that this bound is much lower than the one that follows from the argument above or the result of Fishburn and Graham.

Definition 2.8. We call a vertex v of an edge-ordered graph *close* if the edges incident to v are consecutive in the edge-ordering, that is, they form an interval in the edge-order.

Proposition 2.9. *If an edge-ordered graph G is contained in the max-labeling or inverse max-labeling of some complete graph, then one of the end vertices of the maximal edge in G is close in G. Symmetrically, if G is contained in the min-labeling or inverse min-labeling of some complete graph, then one of the end vertices of the minimal edge in G is close in G.*

If $\chi_<(G^L) = 2$ for some labeling of a simple graph G, then G has a proper 2-coloring with all vertices in one color class being close in G^L. The converse also holds if G is a forest, namely if the forest G has a labeling L and a proper 2-coloring in which all vertices of one of the color classes are close in G^L, then $\chi_<(G^L) = 2$.

Note that the requirement of G being a forest is necessary in the last statement. The edge-ordered cycle C_4^{1234} is bipartite, and all but one of its vertices are close, yet it is not contained in either the min-labeling or max-labeling of any complete graph, so $\chi_<(C_4^{1234}) = \infty$.

Proof. Consider any subgraph G of K_n with either the max-labeling or the inverse max-labeling. The larger-indexed end vertex of the maximal edge in G is close in G. Similarly, in a subgraph G of the min-labeling or inverse min-labeling of K_n the smaller-indexed end vertex of the minimal edge in G is close in G. This proves the first two statements of the proposition.

By Corollary 2.6 we have $\chi_<(G^L) = 2$ if and only if G^L is contained in $K_{n,n}^{\mathrm{can}}$ for some n. We think of $K_{n,n}^{\mathrm{can}}$ as the complete bipartite graph on vertices u_i and v_i with $1 \leq i \leq n$ and with the label of edge u_iv_j being $ni + j$. Clearly, the vertices u_i are close in this graph and they form one color class of the only bipartition of $K_{n,n}$. These vertices remain close in any subgraph of $K_{n,n}^{\mathrm{can}}$ proving the third statement of the observation.

For the final statement we need to embed G^L isomorphically to $K_{n,n}^{\mathrm{can}}$, where n is the number of vertices in G. Let us fix a proper 2-coloring of G with all vertices in one color class (say red) being close. The red vertices are linearly ordered by the labeling of the incident edges (except for isolated vertices). We map the red vertices to vertices u_i respecting this ordering, that is, if for red vertices x and y the edges incident to x are lower than those incident to y, then we map x to u_{i_x} and y to u_{i_y} with $i_x < i_y$. Isolated red vertices can be mapped to any remaining vertices u_i. To obtain an isomorphic embedding all we have to do is map the vertices of the other color class to vertices v_j in $K_{n,n}^{\mathrm{can}}$ such that for any red vertex x the mapping of its neighbors y respect the order of the labels $L(xy)$. As G is a forest these requirements do not form a directed cycle, so all can be satisfied simultaneously. □

We finish this section by highlighting two aspects of the order chromatic number not shared by either the ordinary chromatic number of simple graphs or the interval chromatic number of vertex ordered graphs.

Firstly, we show that the order chromatic number of a family of edge-ordered graphs can indeed be smaller than that of any member. The jump we exhibit here is the largest allowed by Corollary 2.6. Recall that P_5^{1423} denotes the path on five vertices, say a, b, c, d, e, with its edges ordered as $ab < cd < de < bc$. Similarly, P_5^{2314} is the path on five vertices, with edges ordered as $cd < ab < bc < de$.

Proposition 2.10. $\chi_<(P_5^{1423}) = \chi_<(P_5^{2314}) = \infty$, *but* $\chi_<(\{P_5^{1423}, P_5^{2314}\}) = 3$.

Proof. Notice that reversing the edge-order in P_5^{1423} yields an edge-ordered graph isomorphic to P_5^{2314} (by reversing the vertices) giving a certain symmetry to the statements of the proposition.

Neither endpoint of the smallest edge in P_5^{2314} is close, so by Proposition 2.9, P_5^{2314} is not contained in either the min-labeling or the inverse min-labeling of a complete graph. Symmetrically, the max-labeling and the inverse max-labeling avoid P_5^{1423}. This shows that both of these edge-ordered paths have order chromatic number infinity if considered separately.

One can observe that both the max labeling and the inverse max-labeling of K_n contain P_5^{2314} as long as $n \geq 5$ and symmetrically the min-labeling and the inverse min-labeling of K_n contain P_5^{1423}. By Theorem 2.5 this proves that that the order chromatic number of the pair $\{P_5^{1423}, P_5^{2314}\}$ is finite. We want to prove specifically that it is 3. By part 2 of Corollary 2.6 it cannot be 2, so we need only to show that it is at most 3. Instead of exhibiting an explicit 3-chromatic graph that cannot avoid the pair, we use Theorem 2.5 again and show that all canonical edge-orders of $K_{3\times 2}$ contain one of the two edge-ordered paths. Let us recall that the classes of $K_{3\times 2}$ are the pairs $V_i = \{v_{i,1}, v_{i,2}\}$ for $1 \leq i \leq 3$ and the parts of $K_{3\times 2}$ are the complete bipartite subgraphs induced by two classes. As we have observed, in any canonical edge-order of $K_{3\times 2}$ (or of $K_{3\times n}$ in general) there is a smallest part preceding the other two parts or there is a largest part that is preceded by the other two parts. In the former case we can find an isomorphic copy of P_4^{231} in the smallest part and then we can extend it with an edge from another part to get an isomorphic copy of P_4^{2314}. In the latter case we find an isomorphic copy of P_4^{423} in the largest part and extend it with an edge from another part to obtain an isomorphic copy of P_5^{1423}. Thus, no canonical edge-order of $K_{3\times 2}$ avoids both P_5^{1423} and P_5^{2314}. This finishes the proof of the proposition. □

Secondly, recall that the order chromatic number of finite edge-ordered graphs can be infinite. In Section 2.2, we will show the existence of edge-ordered graphs for which the order chromatic number is finite but significantly larger than its number of vertices (or even its number of edges). In particular, we will construct edge-ordered graphs D_n on n vertices for which the order chromatic number is finite but it still grows exponentially with n (see Theorem 2.14).

2.2. **How large can the order chromatic number be?** We saw examples of rather small edge-ordered graphs with order chromatic number infinity. In fact, Theorem 2.18 below claims that *every* simple graph with more than 3 edges that is not a star forest has such an edge-ordering. Here we consider edge-ordered graphs with finite order chromatic number. More specifically, we ask how large the order chromatic number of an n-vertex edge-ordered graph (or of a family of edge-ordered graphs with at most n vertices in each) can be if it is finite.

Let $\mathcal{K}_n$ stand for the family of the four canonical labelings of the complete graph K_n. By the Ramsey theoretic theorem of Leeb that appeared in the paper [35] and stated here as the first half of Theorem 2.5, there exists m for any n such that K_m cannot avoid $\mathcal{K}_n$. Let $f_{\mathrm{Leeb}}(n)$ be the smallest integer m with this property.

Proposition 2.11. *If a family $\mathcal{H}$ of edge-ordered graphs on at most n vertices has finite order chromatic number, then*

$$\chi_<(\mathcal{H}) \leq \chi_<(\mathcal{K}_n) \leq f_{\mathrm{Leeb}}(n).$$

Proof. By Theorem 2.5 and as the order chromatic number of $\mathcal{H}$ is finite, none of the four canonical edge-orders of K_m avoid $\mathcal{H}$ for every m. But then none of the four edge-ordered graphs in $\mathcal{K}_n$ avoid $\mathcal{H}$ as otherwise the corresponding element of $\mathcal{K}_m$ would also avoid $\mathcal{H}$ for all m. (Note that we use here that all elements of $\mathcal{H}$ have at most n vertices.)

The claim above implies that all edge-ordered graphs avoiding $\mathcal{H}$ also avoid $\mathcal{K}_n$ and therefore all simple graphs that cannot avoid $\mathcal{K}_n$ cannot avoid $\mathcal{H}$ either. This proves the first inequality.

To see the second inequality notice that for $m = f_{\text{Leeb}}(n)$, K_m cannot avoid $\mathcal{K}_n$ by definition, so we have $\chi_<(\mathcal{K}_n) \leq \chi(K_m) = m$. □

The upper bound on $f_{\text{Leeb}}(n)$ coming from the argument presented in the proof Theorem 2.5 is a Ramsey number for coloring 4-uniform hypergraphs of which the best available bound is triply exponential in n. However, a better upper bound that is doubly exponential in a polynomial of n has been recently claimed by C. Reiher, V. Rödl, M. Sales, K. Sames, and M. Schacht, [39] and independently also by D. Conlon, J. Fox and B. Sudakov (unpublished). These recent results also contain a doubly exponential lower bound for $f_{\text{Leeb}}(n)$. However, the lower bound does not seem to directly translate to a lower bound on $\chi_<(\mathcal{K}_n)$ because it is possible that a very large graph with a small chromatic number cannot avoid the family $\mathcal{K}_n$.

Now we construct a sequence of edge-ordered graphs D_n to show that the order chromatic number can grow exponentially in the number of vertices and still remain finite: For $n \geq 2$, let D_n be the edge-ordered graph with vertices $x_1, \ldots, x_n$ and the $2n-3$ edges incident to x_1 or x_n with the edge-order $x_1x_2 < x_1x_3 < \cdots < x_1x_n < x_2x_n < \cdots < x_{n-1}x_n$.

Proposition 2.12. *$\chi_<(D_n) < \infty$ for any $n \geq 2$ but $\chi_<(D^*) = \infty$ for every edge-ordering D^* of the underlying simple graph of D_n that is not isomorphic to D_n.*

Proof. We show that D_n is contained in all canonical edge-orders of K_n. By Theorem 2.5 this is enough to see that $\chi_<(D_n) < \infty$.

We embed the vertices of D_n in the min-labeled and the max-labeled K_n in their natural order to show the containment. For the inverse min-labeled K_n we use the order $x_1, x_n, x_{n-1}, \ldots, x_2$. For the inverse max-labeled K_n we use the order $x_{n-1}, x_{n-2}, \ldots, x_1, x_n$.

Now let D^* be an edge-ordering of the underlying graph of D_n with $\chi_<(D^*) < \infty$. We need to show that D^* is isomorphic to D_n.

For $n \leq 3$ the underlying graph of D_n is K_n and all its edge-orderings are isomorphic to D_n. Assume next that $n = 4$. By Proposition 2.9, D^* must have a close vertex incident to the maximal edge and another incident to the minimal edge. These two vertices either need to be non-adjacent, or they need to be incident to all edges. In our case, this means that they could only be x_2 and x_3 (the only non-adjacent pair) or x_1 and x_4 (the only pair incident to all edges). The former case yields two non-isomorphic edge-orderings: $x_1x_2 < x_2x_4 < x_1x_4 < x_1x_3 < x_3x_4$ and $x_1x_2 < x_2x_4 < x_1x_4 < x_3x_4 < x_1x_3$. The first of these is avoided by the inverse min-labeling, while the second is avoided by the min-labeling, so both have infinite order chromatic number. There are also two non-isomorphic edge-orders in which x_1 and x_4 are the close vertices incident to the minimal and maximal edges. One of them is the edge-ordering of D_4, while the other is

$x_1x_2 < x_1x_3 < x_1x_4 < x_3x_4 < x_2x_4$, yielding an edge-ordered graph avoided by all four canonical clique-labelings, so the order chromatic number of this edge-ordered graph is also infinite.

Finally for $n > 4$ we use that the subgraph of D^* induced by x_1, x_n and any two other vertices (being an edge-ordering of the underlying graph of D_4) must be isomorphic to D_4 or the order chromatic number of the subgraph, and hence D^* itself is infinite. This implies that D^* itself is isomorphic to D_n. □

To prove an exponential lower bound on $\chi_<(D_n)$ (in Theorem 2.14), we need the following lemma.

Lemma 2.13. *Let $n \geq 2$ and $m \geq 2$ be integers. The relation $\chi_<(D_n) > m$ holds if and only if there is an edge-ordering K of K_m avoiding D_n such that the auxiliary graphs G_x are bipartite for all vertices x of K_m. Here $V(G_x) = V(K_m) \setminus \{x\}$ and $E(G_x)$ consists of the edges yz such that $xy < yz < zx$ in the edge-ordering of K.*

Proof. By Theorem 2.5 we have $\chi_<(D_n) > m$ if and only if there is a canonical edge-ordering of $K_{m\times n}$ avoiding D_n. To prove the "only if" part of the lemma let us assume that K^* is a canonical edge-ordering of $K_{m\times n}$ that avoids D_n. Recall that the vertices of $K_{m\times n}$ are $v_{i,j}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$. The vertices $v_{i,j}$ with a fixed i form an independent set that we call a *class*. The *parts* of $K_{m\times n}$ are the complete bipartite graphs connecting two classes.

Let K be the subgraph of K^* induced by the m vertices $v_{i,1}$. It is an edge-ordering of the complete graph K_m. We claim that K satisfies the conditions in the lemma, namely it also avoids D_n and the auxiliary graphs G_x are all bipartite.

As a subgraph of K^*, K avoids D_n. We need to prove that the auxiliary graphs are bipartite. So let $x = v_{i,1}$ be a fixed vertex of K. Consider another vertex y of K and the order of the edges $yv_{i,j}$. As on K^* the edge-order is canonical, this is either monotone increasing in j or monotone decreasing in j. We call the vertex y *increasing* or *decreasing* accordingly. We claim that G_x is bipartite because all its edges connect an increasing vertex with a decreasing vertex. Assume for a contradiction yz is an edge of G_x with both y and z being increasing (or both being decreasing). Now consider the subgraph of K^* induced by the vertices y, z and $n-2$ of the vertices in the class of x. This subgraph is isomorphic to D_n contradicting the assumption that K^* avoids D_n. The contradiction proves the "only if" part of the lemma.

For the "if" part let K be an edge-ordering of K_m satisfying the conditions of the lemma. We need to find a canonical edge-ordering K^* of $K_{m\times n}$ avoiding D_n. We identify the vertices of K with the classes in $K_{m\times n}$. This way the parts of $K_{m\times n}$ correspond to the edges of K. In our canonical edge-ordering no pair of parts are interleaved and one part precedes another if the edge in K corresponding to the former part is smaller than the edge corresponding to the latter part. Now we consider the bipartite auxiliary graphs G_x and fix a bipartition to "increasing" and "decreasing" vertices. Note that the same vertex can be designated increasing in one auxiliary graph and decreasing in another. To specify the canonical edge-order of K^* we have to further specify one of the eight canonical orders for each of the $\binom{m}{2}$ parts. Assume the classes $V_{i_1} = \{v_{i_1,j} \mid 1 \leq j \leq n\}$ and $V_{i_2} = \{v_{i_2,j} \mid 1 \leq j \leq n\}$ correspond to vertices x and y in K. We choose the canonical order of the part between these two classes such that the order of the edges $v_{i_1,j_1}v_{i_2,j_2}$ is increasing or decreasing in j_2 (for fixed j_1) according to whether x is increasing or decreasing in G_y. That is, if x is increasing in G_y, then $v_{i_1,j_1}v_{i_2,1} < v_{i_1,j_1}v_{i_2,2} < \ldots < v_{i_1,j_1}v_{i_2,n}$, while if x is decreasing, then $v_{i_1,j_1}v_{i_2,1} > v_{i_1,j_1}v_{i_2,2} > \ldots > v_{i_1,j_1}v_{i_2,n}$. Similarly, we choose the canonical order of the part of K^* induced by $V_{i_1} \cup V_{i_2}$ such that the order of these edges is increasing or decreasing in j_1 (for a fixed j_2) according to whether y is increasing or decreasing in G_x. For any of the four possible cases above, we still have two canonical edge-orders to choose from; we can choose any of these two options arbitrarily.

We claim that K^* avoids D_n. Assume for a contradiction that K^* contains D_n. If the n vertices of the isomorphic copy of D_n come from n different classes in K^*, then this subgraph of K^* would correspond to an isomorphic subgraph of K. This contradicts our assumption that K avoids D_n.

Now assume that two vertices a and b of the subgraph of K^* isomorphic to D_n come from the same class. As any class is independent in K^*, a and b must correspond to two non-adjacent vertices in D_n. Let c and d be the vertices in the subgraph corresponding to the full degree first and last vertex of D_n. As c and d are connected to every vertex, a, b, c, d are four different vertices. Clearly, a, c and d must come from three distinct classes of $K_{m\times n}$. Let x, y and z be the corresponding vertices in K. The subgraph of K^* induced by the four vertices a, b, c and d must be isomorphic to D_4 and this implies that yz is an edge in G_x. So one of y and z must be a decreasing vertex in G_x, the other an increasing vertex. But that means that either $ac < bc$ and $ad > bd$ in K^* or vice versa: $ac > bc$ and $ad < bd$ in K^*. Both cases contradict the isomorphism of the induced subgraph to D_4. The contradiction finishes the proof of the lemma. □

Now we are ready to prove the exponential lower bound on $\chi_<(D_n)$.

Theorem 2.14. $\chi_<(D_n) > \binom{2n-4}{n-2} = \Omega(4^n/\sqrt{n})$.

Proof. Using Lemma 2.13, it is enough to give an edge-ordering K of $K_{\binom{2n-4}{n-2}}$ that avoids D_n such that the auxiliary graphs G_x are bipartite for all vertices x of K. Each vertex of K will correspond to a binary sequence $\{0,1\}^{2n-4}$ containing $n-2$ 0's and $n-2$ 1's. We write $u < v$ if u comes before v in the lexicographic order. It is convenient to think about these sequences as root to leaf paths in a (partial) binary tree of depth $2n-4$, which is drawn in the "usual way", i.e., its leaves are on a line in lexicographic order. When referring to the position where "u diverges from v" we mean the first position where the two sequences differ.

To order the edges we consider for an edge uv where u and v diverge: longer common prefix makes for a larger edge. If these longest common prefixes have the same length for two edges, then these edges are ordered by the distance of their endpoints in the lexicographic ordering: larger distance makes for a larger edge. Finally, edges having a tie in both values are ordered arbitrarily.

Now we need to prove that the edge-ordered complete graph we have just constructed avoids D_n and each auxiliary graph G_x is bipartite.

We start with the latter claim. Among any three vertices there is one that diverges from the other two at the same position. In this case the other two vertices diverge later and therefore the edge connecting them is the largest in the triangle induced by these three vertices. The auxiliary graph G_x consists of the edges uv satisfying $xu < uv < xv$. In this case v and x must diverge from u at the same position. But then they are on the same side of u and $xu < uv$ implies that x is closer to u than v, so u and v are on different sides of x. Thus, every edge of G_x connects vertices from opposite sides of x making G_x bipartite, as claimed.

Now suppose that K contains a D_n, that is, it has vertices $x_1, \dots, x_n$ satisfying $x_1x_2 < x_1x_3 < \dots < x_1x_n < x_2x_n < \dots < x_{n-1}x_n$. For $1 < i < n$ the largest edge spanned by x_1, x_i and x_n is x_ix_n, so x_1 must diverge from x_i and x_n at the same position. Therefore, x_1 must diverge from $x_2, \dots, x_n$ at the same position. Therefore, the relative order of the edges x_1x_i $(1 < i \le n)$ is determined by the distance of their endpoints, and all of the x_i are on the same side of x_1, so we have either $x_1 < x_2 < \dots < x_n$ or $x_1 > x_2 > \dots > x_n$ in the lexicographic order. If for some $1 \le i < j < n$, x_i and x_j diverge from x_n at the same place, then we would have $x_ix_n > x_jx_n$, which is not the case in D_n. Thus, for $i < n$ each x_i diverges at a different place from x_n, and (as all x_i are on the same side of x_n) x_n must have the same digit in all these $n-1$ positions. But this is not possible because x_n has only $n-2$ digits of either type. □

Note that the bound of Theorem 2.14 is trivially sharp if $n \le 3$, but the proposition below shows that it is not sharp for $n = 4$.

Proposition 2.15.

$$10 \le \chi_<(D_4) < \infty$$

Proof of Proposition 2.15. We have already seen in Proposition 2.12 that $\chi_<(D_4)$ is finite.

For the lower bound we use Lemma 2.13. The edge labeling K_9^L satisfies the conditions of that lemma, where the vertices of K_9 are $v_1, v_2, \dots, v_9$ and the label of the edge v_iv_j is given as the j'th entry in the i'th row of the following symmetric matrix. The entries in the diagonal are left blank. A short case analysis is enough to verify that K_9^L satisfies the properties required in Lemma 2.13, but we found the labeling itself by computer search.

$$\begin{pmatrix}
 & 1 & 2 & 3 & 4 & 33 & 34 & 35 & 36 \\
1 & & 26 & 25 & 29 & 30 & 5 & 27 & 8 \\
2 & 26 & & 24 & 15 & 7 & 20 & 18 & 17 \\
3 & 25 & 24 & & 11 & 31 & 22 & 23 & 9 \\
4 & 29 & 15 & 11 & & 10 & 12 & 13 & 16 \\
33 & 30 & 7 & 31 & 10 & & 32 & 28 & 6 \\
34 & 5 & 20 & 22 & 12 & 32 & & 21 & 14 \\
35 & 27 & 18 & 23 & 13 & 28 & 21 & & 19 \\
36 & 8 & 17 & 9 & 16 & 6 & 14 & 19 &
\end{pmatrix}$$

□

With some simple observations combined with a computer search we could also prove $\chi_<(D_4) \le 31$. More precisely, we checked by computer that the edges of a K_6 on $v_1, \dots v_6$ cannot be ordered to avoid D_4 if $v_1v_i < v_1v_j$ for $i < j$, and for each i the bipartite G_{v_i} is empty on $\{v_j \mid j > i\}$, and this can always be guaranteed for some 6 vertices of K_{32} using that each G_x has an independent set of size 16, since G_x is bipartite. With similar tricks the bound 31 can be certainly reduced, but proving the conjectured $\chi_<(D_4) = 10$ is out of reach with such methods.

A strongly related question is that whether in a vertex- and edge-ordered complete graph on $N = 2^n$ vertices, can we always find an n vertex subgraph where the left going edges from any vertex are all smaller than the right going edges from the same vertex, or vice versa? Note that $N \le R_3(n) \le 2^{2^n}$ follows from a simple Ramsey argument: just two-color each triple $u < v < w$ depending on whether $uv < vw$ or not.

2.3. **The best and worst edge-orders of a graph.** For a non-empty finite graph G, let $\chi^-(G) = \min_L \chi_<(G^L)$, where the minimum is taken over all labelings L of G. Similarly, let $\chi^+(G) = \max_L \chi_<(G^L)$. In this subsection we determine $\chi^+(G)$ for all graphs G, and prove the following simple result concerning $\chi^-(G)$.

Proposition 2.16. *$\chi^-(G) \ge \chi(G)$ for any graph G.*
$\chi^-(G) = 2$ if and only if $\chi(G) = 2$.

Proof. If a graph H does not contain the graph G as a subgraph, then clearly all labelings of H avoid all labelings of G. This proves the first statement and also the only if part of the second statement.

If G is bipartite, then it is contained in the complete bipartite graph $K_{n,n}$ for an appropriate n. The canonical labeling $K_{n,n}^{\mathrm{can}}$ induces a labeling G^L of G that is contained in $K_{n,n}^{\mathrm{can}}$, so by Corollary 2.6 we have $\chi_<(G^L) = 2$. This finishes the proof of the proposition. □

The following proposition shows that even the "minimal order chromatic number" χ^- can be infinite for small simple graphs:

Proposition 2.17. *$\chi^-(K_4) = \infty$.*

Proof. Consider a labeling L of K_4. If the three largest edges of K_4^L do not form a star, then neither endpoint of the largest edge in K_4^L is close, so K_4^L is not contained in the max-labeling or inverse max-labeling of a complete graph by Proposition 2.9. If the three largest edges in K_4^L do form a star, then the three smallest ones do not form a star, so (again by Proposition 2.9) K_4^L is not contained in the min-labeling or inverse min-labeling of a complete graph. Therefore, an appropriate edge-ordering of K_n always avoids K_4^L. This finishes the proof.

A closer inspection reveals that every labeling of K_4 is avoided by at least three of the four canonical labelings of K_n. Indeed, a subgraph induced by four vertices of a canonical labeling of K_n is always isomorphic to the corresponding canonical labeling of K_4 and the four canonical labelings of K_4 are pairwise non-isomorphic. □

We call a simple non-empty graph a *star forest* if all connected components are stars. We will study the Turán numbers of edge-ordered star forests in more detail in the next section. As isolated vertices do not affect the order chromatic number we only consider simple graphs without isolated vertices.

Theorem 2.18. *If the simple graph G is a star forest or a 3-edge path, then $\chi^+(G) = 2$. If G is a triangle, then $\chi^+(G) = 3$. All remaining finite simple graphs G without isolated vertices satisfy $\chi^+(G) = \infty$.*

Proof. We prove the first statement using Proposition 2.9. Any star forest has a proper 2-coloring with all vertices in one color class having degree one. These vertices are close in all labelings. Both color classes of P_4 contain a degree 2 vertex, but at any edge-ordering of P_4 makes one of them close, so the last statement of Proposition 2.9 applies again.

K_3 is not bipartite, so $\chi_<(K_3^L) \geq 3$ for all labelings L. But all labelings of K_3 yield isomorphic edge-ordered graphs, so K_3 cannot avoid K_3^L for any L. This makes $\chi^-(K_3) = \chi^+(K_3) = 3$.

Any remaining non-empty graph G without an isolated vertex contains an edge $e_1 = uv$ such that both u and v have degree more than 1. We find a labeling of G that is avoided by both the max-labeling and the inverse max-labeling of any complete graph by making e_1 the maximal edge and ensuring neither u nor v is close, see Proposition 2.9. If there exists an edge not adjacent to e_1 we are done by making it the second largest. If all edges are adjacent to e_1, then one of u or v must have degree at least 3 as G has at least 4 edges. Say e_2 and e_3 are both incident to u. Making e_2 the second largest we ensure v is not close and making e_3 the smallest we ensure u is not close either. □

Another natural question to study is how $\text{ex}_<(n, G^L)$ behaves for the best and worst edge-orderings of a given graph G. By Theorem 2.1, $\text{ex}_<(n, G^L)$ is asymptotically determined by $\chi_<(G^L)$ if $\chi_<(G^L) > 2$. Proposition 2.17 and Theorem 2.18 imply that for many graphs $\chi^-(G) = \chi^+(G) = \infty$, so even for the best edge-order, $\text{ex}_<(n, G^L) = \binom{n}{2}$ because of Theorem 2.5. We have also seen in Section 2.2 that even $\chi^-(D_k)$ can grow exponentially in k, while $\chi(D_k) = 3$. In fact, if we denote by $K_{2,3}^+$ the graph obtained by adding an edge connecting two vertices on the larger side of $K_{2,3}$, then we have $\chi(K_{2,3}^+) = 3$, but $\chi^-(K_{2,3}^+) = \infty$. (This can be proved with a case analysis similar to the proof of Proposition 2.17.)

Proposition 2.16 shows that $\chi(G) = 2$ implies $\chi^-(G) = 2$. Is it in fact possible that for every bipartite G there an edge-ordering L such that $\text{ex}_<(n, G^L) = O(\text{ex}(n, G))$? As we have discussed in the Introduction, this is true when G is a path, because we can pick the monotone increasing edge-labeling for which $\text{ex}_<(n, P_k^{\text{inc}}) = O(n)$. It, however, fails for most trees.

Proposition 2.19. *If a tree T has a vertex from which 3 paths of length 3 start, then $\text{ex}_<(n, T^L) = \Omega(n \log n)$ for any edge-ordering T^L of T.*

The proof of Proposition 2.19 follows from a simple case analysis which shows that such trees T^L always contain a path P_5 of length 4 such that the restriction of the edge-ordering of T^L to this path, yields an edge-ordered path P_5^L for which $\text{ex}_<(n, P_5^L) = \Omega(n \log n)$. (For the characterization of length 4 paths, see Section 4.2.)

3. Star forests

Recall that a *star* is a simple, connected graph in which all edges share a common vertex and a *star forest* is a non-empty graph whose connected components are all stars. In this section we study the Turán numbers $\mathrm{ex}_<(n, F)$ for edge-ordered star forests F. We will show that this problem is closely related to Davenport-Schinzel theory, so let us recall the basic definitions. For a more thorough introduction on Davenport-Schinzel theory see e.g., [25].

A *word* is a finite sequence. We will refer the elements of the sequence as *letters*, but we are not interested in what the actual letters are, we only care about where the same letters repeat. Accordingly, we say that the words $u = a_1 \dots a_n$ and $v = b_1 \dots b_m$ are *equivalent* if $n = m$ and for all $1 \le i, j \le n$ we have $a_i = a_j$ if and only if $b_i = b_j$. We denote the length of the word u by $|u|$, so we have $|u| = n$ in this example. We write $||u||$ for the number of distinct letters in u. A word u is k*-regular* (for some positive integer k) if every k consecutive letters in u are distinct (in case $|u| < k$ we require all letters of u to be distinct). A *subword* is obtained by deleting any number of letters from a word and considering the word formed by the remaining letters in their original order. We say that a word u *contains* another word f if f is equivalent to a subword of u. If this is not the case we say that u *avoids* f. For a non-empty word f and a positive integer n we write $\mathrm{ex}_{DS}(n, f)$ for the length $|u|$ of the longest $||f||$-regular word u on at most n letters (that is $||u|| \le n$) avoiding f. The central problem of Davenport-Schinzel theory is to calculate or estimate this extremal function.

To apply the results of Davenport-Schinzel theory we need to relate edge-ordered graphs to words. We do this in two different ways. First, let F be an edge-ordered star forest. We represent each component of F with a unique letter. We define the corresponding word $w(F)$ to be $w(F) = a_1 \dots a_m$, where m is the number of edges in F and a_i is the letter representing the component of F containing the i'th edge in the edge-ordering of F. We obtain the longer word $w'(F) = a_1^{2m} \dots a_m^{2m}$ by repeating each letter in $w(F)$ $2m$ times. (Here we use exponentiation to denote repetitions.) For our second connection between graphs and words consider an arbitrary edge-ordered graph G. We build a corresponding word over the set of vertices of G as letters by listing the two end vertices of each edge. We list the edges according to their edge-order but we choose the order of the two end vertices of the same edge arbitrarily. We write $u(G)$ for the family of words one can obtain this way. For example, if G is a graph with edges ab, ac, bc, ad with the edge-order $ab < ac < bc < ad$, then $u(G)$ contains the word $abaccbda$ among 15 other words. The length of any word in $u(G)$ is twice the number of edges in G.

The main connection between the containments in these two different contexts is provided by the following lemma.

Lemma 3.1. *Let F be an edge-ordered star forest and let G be an edge-ordered graph. If a word in $u(G)$ contains $w'(F)$ and G has at least as many vertices as F, then G contains F.*

Proof. Let $w(F) = a_1 \dots a_m$. Let u be the subword of an element u_0 of $u(G)$ equivalent to $w'(F) = a_1^{2m} \dots a_m^{2m}$. We have $u = b_1^{2m} \dots b_m^{2m}$ with $b_i = b_j$ if and only if $a_i = a_j$. Each of the letters in u were inserted in u_0 as an end vertex of an edge in G, thus b_i^{2m} must come from $2m$ distinct edges of G, each incident to the vertex b_i. For each $i = 1, \dots, m$ we select one of these edges, $e_i = b_i c_i$ such that the vertices c_i are pairwise distinct and none of them coincides with any of the vertices b_j. We can achieve this (even in a greedy manner) as out of the $2m$ possibilities for the choice of c_i, less than $2m$ are forbidden.

It is easy to see that the subgraph of G consisting of the vertices b_i, c_i (for $i = 1, \dots, m$) and the edges e_i for $i = 1, \dots, m$ is isomorphic to the edge-ordered graph obtained from F by deleting its isolated vertices. The isomorphism can be extended to the isolated vertices of F as G has enough vertices. □

Davenport-Schinzel theory bounds the length of the $||f||$-regular words avoiding a forbidden word f. We will use this bound for $f = w'(F)$ together with Lemma 3.1 to bound the length of any word in $u(G)$ (and with that the number of edges in G) for edge-ordered graphs G avoiding the edge-ordered star forest F. The only obstacle here is that elements of $u(G)$ do not have to be $||w'(F)||$-regular. In fact, they do not even have to be 2-regular. The next lemma helps us overcome this difficulty.

Lemma 3.2. *Let $k > 1$ be an integer and let G be an edge-ordered graph with m edges. Any word in $u(G)$ has a k-regular subword of length larger than $m/(k-1)$.*

Proof. Recall that a word in $u(G)$ can be written as $u = a_1 a_2 \dots a_{2m}$, where $a_{2i-1}a_{2i}$ is the i'th edge of G. We apply the following (standard) greedy procedure to obtain a k-regular subword. We start with the empty word u_0 and for $1 \le i \le 2m$ define $u_i = u_{i-1}a_i$ if $u_{i-1}a_i$ is k-regular, or $u_i = u_{i-1}$ otherwise. Clearly $v = u_{2m}$ is a k-regular subword of u.

Consider any edge $e = a_{2i-1}a_{2i}$ of G. Both of the endpoints a_{2i-1}, a_{2i} must appear among the last k letters of u_{2i}, either because we inserted a_{2i-1} or a_{2i} (or both) after u_{2i-2} or because we did not insert them, so they were already among the last $k-1$ letters in u_{2i-2}. Thus e connects two vertices that appear in v at distance at most $k-1$ from each other. As there are fewer than $(k-1)|v|$ pairs of this type, we have $m < (k-1)|v|$ and $|v| > m/(k-1)$ as needed. □

Theorem 3.3. *Any edge-ordered star forest F with $k > 1$ components satisfies*

$$\mathrm{ex}_<(n, F) \le (k-1)\mathrm{ex}_{DS}(n, w'(F)).$$

Proof. Let G be an edge-ordered graph with n vertices and $m = \mathrm{ex}_<(n, F)$ edges that does not contain F. By Lemma 3.1, any word in $u(G)$ avoids $w'(F)$. Any subword of an element of $u(G)$ must also avoid $w'(F)$, among them the k-regular subword of length at least $m/(k-1)$ guaranteed by Lemma 3.2. Note that $k = ||w'(F)||$ and $||u(G)|| \le n$. By the definition of the extremal function $\mathrm{ex}_{DS}(n, w'(F))$ this means that $m/(k-1) \le \mathrm{ex}_{DS}(n, w'(F))$ as required. □

We use this last theorem to prove an almost linear upper bound on $\mathrm{ex}_<(n, F)$ for an arbitrary edge-ordered star forest F and linear upper bound for certain special edge-ordered star forests.

Corollary 3.4. *Any edge-ordered star forest F satisfies*

$$\mathrm{ex}_<(n, F) \le n2^{(\alpha(n))^c},$$

where $\alpha(n)$ is the extremely slow growing inverse Ackermann function and the exponent c depends on F, but not on n.

Further, if $w(F)$ is of the form $a^i b^j a^k b^l$ for two distinct letters a and b and non-negative exponents i, j, k and l, then

$$\mathrm{ex}_<(n, F) = O(n).$$

Proof. We apply Theorem 3.3 for both bounds. The first bound follows because the stated upper bound holds for $\mathrm{ex}_{DS}(n, w)$ for any word w, see [25].

The second bound follows from the fact if $w(F)$ has the form claimed, then $w'(F)$ must also have this form (with different exponents) and by the paper [1] $\mathrm{ex}_{DS}(n, w)$ is linear for such words w.

Note that Theorem 3.3 does not apply if F is a single star, but in this case an edge-ordered graph avoids F if and only if its maximal degree is below the number m of edges in F, so we have $\mathrm{ex}_<(n, F) = \lfloor (m-1)n/2 \rfloor = O(n)$. □

Note that the Turán number of a graph with at least two edges – even without an edge-ordering – is at least $\lfloor n/2 \rfloor$. So the linear upper bound in Corollary 3.4 is tight. It applies to every star forest with two star components and at most four edges. We finish the section by showing that a linear upper bound does not hold for a certain edge-ordering of the star forest consisting of a 2-edge star and a 3-edge star. The result is closely connected to the celebrated result of Hart and Sharir [22] that we can state as $\mathrm{ex}_{DS}(n, ababa) = \Theta(n\alpha(n))$. It is simpler for us, however, to derive our lower bound from a related result of Füredi and Hajnal [18].

Theorem 3.5. *The edge-ordered star forest F consisting of five edges such that the first, third and fifth edges form a star component and the second and fourth edges form another component satisfies*

$$\mathrm{ex}_<(n, F) = \Omega(n\alpha(n)),$$

where $\alpha(n)$ is the inverse Ackermann function.

Proof. Füredi and Hajnal proved in Corollary 7.5 of [18] that there exists an n by n 0-1 matrix A_n with $\Theta(n\alpha(n))$ 1-entries that does not contain a submatrix of the form $\begin{bmatrix} & 1 & & 1 \\ 1 & & 1 & \end{bmatrix}$, where the positions left blank could be arbitrary.

We build a bipartite graph G_n such that A_n is its adjacency matrix. G_n has $2n$ vertices, n of them (the row vertices) corresponding to the rows of A_n, and another n (the column vertices) corresponding to the columns. The edges of G_n correspond to the 1 entries in A_n, so G_n has $\Theta(n\alpha(n))$ edges.

We order the edges of G_n left to right according to the column where the corresponding 1 entry appears. More precisely, an edge e is less than another edge e' if the 1 entry corresponding to e is in a column that is to the left of the column containing the 1 entry corresponding to e'. We order the edges within the same column arbitrarily. We claim that the edge-ordered graph so obtained does not contain F. Assume for a contradiction that it contains F, so a subgraph of G_n (as an edge-ordered graph) is isomorphic to F. We denote the vertices of F by a, b, c_1, c_2, c_3, c_4 and c_5 as depicted in Figure 1.

We denote the corresponding vertices in the subgraph of G_n by the corresponding upper case letters $A, B, C_1, C_2, C_3, C_3, C_4$ and C_5. Notice that the column vertices of G_n are close, but neither central vertex a or b of F is close, therefore A and B must be row vertices. The vertices C_i are adjacent to A or B, so they are column vertices. As the isomorphism preserves the edge-ordering, these columns C_i must appear left to right in order of increasing indices. Rows A and

FIGURE 1. The edge-ordered star forest F

B can be in either order. If row A is below row B, then consider the 2 by 4 submatrix of A_n formed by the rows A and B and the columns $C_1, \dots, C_4$. It is easy to see that this submatrix has a 1 entry in the four specified positions, contradicting the defining property of A_n. In case row A is above row B, a similar contradiction comes from the 2 by 4 submatrix of A_n formed by the rows A and B and the columns $C_2, \dots, C_5$.

The contradiction proves our claim that G_n does not contain F and thus shows that $\mathrm{ex}_<(2n, F)$ is at least the number of edges in G_n, so $\mathrm{ex}_<(2n, F) = \Omega(n\alpha(n))$. Using the monotonicity this implies the stated lower bound on $\mathrm{ex}_<(n, F)$. □

4. PATHS

Let us start with introducing avoidance in an asymmetric bipartite context. It will play an important role in several of our results in this section.

By *edge-ordered bipartite graphs* we mean an edge-ordered graph whose underlying graph is bipartite with a specified bipartition to *left vertices* and *right vertices*. If the edge-ordered graph H has a specified root $x \in V(H)$, then we can distinguish if an edge-ordered bipartite graph G contains H with the root of H being a left vertex or a right vertex. Accordingly, we say that G *left-contains* H if a subgraph of G is isomorphic to H and the vertex corresponding to the root of H is a left vertex in G. Otherwise we say, G *left-avoids* H. Similarly, we say G *right-contains* (or *right-avoids*) H, according to whether G has a subgraph isomorphic to H in which the vertex corresponding to the root of H is a right vertex. For this definition we consider the starting vertex of an edge labeled paths P_k^L to be its root. Note that this definition depends on the presentation of P_k^L, for example P_4^{132} and P_4^{231} are isomorphic, but have different roots, so left-avoiding P_4^{132} is the same as right-avoiding P_4^{231}.

As we have mentioned in the introduction, known results on the altitude of graphs imply a linear upper bound on the number of edges if a monotone labeling of the path P_k is forbidden. First we prove a similar statement for any trees. We will use this to prove Theorem 4.2, which gives a useful upper bound for the Turán numbers of several edge-ordered paths.

We say that a labeling of a rooted tree T is *decreasing* if the labels are decreasing on every branch (that is, on every path starting at the root). We call the labeling *increasing* if the labels are increasing along every branch.

Note that in the next lemma we forbid all increasing (or all decreasing) labelings of a tree, rather than a specific one.

Lemma 4.1. *Let T be a rooted tree of height h with t vertices. If an edge-ordered graph on n vertices does not contain* any *decreasing labeling of T, then it has fewer than htn edges. Moreover, if an edge-ordered bipartite graph G does not left-contain any decreasing labeling of T, then it also has fewer than htn edges.*

The same bounds hold for edge-ordered graphs avoiding (or edge-ordered bipartite graphs left-avoiding) all increasing labelings of T.

Proof. By symmetry, it is enough to deal with graphs avoiding the decreasing labelings of T. Let G be an edge-ordered graph (or edge-ordered bipartite graph, respectively) with n vertices and htn edges. We will prove that G contains (left-contains, respectively) a decreasing labeling of T by induction on h. In case $h = 1$, the average degree is larger than t, so G contains the star T with some labeling, but every labeling is decreasing. In the bipartite case the average degree of left vertices is larger than t, so G left-contains T as well.

For $h > 1$ we delete the t edges with smallest labels incident to every vertex v of G and let G' be the resulting edge-ordered graph. In case a vertex of G has degree less than t, we delete all incident edges. Let us delete the last level from T (the vertices farthest from the root x) and let T' be the resulting tree of height $h - 1$.

T' has fewer than t vertices and G' has at least $htn - tn = (h - 1)tn$ edges. By induction, we can find a decreasing copy of T' in G' (with the root being a left vertex in the bipartite case). We extend this copy of T' to obtain a copy of T in G. At every vertex where we need to add an edge, we select an edge from the t lowest edges at that vertex. These were deleted from G, so the monotonicity will be maintained. We add the edges one by one making sure we do not create cycles: this is possible because at most $t - 1$ of the smallest edges lead to a vertex of G already used. □

Remark. A more careful analysis of the proof gives that if we have t_i vertices on level i, then the upper bound on the number of edges in G can be improved to $n \sum_i (h - i + 1)t_i$.

Using the above lemma, we give a weaker bound for a couple specific orderings of paths. We call a labeling of a path P *monotone* if it is increasing or decreasing when considered with a root at one of the degree 1 vertices.

Theorem 4.2. *Let P be an edge-ordered path with a vertex v that cuts it into two monotone paths P' and P'', such that all labels of P' are smaller than all labels of P''. Then $\mathrm{ex}_<(n, P_k^L) = O(n \log n)$.*

Proof. Let us set $c = 4k^3$, where k is the number of vertices in P. We use induction on n to prove that any edge-ordered graph G with n vertices and more than $cn \log n$ edges contains P.

Assume that our statement holds for smaller values of n and let G be an edge-ordered graph on n vertices and more than $cn \log n$ edges. Our goal is to show that G contains P. Let G_1 be the subgraph of G formed by the set of the $\lceil \frac{c}{2} n \log n \rceil$ smallest edges of G and let G_2 be the subgraph of G formed by the remaining edges.

We consider both P' and P'' as rooted trees with root v. Let T be the rooted tree obtained by identifying the roots of k pairwise disjoint copies of the path underlying of P'. We call a labeling of T *appropriate* if it is a decreasing labeling and the labeling of P' is also decreasing or if it is an increasing labeling and the labeling of P' is also increasing.

Let V_1 be the set of vertices that are roots of appropriately labeled copies of T in G_1. We designate them as right vertices and the rest of the vertices as left vertices. Observe that there are at most $2k^3n$ edges of G_1 that are incident to a left vertex. Indeed, the subgraph of G_1 induced by the left vertices avoids all appropriate labelings of T, so it has at most k^3n edges by Lemma 4.1, while the edge-ordered bipartite graph formed by the edges of G_1 between left and right vertices left-avoids all appropriate labelings of T, so it has also at most k^3n edges by the same lemma.

This implies that the subgraph of G_1 induced by V_1 has at least $\frac{c}{2}n\log n - 2k^3n$ edges. It avoids P, so by induction it has at most $c|V_1|\log|V_1|$ edges. Therefore, we must have $|V_1| > n/2$.

Let V_2 be the set of vertices that are roots of an isomorphic copy of P'' in G_2. A similar argument shows that we must have $|V_2| > n/2$. This implies there is a vertex $x \in V_1 \cap V_2$. Consider an isomorphic copy P^* of P'' in G_2 rooted at x. Also, consider an appropriately labeled copy T^* of T in G_1 rooted at x. T^* has k branches, at least one of them does not meet P^* outside the common root. Clearly, the union of this branch with P^* is an isomorphic copy of P in G. □

Remark. Let T be an edge-ordered tree with a single vertex v of degree larger than 2. We call the maximal paths starting at v the branches of T. A similar proof shows that if the branches are monotone and the edges of the branches form intervals in the edge-ordering, then $\mathrm{ex}_<(n,T) = O(n\log n)$.

4.1. **Edge-ordered paths with three edges.** The path P_4 has three non-isomorphic labelings: P_4^{123}, P_4^{132} and P_4^{213}. This section is about their Turán numbers. We determine $\mathrm{ex}_<(n, P_4^{132})$ and $\mathrm{ex}_<(n, P_4^{213})$ exactly and $\mathrm{ex}_<(n, P_4^{123})$ up to an additive constant. First we prove a simple graph theoretical lemma that will be used for the proof of both results.

Lemma 4.3. *Let G be a simple graph with $n \geq 1$ vertices and m edges that does not contain a cycle of length 4 or more. Then $m \leq \frac{3}{2}(n-1)$.*

Proof. We use induction by n. If there is no triangle in G, then G is a forest and therefore $m \leq n-1$ and we are done. Otherwise, it has a triangle abc. Let G' be the graph obtained by removing the edges of this triangle from G. The vertices a, b and c fall in distinct components of G' as any path connecting them in G' could be extended by two edges of the triangle to a cycle of length at least four in G. Let G_a and G_b denote the connected component of the vertices a and b in G', respectively, and let G_c be the subgraph of G' formed the remaining components. By the inductive hypothesis on these graphs we have

$$m = |E(G_a)|+|E(G_b)|+|E(G_c)|+3 \leq \frac{3}{2}(|V(G_a)|-1)+\frac{3}{2}(|V(G_b)|-1)+\frac{3}{2}(|V(G_c)|-1)+3 = \frac{3}{2}(n-1).$$

□

Theorem 4.4. $\mathrm{ex}_<(n, P_4^{132}) = \mathrm{ex}_<(n, P_4^{213}) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor$

Proof. By symmetry (reversing the edge-order) it is enough to deal with P_4^{132}.

Consider any labeling of a cycle of length at least four. The subgraph formed by the largest edge in the cycle and its two adjacent edges is isomorphic to P_4^{132}. Thus, if an edge-ordered graph avoids P_4^{132}, then its underlying simple graph has no cycle of length at least four. The upper bound follows from Lemma 4.3.

Now we will show that for every n, there is an edge labeled graph G with $\lfloor \frac{3}{2}(n-1) \rfloor$ edges that avoids P_4^{132}. Let us obtain G from an n-vertex star by adding to it a matching of size $\lfloor \frac{n-1}{2} \rfloor$ connecting leaves of the star. We have $|E(G)| = n-1+\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{3}{2}(n-1) \rfloor$ as needed. Label the edges in such a way that the edges of the original star receive the smallest labels. It is easy to check that the middle edge of any 3-edge path in G is from the original star but the path has to also contain an edge outside this star. Therefore, G avoids P_4^{132}. ☐

Theorem 4.5. $\mathrm{ex}_<(n, P_4^{123}) \le \frac{3n}{2}$ *holds for any* $n \ge 1$ *and it holds with equality if and only if* n *is divisible by* 4.

Proof. We start by describing two classes of graphs that have a monotone path of length 3 in any labeling. Odd cycles of length 5 or more are like that. Indeed, going around the cycle we can note if the label increases or decreases going from one edge to the next. This two cannot alternate because the cycle is odd, so we have two consecutive increases or two consecutive decreases. The three edges involved form a monotone path.

Now assume that a graph G has four vertices A, B_1, B_2 and B_3 such that A is connected to all three of $\{B_1, B_2, B_3\}$ and all three of $\{B_1, B_2, B_3\}$ has a neighbor not in $\{A, B_1, B_2, B_3\}$. (These neighbors may or may not coincide.) Any labeling G^L contains a 3-edge monotone path. Indeed, we can assume by symmetry that $L(AB_1) < L(AB_2) < L(AB_3)$. Let C be a neighbor of B_2 with $C \notin \{A, B_1, B_3\}$. If $L(B_2C) < L(AB_2)$, then B_3AB_2C is a monotone path, otherwise B_1AB_2C is.

We use induction on n to prove the upper bound. The statement is trivial for $n \le 4$. Now assume that $n \ge 5$. Let G be a graph with n vertices and m labeled edges with no monotone path of length 3.

If there is no cycle of length at least 4 in G, then Lemma 4.3 implies $m \le \frac{3}{2}(n-1) < \frac{3n}{2}$. So G contains a cycle of length at least 4. Let C be a such a cycle of minimal length t. By our first observation, t cannot be odd, so it is even.

First, assume that there is a vertex $A \in C$ connected to some vertex $B_1 \notin C$. Let B_2 and B_3 be the neighbors of A in C. Then B_1 cannot be connected to B_2 or B_3, since that would create an odd cycle of length $t+1$. B_1 cannot be connected to a vertex not in $\{A, B_2, B_3\}$ either, since this would create the other type of forbidden subgraph we described before. Therefore the only neighbor of B_1 is A. Let us delete B_1 from G. By induction the remaining graph has at most $\frac{3}{2}(n-1)$ edges, therefore $m \le \frac{3}{2}(n-1)+1 < \frac{3n}{2}$.

Now assume that there is no edge connecting a vertex of C to a vertex not in C, that is the vertices of C form a component of G. The rest of the graph contains at most $\frac{3}{2}(n-t)$ edges by induction. If $t > 4$, then C must be an induced cycle as a chord in C would create a shorter cycle still of length at least 4, so we have $m \le t + \frac{3}{2}(n-t) < \frac{3}{2}n$. Finally if $t = 4$, then the component of C can contain at most 6 edges and we have $m \leq 6 + \frac{3}{2}(n-4) = \frac{3}{2}m$. We can only have equality in this case and (by induction) only if all components of G are cliques of size 4.

To completely characterize the cases of equality in the theorem, it is enough to show that a disjoint union of copies of K_4 can be labeled in a way avoiding P_4^{123}. Clearly, it is enough to label one component. A labeling of K_4 avoids P_4^{123} if and only if both the two smallest and the two largest labels are given to pairs of independent edges. ☐

Corollary 4.6. $6\lfloor \frac{n}{4} \rfloor = \mathrm{ex}_<(4\lfloor \frac{n}{4} \rfloor, P_4^{123}) \leq \mathrm{ex}_<(n, P_4^{123}) \leq \frac{3n}{2}$, *which determines* $\mathrm{ex}_<(n, P_4^{123})$ *up to an additive constant.*

We remark that the additive constant in the above corollary can be removed with a small additional effort to obtain an exact result. We just sketch the proof here. The lower bound in Corollary 4.6 comes from the edge-ordered graph which makes the bound in Theorem 4.5 tight on $4\lfloor n/4 \rfloor$ vertices and up to 3 isolated vertices. We can add a complete graph instead of the isolated vertices and this improves the lower bound by 1 or 3 if $n = 4k+2$ or $n = 4k+3$, respectively. (The edge-order of the additional edges do not matter.) To prove that this improved lower example is tight, it is enough to realize that the simple graph K_4^+ consisting of a complete graph K_4 and a single additional edge connecting a vertex of K_4 with a new leaf must contain P_4^{123}.

4.2. **Edge-ordered paths with four edges.** The labelings (or edge-orderings) of P_5 are given by permutations of $\{1,2,3,4\}$. However, two reverse permutations (e.g., 1324 and 4231) yield isomorphic labeled graphs. Also, the Turán number remains the same if we reverse the edge-ordering. For example if G is a P_5^{1243}-free labeled graph, then reversing the edge-ordering in G gives a P_5^{4312}-free graph. Therefore, the Turán numbers of the two or four labelings are equal in each of the eight classes in the following table. For each of these equivalence classes, we summarize the upper and lower bound we prove on $\mathrm{ex}_<(n, P_5^L)$.

Turán numbers of edge-ordered paths with four edges			
Labeling	Lower bound	Upper bound	Proved in
$\{1234, 4321\}$	$\Omega(n)$	$O(n)$	Prop. 4.7 (i)
$\{1243, 3421, 4312, 2134\}$	$\Omega(n)$	$O(n)$	Prop. 4.7 (ii)
$\{1324, 4231\}$	$\Omega(n \log n)$	$O(n \log n)$	Thm. 4.12
$\{1432, 2341, 4123, 3214\}$	$\Omega(n \log n)$	$O(n \log n)$	Thm. 4.10
$\{2143, 3412\}$	$\Omega(n \log n)$	$O(n \log n)$	Thm. 4.9 (ii)
$\{1342, 2431, 4213, 3124\}$	$\Omega(n \log n)$	$O(n \log^2 n)$	Thms. 4.9 (i), 4.14
$\{2413, 3142\}$	$\binom{n}{2}$	$\binom{n}{2}$	Prop. 4.8 (ii)
$\{1423, 3241, 4132, 2314\}$	$\binom{n}{2}$	$\binom{n}{2}$	Prop. 4.8 (i)

Proposition 4.7. ***(i)*** $\mathrm{ex}_<(n, P_5^{1234}) = \Theta(n)$

(ii) $\mathrm{ex}_<(n, P_5^{1243}) = \Theta(n)$

Proof. The lower bounds are obvious in both cases. We mentioned earlier the linear upper bound for monotone paths of any length. That implies the upper bound in **(i)** but we prove it together with **(ii)** to obtain the same upper bound of $9n/2$ that is stronger than what follows from the earlier proof.

Let us consider an edge-ordered graph G on n vertices with more than $9n/2$ edges. Our goal is to prove that G contains both P_5^{1243} and P_5^{1234}. For every vertex v of G, we remove the smallest three edges incident to v (or all incident edges if the degree of v is less than 3). This way we remove at most $3n$ edges, thus the resulting graph G' has more that $3n/2$ edges.

By Theorem 4.4, G' contains P_4^{132}. Let v_1, v_2, v_3, v_4 be the vertices of a subgraph of G' isomorphic to P_4^{132}, so v_1v_2, v_3v_4 and v_2v_3 are edges of G' ordered in this order. Recall that we removed the three smallest edges incident to v_1 from G. The other endpoint of at least one of these three edges is different from v_3 and v_4. Choosing such a vertex u as a starting vertex, we obtain the path $uv_1v_2v_3v_4$ in G, and its labeling makes it isomorphic to P_5^{1243}.

Observe that G' also contains a P_4^{123} by Theorem 4.5, and then the same reasoning as above yields that G also contains P_5^{1234}. ☐

Note that by Theorem 2.3 the next statement is equivalent to $\chi_<(P_5^{1423}) = \chi_<(P_5^{2413}) = \infty$.

Proposition 4.8. ***(i)*** $\mathrm{ex}_<(n, P_5^{1423}) = \binom{n}{2}$

(ii) $\mathrm{ex}_<(n, P_5^{2413}) = \binom{n}{2}$

Proof. This follows directly from the fact that the max-labeling of K_n avoids both P_5^{1423} and P_5^{2413}. This last statement follows from Proposition 2.9 as neither end vertex of the largest edge is close in either of the edge-ordered paths P_5^{1423} and P_5^{2413}. ☐

We prove several of the lower bounds of the form $\mathrm{ex}_<(n, P) = \Omega(n \log n)$ by constructing edge-ordered graphs G_i avoiding P such that G_i has 2^i vertices and $\Omega(i2^i)$ edges. This is enough by the monotonicity of $\mathrm{ex}(n, P)$. Indeed, if P has no isolated vertices than one can add isolated vertices to any edge-ordered graph avoiding P to obtain an edge-ordered graph on more vertices and the same number of edges, still avoiding P. So in the situation above we have $\mathrm{ex}_<(n, P) \geq \mathrm{ex}_<(2^{\lfloor \log n \rfloor}, P) = \Omega(n \log n)$.

Theorem 4.9. ***(i)*** $\mathrm{ex}_<(n, P_5^{1342}) = \Omega(n \log n)$

(ii) $\mathrm{ex}_<(n, P_5^{2143}) = \Theta(n \log n)$

Proof. The upper bound of **(ii)** follows from Theorem 4.2.

To prove **(i)**, we build the P_5^{1342}-free edge-ordered graphs G_i recursively. Let G_0 be a single vertex. To construct G_{i+1} we take two copies of G_i and add a perfect matching M between the two copies. Note that we can take an arbitrary perfect matching. We keep the order of the edges within both copies of G_i, but make all edges in one copy (the large copy) larger than any edge in the other copy (the small copy). We further make all edges in the matching M larger than any other edge. The order among the matching edges is arbitrary.

Clearly, G_i has 2^i vertices and $i2^{i-1}$ edges. It remains to prove that it avoids P_5^{1342}. We do this by induction on i. The statement trivially holds for G_0, so assume it holds for G_i and assume for a contradiction that an isomorphic copy P_5^{1342} shows up in G_{i+1} formed by edges e_1, e_2, e_3 and e_4 with the edge-ordering $e_1 < e_4 < e_2 < e_3$. It cannot be completely inside a copy of G_i by the inductive hypothesis, but it is connected, so it has to contain an edge from M. As e_3 is the largest of the four edges it must come from M and therefore e_2 and e_4 (being incident to distinct end points of e_3) must come from the two separate copies of G_i, e_2 coming from the large copy and e_4 from the small copy. The edge e_1 is adjacent to e_2 from the large copy but smaller than e_4 from the small copy, a contradiction.

The lower bound of **(ii)** is given by a similar recursive construction. We construct the P_5^{2143}-free edge-ordered graphs G'_i similarly. G'_0 is a single vertex, and G'_{i+1} is obtained by connecting two disjoint copies of G'_i by a perfect matching M, but the edge-ordering is different. We still keep the edge-orderings inside both copies of G'_i and make all edges of one copy larger than any edge of the other copy, but this time the edges of M will be intermediate: larger than the edges in the small copy of G'_i and smaller than the edges in the large copy. We can choose the perfect matching M arbitrarily and the order of the edges inside M is arbitrary too.

We still have that G'_i has 2^i vertices and $i2^{i-1}$ edges. For the inductive proof that these graphs avoid P_5^{2143} assume for a contradiction that G'_i avoids it but G'_{i+1} has an isomorphic copy formed by the edges e_1, e_2 e_3 and e_4 with the edge-ordering $e_2 < e_1 < e_4 < e_3$. Here G'_{i+1} consist of two copies of G'_i connected by a matching M. If $e_2 \in M$, then two adjacent edges e_1 and e_3 are in different copies of G'_i, thus one of them should be smaller than e_1, a contradiction. Similarly, if $e_3 \in M$, then one of its adjacent edges e_2 or e_4 should be larger than e_4, a contradiction. So e_2 and e_3 are in one of the copies of G'_i and as they are adjacent, it is the same copy. As the other two edges are in between them in the ordering they should also be in the same copy of G'_i contradicting the inductive assumption that G'_i avoids P_5^{2143}. □

The edge-ordered graphs G_i and G'_i in the proof above are not well defined as we made several arbitrary choices in their constructions. But it is instructive to observe that if we choose the connecting matchings in the most natural way in each step (namely, connecting the same vertex in the two copies of G_{i-1} or G'_{i-1}), then the underlying simple graphs of both G_i and G'_i are the i-dimensional hypercube.

Theorem 4.10. $\mathrm{ex}_<(n, P_5^{1432}) = \Theta(n \log n)$

Proof. The upper bound follows from Theorem 4.2. It can also be derived from Lemma 4.1(c) in [41]. To prove the lower bound we use a result of Füredi [17]: there exist $n \times n$ 0-1 matrices A_n that do not contain the submatrix $\begin{bmatrix} & 1 & 1 \\ 1 & & 1 \end{bmatrix}$ (with arbitrary entries in the two places left blank) and have $\Omega(n \log n)$ 1 entries.

We build a bipartite graph G_n such that A_n is its adjacency matrix. G_n has $2n$ vertices, n of them (the row vertices) corresponding to the rows of A_n, and another n (the column vertices) corresponding to the columns. The edges of G_n correspond to the 1 entries in A_n, so G_n has $\Omega(n \log n)$ edges. Now we make G_n into an edge-ordered graph by ordering its edges according to the corresponding 1 entries in A_n. If two 1 entries are in distinct columns, the one to the right is larger. If they are in the same column, then the one lower in the column is larger.

It remains to prove G_n avoids P_5^{1432}. Assume for a contradiction that an isomorphic copy of P_5^{1432} shows up in G_n. Notice that column vertices are close, but the common vertex of the first two edges in P_5^{1432} is not close, so it must correspond to a row vertex. This means that two rows and three columns corresponding to the five vertices of the copy of P_5^{1432} form exactly the forbidden type of submatrix, a contradiction. □

Our next lemma will be used in proving Theorem 4.12. Recall that we introduced edge-ordered bipartite graphs as edge-ordered graphs whose underlying graph has a specified bipartition to left and right vertices. When an edge-ordered bipartite graph G contains an edge-ordered graph H with a specified root (in case of paths it is just the first vertex), we distinguish left-containment or right-containment according to the position of the root of H in G.

Lemma 4.11. *The maximal number of edges an edge-ordered bipartite graph on n vertices that right-avoids both P_4^{132} and P_4^{213} can have is $\Theta(n \log n)$.*

Proof. The proof of the lower bound is similar to the recursive construction of the lower bounds in Theorem 4.9. We build the edge-ordered bipartite graphs G_i recursively. The graph G_i will have 2^{i-1} left vertices, 2^{i-1} right vertices and $(i+1)2^{i-2}$ edges.

We start with G_1 being (the only edge-ordering of) K_2 and we designate one of the vertices left, the other right. For $i \geq 1$ we construct G_{i+1} as follows. We take two disjoint copies of G_i and connect them by a perfect matching M between the left vertices of the first copy and the right vertices of the second copy. We keep the order of the edges within either of the two copies of G_i and make the edges of M larger than the edges in the first copy of G_i and smaller than the edges in the second copy of G_i. Note that the matching M and the order of edges within M is arbitrary. A left vertex of either copy of G_i will also be a left vertex of G_{i+1}, while a right vertex of either copy of G_i is also a right vertex of G_{i+1}.

We claim that G_i right-avoids P_4^{132} for all i. We prove this by induction on i. It trivially holds for G_1. Assume that G_i right-avoids P_4^{132} but we still find a copy P of P_4^{132} in G_{i+1} formed by the edges e_1, e_2 and e_3 ordered as $e_1 < e_3 < e_2$. We need to prove it starts at a left vertex. If P is contained in one of the copies of G_i, then it starts as a left vertex by the inductive hypothesis. Otherwise one of the edges in P is in the matching M. Notice that M is an induced matching (no edge of G_{i+1} connects two edges in M), so exactly one edge of P is in M. It cannot be e_2 as one of the other two edges of P would then be in the second copy of G_i and would be larger than e_2. The matching edge cannot be e_3 either as then e_1 and e_2 (being smaller and larger than e_3, respectively) would be in different copies of G_i and could not be adjacent. So e_1 must be in M, and then then e_2 and e_3 (being larger than e_1) are in the second copy of G_i, so P starts in the left side of the first copy of G_i proving the claim.

A similar inductive proof shows that G_i right-avoids P_4^{213} for all i. Indeed, if G_i right-avoids P_4^{213} but an isomorphic copy P of P_4^{213} consisting of the edges e_1, e_2, and e_3 in the edge-ordering $e_2 < e_1 < e_3$ shows up in G_{i+1}, then either P is contained in a single copy of G_i and then it starts at a left vertex or exactly one of its edges is in the matching M. As above, the edge in M cannot be e_1 or e_2, so it must be e_3 with e_1 and e_2 from the first copy of G_i, but then P starts at a left vertex again. This finishes the proof of the lower bound.

For the upper bound we consider an edge labeled bipartite graph G^L on n vertices that right-avoids both P_4^{132} and P_4^{213}. That is, G is a bipartite graph with a given bipartition to left and right vertices and L is an injective labeling $L : E(G) \to \mathbb{R}$ specifying the edge-ordering. As only the relative order of the labels matters we can assume without loss of generality that the labels are integers between 1 and $|E(G)|$. This assumption is needed because in the proof below we compare not only the labels but also distances between labels.

For a non-isolated left vertex x in G we write $m(x)$ for the minimal label $L(e)$ of an edge e incident to x. For a non-isolated right vertex y in G we write $M(y)$ for the maximal label $L(e)$ for an edge e incident to y. For an edge $e = xy$ of G with x a left vertex and y a right vertex we call $L(e) - m(x)$ the *left-weight* of e and $M(y) - L(e)$ is the *right-weight* of e. Note that these are non-negative integers less than $|E(G)| < n^2$. We call the edge *left-leaning* if its left-weight is larger than its right-weight, otherwise it is *right-leaning*.

Let xy_1 and xy_2 be two edges of G incident to the same left vertex x and assume $L(xy_2) > L(xy_1)$. Then the left weight of xy_2 is larger than the left-weight of xy_1. We must further have $M(y_1) < L(xy_2)$ as otherwise the path y_2xy_1 followed by the edge of label $M(y_1)$ would show that G right-contains P_4^{213}, contrary to our assumption. If we further assume that xy_1 is right-leaning, then we have $L(xy_2) - m(x) > M(y_1) - m(x) \geq 2(L(xy_1) - m(x))$, so the left-weight of xy_2 is more than twice of that of xy_1. As all these left-weights are non-negative integers below n^2, this implies that there are $O(\log n)$ right-leaning edges of G incident to x. The total number of right-leaning edges is therefore $O(n \log n)$.

To obtain a similar bound for left-leaning edges, let x_1y and x_2y be two edges of G incident to the same right vertex y with $L(x_1y) < L(x_2y)$. We have $m(x_2) > L(x_1y)$ as otherwise the path starting with the edge of label $m(x_2)$ continued by x_2yx_1 would show that G right-contains P_4^{132}. So if x_2y is left-leaning, then we have $M(y) - L(x_1y) > M(y) - m(x_2) \geq 2(M(y) - L(x_2y))$, so the right-weight of x_1y is more than twice the right-weight of x_2y. As before, this implies that the number of left-leaning edges incident to y is $O(\log n)$ and the total number of left-leaning edges in G is $O(n \log n)$. As every edge of G is either left- or right-leaning, G has $O(n \log n)$ edges. This finishes the proof of the upper bound. □

Theorem 4.12. $\mathrm{ex}_<(n, P_5^{1324}) = \Theta(n \log n)$

Proof. Consider an edge-ordered bipartite graph G that right-contains P_5^{1324}. The first three edges of the isomorphic copy of P_5^{1324} forms an isomorphic copy of P_4^{132}, so G also right-contains P_4^{132}. Similarly, if G left-contains P_5^{1324}, then the last three edges of the isomorphic copy of P_5^{1324} is isomorphic to P_4^{213}, so G right-contains P_4^{213}. Therefore, if an edge-ordered bipartite graph G right-avoids both P_4^{132} and P_4^{213}, then G both right- and left-avoids P_5^{1324}, so it avoids P_5^{1324}. By Lemma 4.11 such edge-ordered bipartite graphs G exist with n vertices and $\Omega(n \log n)$ edges proving the lower bound in the theorem.

For the upper bound we will also use Lemma 4.11 but we need a more involved deduction. Let G^L be an edge-ordered graph with n vertices and m edges avoiding P_5^{1324}. Our goal is to prove $m = O(n \log n)$.

First we partition the vertex set of G to left and right vertices and consider the bipartite subgraph G' of G formed by the edges between the left and the right vertices. We can do the partition in such a way, that G' contains at least $m/2$ edges. We remove the edge with the minimal label incident to every vertex, and obtain the subgraph G'' with the edge set E. We clearly have $|E| \geq m/2-n$. When saying that E or a subset of E avoids (or left- or right-avoids) a pattern we mean the statement for the edge-ordered bipartite subgraph of G' formed by those edges. In particular, E avoids both P_5^{1324} and C_4^{1324}. It avoids the former because the entire edge-ordered graph G avoids it. Assume a copy of C_4^{1324} shows up in E. Let x be the vertex incident with the edges with the smallest and third smallest label in this cycle. We have deleted from G' the edge xy with the minimal label incident to x. As G' is bipartite, y is not on the four-cycle, thus we can replace the edge with the lowest label in the cycle with xy to obtain a copy P_5^{1324} in G', a contradiction.

For a non-isolated vertex x of G'' let $m(x)$ (respectively, $M(x)$) stand for the minimal (respectively, maximal) label $L(e)$ of an edge $e \in E$ incident to x. For an edge $xy \in E$ with x a left vertex and y a right vertex we write $S(xy)$ (respectively, $T(xy)$) for the set of edges $x'y \in E$ with $m(x) < L(x'y) < L(xy)$ (respectively, with $L(xy) < L(x'y) < M(x)$). Let $S = \{e \in E \mid |S(e)| \leq 1\}$ and $T = \{e \in E \mid |T(e)| \leq 1\}$.

Let us form an auxiliary graph with the vertex set S by connecting $e \in S$ to the at most one element in $S(e) \cap S$. Now e is connected to at most one other edge of label less than $L(e)$, so this auxiliary graph is a forest. Forests are bipartite, so we can partition S into the independent sets S_1 and S_2. Note that if $y'xyx'$ is an isomorphic copy of P_4^{132} starting at a right vertex y', then $m(x) \leq L(xy') < L(x'y) < L(xy)$, therefore $x'y \in S(xy)$. This means, that S_1 and S_2, being independent sets in the auxiliary graph cannot contain such a path, so both S_1 and S_2 right-avoids P_4^{132}.

Similarly, the auxiliary graph on the vertex set T, where $e \in T$ is connected to the at most one element of $T(e) \cap T$ is a forest, so T can be partitioned into the independent sets S_3 and S_4. As above, both S_3 and S_4 left-avoid P_4^{213} because the first two edges of any left-starting copy of P_4^{213} in T are connected in this auxiliary graph.

Let y be a right vertex and let xy and $x'y$ be edges in E with $L(xy) < L(x'y)$. Extend the path $x'yx$ at x' with the edge of label $m(x')$ and at x with the edge of label $M(x)$. Unless $m(x') > L(xy)$ or $M(x) < L(x'y)$ we obtain a copy of P_5^{1324} or (if the two edges added are adjacent) a copy of C_4^{1324}. As E avoids both of these patterns we must have $m(x') > L(xy)$ or $M(x) < L(x'y)$. Let $H = \{x''y \in E \mid L(xy) < L(x''y) < L(x'y)\}$. If $m(x') > L(xy)$, then $S(x'y) \subseteq H$, while if $M(x) < L(x'y)$, then $T(xy) \subseteq H$. Assume now that $|H| \leq 1$ (that is, xy and $x'y$ are neighbors or second neighbors in the ordering of the edges at y), and we conclude that $x'y \in S$ or $xy \in T$.

Arrange the edges in E incident to the right vertex y according to their labels. By the previous paragraph, if two edges are consecutive or second neighbors in this list, then one of them must be in $S \cup T$. As a consequence we have $|S \cup T| \geq 2|E|/3 - n$. Note that $S \cup T$ can be covered by four sets (namely S_1, S_2, S_3 and S_4), each of which either right-avoids P_4^{132} or left-avoids P_4^{213}.

Notice that we kept the left-right symmetry when defining the edge set E, so the statements in the last paragraph have their mirror images too. In particular, there exists a set $U \subseteq E$ with $|U| \geq 2|E|/3 - n$ such that $U = U_1 \cup U_2 \cup U_3 \cup U_4$ and each set U_i either left-avoids P_4^{132} or right-avoids P_4^{213}.

But now we have $|(S \cup T) \cap U| \geq |E|/3 - 2n$ and $(S \cup T) \cap U = \bigcup_{i,j}(S_i \cap U_j)$. Here 8 out of the 16 intersections satisfies that $S_i \cap U_j$ (left- and right-avoids and therefore) avoids either P_4^{132} or P_4^{213}, in which case $|S_i \cap U_j| = O(n)$ by Theorem 4.4, while in another 8 cases $S_i \cap U_j$ either right-avoids both of P_4^{132} and P_4^{213} or left-avoids both of them. We have $|S_i \cap U_j| = O(n \log n)$ in the right-avoiding case by Lemma 4.11 and the same bound holds by symmetry in the left-avoiding case.

Summarizing, we must have $|(S \cup T) \cap U| = O(n \log n)$ and therefore $|E| = O(n \log n)$ and finally we must also have $m = O(n \log n)$ for the number m of edges in G. This finishes the proof of the upper bound. □

To prepare for our final result about four edge paths, namely Theorem 4.14, we start with the following lemma. Note that while we do not expect Theorem 4.14 to be tight, this lemma is tight. Indeed, a slight modification of the construction given in the proof of Theorem 4.9(i) for the edge-ordered graphs G_i avoiding P_5^{1342} yields edge-ordered bipartite graphs avoiding P_5^{1342} and also right-avoiding P_4^{132}. One only has to maintain a bipartition of the constructed graph G_i to an equal number of left and right vertices and (when constructing G_{i+1} from G_i) to restrict the matching to connect the right vertices in the smaller copy of G_i to the left vertices in the larger copy.

Also note that this lemma follows directly from Lemma 4.1(b) in [41]. We include the proof to be self contained.

Lemma 4.13. *If an edge-ordered bipartite graph on n vertices avoids P_5^{1342} and right-avoids P_4^{132}, then it has $O(n \log n)$ edges.*

Proof. Let H^L be the bipartite edge labeled graph on n vertices that avoids P_5^{1342} and right-avoids P_4^{132}. Our goal is to bound the number of edges in H. As in the proof of the upper bound in Lemma 4.11 we will compare differences between labels of edges, and to make this meaningful we assume L takes integer values between 1 and n^2.

First we delete the smallest labeled edge incident to each non-isolated vertex of H^L to obtain the subgraph H'. We lose less than n edges and H' avoids C_4^{1342}. Indeed, if a copy of C_4^{1342} showed up in H' we could replace the smallest edge in the cycle with one of the edges not in H' to obtain a copy of P_4^{1342} in H^L, a contradiction. (Note, we did exactly the same thing in the proof of Theorem 4.12 to obtain a large subgraph of a graph avoiding P_5^{1324} that avoids C_4^{1324}.)

For each non-isolated right vertex y in H' we define $m(y)$ to be the smallest label of an edge of H' incident to y. When referring to an edge xy of H' we will always assume x is a left vertex and y is a right vertex. With this notation we define the *weight* of an edge xy of H' to be $w(xy) = L(xy) - m(y)$. We call the edge xy of H' *minimal* if $w(xy) = 0$, otherwise we define $n(xy)$ to be the label of the "next smallest label at y", that is $n(xy) = \max L(x'y)$, where the maximum is taken for edges $x'y$ in H' with $L(x'y) < L(xy)$. For a non-minimal edge xy of H'

we compare $L(xy) - n(xy)$ and $n(xy) - m(y)$. If the former is larger, then xy is *light*, otherwise it is *heavy*.

The weight of light edge is more than twice of the weight of any other edge of smaller weight incident to the same right vertex. Therefore, the number of light edges incident to any one right vertex is $O(\log n)$ and the total number of light edges in H' is $O(n \log n)$. Clearly, the number of minimal edges in H' is at most n, while the number of edges of H not in H' is also at most n. To finish the proof of the lemma it remains to limit the number of heavy edges.

Take two heavy edges from the same left vertex x: xy and xy' with $L(xy) < L(xy')$. Extend the 2-edge path yxy' at y' with the edge labeled $n(xy')$. As H^L right-avoids P_4^{132} we must have $n(xy') < L(xy)$. We further extend the 3-edge path at y with the edge labeled $m(y)$. If $m(y) < n(xy')$, we obtain an isomorphic copy of P_5^{1342} or C_4^{1342}. As H' avoids both we must have $m(y) > n(xy')$. As xy' is heavy we have $w(xy') \geq 2(L(xy') - n(xy'))$. But $L(xy') - n(xy') > L(xy) - m(y) = w(xy)$. So the weight doubles from one heavy edge incident to a given left vertex x to the next heavy edge. Therefore, the number of heavy edges incident to x is $O(\log n)$ and the total number of heavy edges in H' is $O(n \log n)$, proving the lemma. □

Theorem 4.14. $\mathrm{ex}_<(n, P_5^{1342}) = O(n \log^2 n)$

Proof. Let G be an edge-ordered graph avoiding P_5^{1342} on n vertices with a maximal number of $m = \mathrm{ex}_<(n, P_5^{1342})$ edges. Let L be the graph formed by the $\lfloor m/2 \rfloor$ smallest edges (lower half) of G and let U be the subgraph formed by the remaining edges (upper half) of G. We call a vertex a *left vertex* it has at least 3 incident edges in L, otherwise it is a *right vertex.* (Note that neither L nor U must be bipartite though.)

Clearly, L has at most $3n$ edges not between two left vertices, so at least $\lfloor m/2 \rfloor - 3n$ edges between left vertices.

We claim that U does not contain an isomorphic copy of P_4^{132} that ends at a left vertex. Indeed, such a copy could be extended at its end with an edge from L. We have at least three choices for this last edge, so at least one yields a simple four edge path and that would be isomorphic to P_5^{1342}, a contradiction.

Let U_{bip} be the edge-ordered bipartite graph consisting of the edges in U between a left and a right vertex. As a subgraph of G it avoids P_5^{1342} and as shown above it also right-avoids P_4^{132}, so by Lemma 4.13, U_{bip} contains $O(n \log n)$ edges.

The edges of U between left vertices form an edge-ordered graph avoiding P_4^{132} by the same claim above. Thus, by Theorem 4.4 $O(n)$ edges of U connect two left vertices.

By the previous two paragraphs, there are $m/2 - O(n \log n)$ edges of U connecting right vertices. They form an edge-ordered graph avoiding P_5^{1342} just as the $m/2 - O(n)$ edges of L between left vertices do. We have either at most $\lfloor n/2 \rfloor$ left vertices or at most at most $\lfloor n/2 \rfloor$ right vertices, and in either case we must have $\mathrm{ex}_<(\lfloor n/2 \rfloor, P_5^{1342}) \geq m/2 - O(n \log n)$. We can rewrite this as $\mathrm{ex}_<(n, P_5^{1342}) \leq 2\mathrm{ex}_<(\lfloor n/2 \rfloor, P_5^{1342}) + O(n \log n)$. This recursion solves to $\mathrm{ex}_<(n, P_5^{1342}) = O(n \log^2 n)$, as claimed. □

4.3. **Longer paths.** Some of our results above directly imply bounds for longer edge-ordered paths as well. For example, we have a linear upper bound for the Turán numbers of monotone paths of any length and Theorem 4.2 gives an upper bound for the Turán numbers of some other edge-orderings of longer paths. Some of our constructions in the previous section can be shown to avoid more edge-ordered paths than what was shown. See more on this in Section 6.2.

Our results in the previous sections imply that all edge-orderings of P_4 have order chromatic number 2 and all edge-orderings of P_5 have order chromatic number 2 or infinity. Here we show that this does not remain the case for the edge-orderings of P_6. By Theorem 2.3, our result below implies that $\mathrm{ex}_<(n, P_6^{14325}) = n^2/4 + o(n^2)$. This shows a very different asymptotic behaviour compared to the Turán numbers for shorter edge-ordered paths.

Theorem 4.15. $\chi_<(P_6^{14325}) = 3$

Proof. The inequality $\chi_<(P_6^{14325}) \geq 3$ follows directly from Proposition 2.9. Indeed P_6^{14325} has a single proper two-coloring and both color class contains a vertex that is not close.

For the inequality in the reverse direction we use Theorem 2.5. It is enough to show that all canonical edge-orders of $K_{3\times3}$ contain P_6^{14325}. As we have observed in Section 2.1, in all canonical edge-orders for $K_{3\times3}$ either one of the three parts precedes the other two parts or one of the three parts is preceded by the other two parts. (We have already used this fact in the proof of Proposition 2.10.) In the former case we can find an isomorphic copy of P_5^{1432} in the minimal part and then we can extend it to P_6^{14325} by an edge outside this part. In the latter case we find an isomorphic copy of P_5^{4325} in the maximal part and extend it to P_6^{14325} using an edge outside this part. $\square$

5. 4-cycles

The four edge cycle C_4 has three non-isomorphic edge-orderings. The only one which embeds into a max-labeled clique is C_4^{1243}, therefore $\chi_<(C_4^{1234}) = \chi_<(C_4^{1324}) = \infty$ and

$$\mathrm{ex}_<(n, C_4^{1234}) = \mathrm{ex}_<(n, C_4^{1324}) = \binom{n}{2}.$$

In this section we improve the upper bound $\mathrm{ex}_<(n, C_4^{1243}) = O(n^{5/3})$ proved in [20]. Our proof is inspired by some ideas of [30]. Note the simple lower bound $\mathrm{ex}_<(n, C_4^{1243}) \geq \mathrm{ex}(n, C_4) = \Theta(n^{3/2})$.

Theorem 5.1. $\mathrm{ex}_<(n, C_4^{1243}) = O(n^{3/2} \log n)$

Proof. Let G^L be an edge-ordered graph with n vertices and m edges avoiding C_4^{1243}. We assume the edges are labeled with the integers 1 through m. Our goal is to bound m.

We call an edge-ordered subgraph of G^L isomorphic to C_4^{1234} an *increasing 4-cycle*. Consider an increasing 4-cycle on vertices a, b, c and d with $L(ab) < L(bc) < L(cd) < L(da)$. We say that the *width* of this increasing 4-cycle is $w(abcd) = L(da) - L(cd) + L(bc) - L(ab)$. Note that $1 < w(abcd) < m$. We say that the increasing 4-cycle $abcd$ contributes the value $v = \log(m/w(abcd))$ to the pair $\{b, d\}$ of vertices and the value $-v$ to the pair $\{a, c\}$. For two distinct vertices x, y of G, let $V(x, y)$ be the total value the pair $\{x, y\}$ received from the contributions of all increasing

4-cycles in G. As each 4-cycle contributes a total of zero value we clearly have

$$\sum_{x,y} V(x,y) = 0, \tag{1}$$

where the summation runs over all unordered pairs of distinct vertices in G.

We will show that $V(x,y)$ is strictly positive unless x and y have only a few common neighbors. This will help us bound the codegrees and the eventually the number of edges in G.

For a pair of distinct vertices x and y of G, let $N(x,y)$ stand for the set of common neighbors of x and y in G. For $z \in N(x,y)$ we write $w_{xy}(z) = L(xz) - L(zy)$. Note that $w_{yx}(z) = -w_{xy}(z)$.

Any contribution to $V(x,y)$ must come from an increasing 4-cycle $xzyt$ with $z,t \in N(x,y)$. Consider first a pair of distinct vertices $z,t \in N(x,y)$ with $w_{xy}(z)$ and $w_{xy}(t)$ having the same sign. We claim that in this case $xzyt$ is an increasing 4-cycle of width $w = w(xzyt) = |w_{xy}(z) - w_{xy}(t)|$ and it contributes $\log(m/w)$ to the pair x,y. By symmetry, it is enough to show this assuming that both $w_{xy}(z)$ and $w_{xy}(t)$ are positive and $L(xz) > L(xt)$. This implies $L(yz) < L(yt)$ as otherwise the 4-cycle $xzyt$ would be isomorphic to the forbidden 4-cycle C_4^{1243}. So we have $L(zy) < L(yt) < L(tx) < L(xz)$ making a $xzyt$ an increasing 4-cycle of width $w = L(xz) - L(tx) + L(yt) - L(zy) = w_{xy}(z) - w_{xy}(t)$ as claimed and contributing $\log(m/w)$ toward $V(x,y)$.

Now consider a pair $z,t \in N(x,y)$ with $w_{xy}(z)$ and $w_{xy}(t)$ having opposite signs, say $w_{xy}(z) < 0 < w_{xy}(t)$. In this case the 4-cycle $xzyt$ is not necessarily increasing, but if it is, its width is again $w = |w_{xy}(z) - w_{xy}(t)|$ and it contributes $-\log(m/w)$ toward $V(x,y)$. Indeed, the 4-cycle $xzyt$ is only increasing if either $L(yt) < L(tx) < L(xz) < L(zy)$ or $L(xz) < L(zy) < L(yt) < L(tx)$ and our assertions hold in either case.

We can calculate $V(x,y)$ by summing the above values for all distinct $z,t \in N(x,y)$:

$$V(x,y) \geq \sum_{z,t} \operatorname{sign}(w_{xy}(z)w_{xy}(t)) \log\left(\frac{m}{|w_{xy}(z) - w_{xy}(t)|}\right), \tag{2}$$

where the summation is for unordered pairs of distinct vertices $z,t \in N(x,y)$. We have inequality and not equality because some of the pairs may yield non-increasing 4-cycles and thus do not contribute to $V(x,y)$, but as we saw this can only happen when $\operatorname{sign}(w_{xy}(z)w_{xy}(t))$ is negative.

It will be easier to deal with $\max(|w_{xy}(z)|, |w_{xy}(t)|)$ in place of $|w_{xy}(z) - w_{xy}(t)|$ in inequality (2). The former is the larger of the two values if the signs of $w_{xy}(z)$ and $w_{xy}(t)$ agree, but the latter is larger otherwise, so we always have

$$\operatorname{sign}(w_{xy}(z)w_{xy}(t)) \log\left(\frac{m}{|w_{xy}(z) - w_{xy}(t)|}\right) > \operatorname{sign}(w_{xy}(z)w_{xy}(t)) \log\left(\frac{m}{\max(|w_{xy}(z)|, |w_{xy}(t)|)}\right). \tag{3}$$

Further, the difference between the two sides of inequality (3) is at least 1 whenever

$$\frac{1}{2} \leq \frac{w_{xy}(z)}{w_{xy}(t)} \leq 2. \tag{4}$$

As $1 \le |w_{xy}(z)| < m$ holds for any $z \in N(x,y)$ we can partition $N(x,y)$ into $2\lceil \log m \rceil$ parts such that whenever z and t are from the same part, condition (4) holds. This means that condition (4) is satisfied for at least $d_{xy}^2/(2\lceil \log m \rceil)$ of the ordered pairs $z,t \in N(x,y)$, where $d_{xy} = |N(x,y)|$ is the codegree of x and y in G. Thus, (4) is also satisfied for at least $d_{xy}^2/(4\lceil \log m \rceil) - d_{xy}$ unordered pairs of distinct vertices $z,t \in N(x,y)$. Substituting inequality (3) in our bound (2) and using the slack in (3) whenever (4) is satisfied, we obtain

$$V(x,y) > \sum_{z,t} \mathrm{sign}(w_{xy}(z)w_{xy}(t)) \log\left(\frac{m}{\max(|w_{xy}(z)|, |w_{xy}(t)|)}\right) + \frac{d_{xy}^2}{4\lceil \log m \rceil} - d_{xy}, \qquad (5)$$

where the summation is for unordered pairs of distinct vertices $z,t \in N(x,y)$.

Consider now the following integral

$$\begin{aligned} 0 &\le \int_1^m \frac{1}{u}\left(\sum_{z\in N(x,y), |w_{xy}(z)|<u} \mathrm{sign}(w_{xy}(z))\right)^2 \mathrm{d}u \\ &= \int_1^m \frac{1}{u} \sum_{z,t\in N(x,y), \max(|w_{xy}(z)|,|w_{xy}(t)|)<u} \mathrm{sign}(w_{xy}(z)w_{xy}(t))\, \mathrm{d}u \\ &= \sum_{z,t\in N(x,y)} \int_{\max(|w_{xy}(z)|,|w_{xy}(t)|)}^m \frac{\mathrm{sign}(w_{xy}(z)w_{xy}(t))}{u}\, \mathrm{d}u \\ &= \sum_{z,t\in N(x,y)} \mathrm{sign}(w_{xy}(z)w_{xy}(t)) \ln\left(\frac{m}{\max(|w_{xy}(z)|, |w_{xy}(t)|)}\right). \end{aligned}$$

The summations here are for ordered pairs z,t and contains terms with $z = t$. We simply bound these latter terms by $\ln m$ and switch to binary logarithm to obtain

$$\sum_{z,t\in N(x,y)} \mathrm{sign}(w_{xy}(z)w_{xy}(t)) \log\left(\frac{m}{\max(|w_{xy}(z)|, |w_{xy}(t)|)}\right) \ge -d_{xy}\log m/2,$$

where the summation is now for unordered pairs of distinct vertices $z,t \in N(x,y)$. With our bound (5) this means

$$V(x,y) > \frac{d_{xy}^2}{4\lceil \log m \rceil} - d_{xy} - d_{xy}\log m/2.$$

It remains to sum this last bound for all unordered pairs of distinct vertices x,y of G. On the left hand side we obtain zero by equality (1). With the notation $D = \sum_{x,y} d_{xy}$ we clearly have $\sum_{x,y} d_{xy}^2 > 2D^2/n^2$ (both summations are for unordered pairs of distinct vertices of G). We obtain:

$$0 > \frac{2D^2}{4n^2\lceil \log m \rceil} - (\log m/2 + 1)D,$$

and therefore

$$D = O(n^2 \log^2 m). \tag{6}$$

Bounding m from this bound on D is straightforward. Let d_x denote the degree of the vertex x of G, then we have $\sum_x d_x = 2m$ and $\sum_x \binom{d_x}{2} = D$. By convexity and assuming $m \geq n$ we also have

$$D = \sum_x \binom{d_x}{2} \geq n\binom{2m/n}{2} = \Omega\left(\frac{m^2}{n}\right).$$

We obtain the bound $m = O(n^{3/2} \log n)$ claimed in the theorem by combining this last well known bound with our bound (6) on D. □

6. Concluding remarks

6.1. **An application: number of unit distances among n planar points in convex position.** Turán theory for edge-ordered graphs is likely to have several applications in other areas, especially in discrete geometry. As an example, we show a simple application of one of our results concerning the Turán number of P_5^{2143} (Theorem 4.9 (ii)). Erdős and Moser asked in 1959 to determine the maximum number of point pairs among n points in the plane in *convex position* that can be exactly unit distance apart. If we denote this quantity by $f(n)$, then the best bounds known are $2n-7 \leq f(n) = O(n \log n)$, due to Edelsbrunner-Hajnal [12], and Füredi [17]. (For a later, simpler proof of the upper bound, see [6].) Here we reprove the upper bound using the theory of forbidden edge-ordered graphs. Füredi [17] used forbidden submatrices, and our argument is inspired by his.

Proposition 6.1 ([17])**.** *The number of unit distances among n points in the plane in convex position is $O(n \log n)$.*

Proof. Define a graph G with the n points in convex position in the plane as its vertices and by connecting those points that are unit distance apart. Represent these edges as straight-line segments of length one. Without loss of generality (rotating the plane if needed), we can assume that at least half of these line-segments have slope between -1 and $+1$. Keep only these edges to form a graph G_1, thus we have $|E(G_1)| \geq \frac{1}{2}|E(G)|$.

Now add an infinite number of vertical lines (each of infinite length) to the plane such that two neighboring vertical lines are $\frac{7}{5}$ units apart. Keep only those edges of G_1 that do not cross any of these vertical lines to form the graph G_2. A simple probabilistic argument shows that the vertical lines can be placed in such a way that $|E(G_2)| \geq \frac{2/5}{7/5}|E(G_1)| = \frac{2}{7}|E(G_1)| \geq \frac{1}{7}|E(G)|$.

We show that any path in G_2 must consist of alternating steps to the left and to the right. Indeed, since any edge of G_2 is a unit line-segment with slope between -1 and +1, the horizontal component (or the x-component) of any edge of G_2 is at least $\frac{1}{\sqrt{2}}$, so two edges in the same direction would not fit between two vertical lines of distance $\frac{7}{5} < \sqrt{2}$.

Order the edges of G_2 by the slope of the respective line-segments, breaking ties arbitrarily.

Claim 1. G_2 is P_5^{2143}-free.

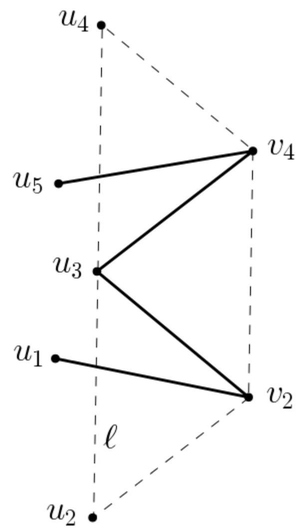

FIGURE 2. Illustration for the proof of Claim 1

Proof. Suppose that we have a path $u_1v_2u_3v_4u_5$ in G_2 such that the slopes of its edges is in the order $v_2u_3 < u_1v_2 < v_4u_5 < u_3v_4$. Draw a line ℓ parallel to v_2v_4 through u_3, and denote the points at distance $|v_2v_4|$ from u_3 on ℓ by u_2 and u_4, respectively, such that u_2v_2 is a translate of u_3v_4 and u_4v_4 is a translate of u_3v_2. Due to the order of the slopes, looking from v_2, u_1 must lie between u_2 and u_3, thus beyond the line ℓ. Similarly, u_5 must also lie on the other side of ℓ as v_4. Thus u_3 is in the convex hull of $\{u_1, v_2, v_4, u_5\}$, contradicting our assumption that the points are in convex position. □

Therefore, by Theorem 4.9 (ii), the number of unit distances among the points in $V(G)$ is $|E(G)| \leq 7|E(G_2)| \leq 7\text{ex}_<(n, P_5^{2143}) = O(n \log n)$. □

Remark. One may try to use that G_2 is also P_5^{3142}-free in the hope of getting a better upper bound. Currently, the best known bound is $n \log_2 n + O(n)$ due to Aggarwal [2]. It is a very interesting question to determine whether a linear upper bound can be obtained by excluding some other edge-ordered graph(s).

Remark. After the first preprint of this paper has appeared on arXiv, a new geometric application of our results has been discovered by Keszegh and the fourth author [24]. They proved that among n pairwise disjoint x-monotone red curves and n pairwise disjoint x-monotone blue curves there can be at most $O(n \log n)$ red-blue tangencies, improving the earlier $O(n \log^2 n)$ bound of Pach, Suk, and Treml [37], who already gave a lower bound of $\Omega(n \log n)$ for the problem. The proof in [24] is based on a simple application of Theorem 4.12 to the graph G whose vertices are the curves and whose edges are the tangencies in which the red curve is above the blue curve, since G is P_5^{1324}-free when the edges are ordered according to the x-coordinates of the corresponding tangencies.

6.2. **Generalizations of our construction techniques.** In the proof of Theorem 4.9 we gave recursive constructions of edge-ordered graphs G_i and G'_i, both with 2^i vertices and $i2^i$ edges. G_i was constructed to avoid P_5^{1342}, G'_i was constructed to avoid P_5^{2143}. But the same argument shows that they avoid other edge-ordered graphs too. We can formulate the following statement.

Proposition 6.2. *Let H be an edge-ordered graph on more than one vertices that has no partition of its vertex set into two non-empty parts A and B such that the edges between A and B form a matching and are larger than any other edges in the graph, while edges within A are smaller than the edges within B. In this case the graphs G_i in the proof of Theorem 4.9 avoid H.*

Similarly, the graphs G'_i in the proof of Theorem 4.9 avoid all edge-ordered connected graphs on more than one vertices that have no partition of their vertex sets into two non-empty sets A and B such that the edges between A and B form a matching and they are larger than any edge within A and smaller than any edge within B.

Proof. We prove the statements of the proposition by induction on i. They trivially hold for the single vertex graph G_0. Assume for a contradiction that G_i avoids an edge-ordered graph H but G_{i+1} contains it. Recall that G_{i+1} is constructed from two disjoint copies of G_i by adding a perfect matching between them. So given an isomorphic copy of H in G_{i+1} we can partition the vertex set of H according to which copy the corresponding vertex in the copy of H belongs to. By the way the edge-ordering of G_{i+1} was defined, this partition violates the assumption on H unless one of the parts is empty. But if one of the parts is empty, then H is contained in a single copy of G_i violating the inductive assumption. The contradiction proves the first statement of the proposition.

For the second statement the same proof works verbatim if we replace G_i and G_{i+1} with G'_i and G'_{i+1}. □

A similar generalized statement can be formulated about the edge-ordered graphs avoided, left-avoided and right-avoided by the edge-ordered bipartite graphs G_i constructed in the proof of Theorem 4.12.

We used a simple connection to the theory of forbidden matrix patterns in the proofs of Theorems 3.5 and 4.10. In general we can make an edge-ordered bipartite graph $G(M)$ from any 0-1 matrix M by having a left vertex for every row, a right vertex for every column, an edge between the corresponding vertices for every 1-entry in the matrix and ordering edges first according to their column and within a column according to the row.

Recall that for 0-1 matrices M and P we say that M *contains* P if P is a submatrix of M or P can be obtained from a submatrix of M by switching a few 1 entries to 0. If M does not contain any pattern P with $G(P)$ isomorphic to a fixed edge-ordered graph H, then $G(M)$ clearly avoids H. In the proof of Theorem 4.10 we used the fact that there is exactly one 0-1 matrix P with $G(P)$ isomorphic to P_5^{1432}, namely $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. The same connection can be used for other patterns as well.

6.3. **Open problems.**

- In the first draft of this paper we asked whether $\mathrm{ex}_<(n,T) = n^{1+o(1)}$ holds for every edge-ordered forest T with $\chi_<(T) = 2$. In a recent development, Gaurav Kucheriya and the fifth author managed to prove this statement, [27]. This represents a characterization as it is easy to see that such a bound never holds if T contains a cycle or its order chromatic number is larger than 2. Our conjecture can be considered as the edge-ordered analogue of a similar conjecture on vertex-ordered trees that is still open. The vertex-ordered version originates from the work of Füredi and Hajnal, [18]. They formulated their conjecture in terms of 0-1 matrices but it is equivalent to the statement that the extremal function of a forbidden vertex-ordered forest with interval chromatic number 2 is always $O(n \log n)$. This conjecture has been refuted by Pettie [38], but the extremal function of his counterexample is only slightly larger. An upper bound of $O(n \log^2 n)$ is still a possibility. An even weaker version of the conjecture was given in [36] with the conjectured bound $n^{1+o(1)}$. This conjecture is still open. The best partial results appeared in [26].

  The bound proved in [27] for $\mathrm{ex}_<(n,T)$ where T is an edge-ordered forest of order chromatic number 2 is $n2^{O(\sqrt{\log n})}$. They also conjecture that the much stronger $n \log^{O(1)} n$ bound also holds. Although we do not know of any edge-ordered forest T with order chromatic number 2 that violates $\mathrm{ex}_<(n,T) = O(n \log n)$, we conjecture that such an edge-ordered forest T exists. Some of the techniques used to construct the counterexample in [38] might be useful in resolving this problem.
- How large can $\chi_<(\mathcal{H})$ be for a family of n-vertex edge-ordered graphs $\mathcal{H}$ if it is finite? It would be interesting to decide whether it grows exponentially or double exponentially in n. In Proposition 2.11 we proved that the optimal family to consider here is the family $\mathcal{K}_n$ consisting of the four canonical edge-orderings of K_n and gave a doubly exponential upper bound for its order chromatic number. We can, however, ask the same question with respect to a single edge-ordered graph: How large can $\chi_<(H)$ be for an n-vertex edge-ordered graph H if it is finite? It is not clear which single n-vertex edge-ordered graph has the largest finite order chromatic number. Theorem 2.14 states an exponential lower bound in case of the edge-ordered graph D_n.
- Theorem 2.3 gives exact asymptotics for $\mathrm{ex}_<(n,H)$ if $\chi_<(H) > 2$. Many of our results give the exact order of magnitude of $\mathrm{ex}_<(n,H)$ for some edge-ordered graphs H of order chromatic number 2. But in a few cases our lower and upper bound do not coincide. In all of these cases it would be interesting to narrow the gap. For instance, is it true that $\mathrm{ex}_<(n, C_4^{1243}) = \Theta(n^{3/2})$? Theorem 5.1 shows an upper bound of $O(n^{3/2} \log n)$ but the only lower bound we have comes from the classical extremal function of C_4 which is $\Theta(n^{3/2})$. Another interesting question is whether $\mathrm{ex}_<(n, P_5^{1342}) = \Theta(n \log n)$? Theorem 4.14 shows an upper bound of $O(n \log^2 n)$. These are the only edge-orderings of C_4 and P_5 for which we do not know the order of magnitude of the Turán number.
- We studied here the Turán numbers of some short paths and cycles, but did not go beyond four edges except for exhibiting a single edge-ordering of P_6 with order chromatic number 3 in Theorem 4.15. It would be interesting to study the Turán numbers of longer

paths. We expect that results in this direction would quickly find applications just as Theorem 4.12 has already found an application in discrete geometry, see [24].

The order chromatic number of paths is also an interesting topic of research. We saw here that edge-ordered paths up to 3 edges have order chromatic number 2, edge-ordered paths with 4 edges have order-chromatic number 2 or infinity, but there exists an edge-ordered 5-edge path with order-chromatic number 3. What is the situation for longer paths? What is the maximum order-chromatic number of an edge-ordering of P_n if it is finite? We expect that the answer grows significantly slower than in the case addressed above, where the forbidden edge-ordered graph was arbitrary. Can the answer in this case exceed n?

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