On directed local chromatic number, shift graphs, and Borsuk-like graphs

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Abstract

We investigate the local chromatic number of shift graphs and prove that it is close to their chromatic number. This implies that the gap between the directed local chromatic number of an oriented graph and the local chromatic number of the underlying undirected graph can be arbitrarily large. We also investigate the minimum possible directed local chromatic number of oriented versions of "topologically t-chromatic" graphs. We show that this minimum for large enough t-chromatic Schrijver graphs and t-chromatic generalized Mycielski graphs of appropriate parameters is $\lceil t/4 \rceil + 1$.

1 Introduction

The local chromatic number of a graph G, defined by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress [6] is a coloring parameter that was further investigated recently in the papers [13, 23, 24]. (See also [3] for some related results.) Denoting the set of neighbors of a vertex v by $N(v)$, it is defined as follows.

Definition 1 ([6]) *The* local chromatic number *of a graph G is*

$$\psi(G) := \min_c \max_{v \in V(G)} |\{c(u) : u \in N(v)\}| + 1,$$

where the minimum is taken over all proper vertex-colorings c of G.

Thus $\psi(G)$ is the minimum number of colors that must appear in the most colorful closed neighborhood of a vertex in any proper coloring that may involve an arbitrary number of colors. It was shown in [6] that there exist graphs G with $\psi(G) = 3$ and $\chi(G) > k$ for any positive integer k, where $\chi(G)$ denotes the chromatic number of G.

Changing "neighborhood" to "outneighborhood" in the previous definition we arrive at the directed local chromatic number (of a digraph) introduced in [13]. For a directed graph F let the set of outneighbors of a vertex v be $N_+(v) = \{u \in V(F) : (v, u) \in E(F)\}$. By a proper vertex-coloring of a directed graph we mean a proper vertex-coloring of the underlying undirected graph.

Definition 2 ([13]) *The directed local chromatic number of a directed graph F is defined as*

$$\psi_{\mathrm{d}}(F) = \min_c \max_{v \in V(F)} \{c(u) : u \in N_+(v)\} + 1,$$

where c runs over all proper vertex-colorings of F.

The directed local chromatic number of a digraph is always less than or equal to the local chromatic number of the underlying undirected graph and we obviously have equality if our digraph is symmetrically directed, i.e., for every ordered pair (u, v) of the vertices (u, v) is an edge if and only if (v, u) is an edge. A digraph $F = (V, E)$ is called *oriented* if the contrary is true: $(u, v) \in E$ implies $(v, u) \notin E$. An *orientation* of an undirected graph G is an oriented graph $\hat{G}$ that has G as its underlying undirected graph.

It is a natural question whether every undirected graph G has an orientation the directed local chromatic number of which achieves the local chromatic number of G. In the first version of this paper we wrote that we knew very little about this question. A recent development is that Ambrus Zsbán [28] showed that the above is not true for all graphs. (See [20] for a problem of similar flavor: the relation of Shannon capacity and the maximum possible Sperner capacity of its orientations.)

In this paper we explore the other extreme: what is the minimum possible directed local chromatic number that an orientation of a graph can attain.

In the following section we give some more definitions and summarize some facts about the investigated parameters. In Section 3 we investigate shift graphs. We observe that they have an orientation with directed local chromatic number 2 and prove that their local chromatic number can be arbitrarily large, in particular, it differs at most 1 from their chromatic number. We also consider the behavior of a symmetrized variant of shift graphs.

In section 4 we concentrate on *Borsuk-like graphs*: these are graphs the chromatic number of which can be determined by applying Lovász's topological method (cf. [15]), while, at the same time they admit optimal colorings where no short odd length walks exist that start and end in the same color class. Several graphs have this property. In [23] we have shown that the local chromatic number of these graphs is around one half of their chromatic number. Here we show that the minimum directed local chromatic number of a Borsuk-like graph of appropriate parameters is about one quarter of its chromatic number.

2 Minimum and maximum directed local chromatic number

It is natural to define the following extreme values of $\psi_{\mathrm{d}}(G)$.

Definition 3 *For an undirected graph G we define the* minimum directed local chromatic number *as*

$$\psi_{\mathrm{d,min}}(G) := \min_{\hat{G}} \psi_{\mathrm{d}}(\hat{G})$$

and the maximum directed local chromatic number *as*

$$\psi_{\mathrm{d,max}}(G) := \max_{\hat{G}} \psi_{\mathrm{d}}(\hat{G}),$$

where $\hat{G}$, in both cases, runs over all orientations of G.

It is obvious that $\psi_{\mathrm{d,max}}(G) \leq \psi(G)$. Equality holds for complete graphs (by the transitive orientation), and more generally, for all graphs with equal chromatic and clique number, thus for all perfect graphs, in particular. A less obvious example for equality is given by Mycielski graphs, see Proposition 19 in Section 4. In the first version of this paper we wrote that we did not know whether equality holds for all graphs. Recently, however, Ambrus Zsbán [28] constructed a graph F with $\psi_{\mathrm{d,max}}(F) < \psi(F)$.

Our main concern here is the behavior of $\psi_{\mathrm{d,min}}(G)$. Clearly, if the graph has any edge, then $\psi_{\mathrm{d,min}}(G)$ is already at least 2. We will see in the next section that there are graphs with $\psi_{\mathrm{d,min}}(G) = 2$ and $\psi(G)$ arbitrarily large.

To conclude this section we give two easy estimates on $\psi_{\mathrm{d,min}}(G)$ in terms of $\chi(G)$ and the clique number $\omega(G)$, respectively. Recall that a homomorphism from graph G to another graph H is a mapping $f : V(G) \to V(H)$ such that whenever $\{a, b\}$ is an edge of G, then $\{f(a), f(b)\}$ is an edge of H. For a detailed introduction to graph homomorphisms, see [12].

The following relation of $\psi_{\mathrm{d,min}}$ to the clique number and the chromatic number is immediate.

Proposition 4 *For every graph G*

$$\left\lfloor \frac{\omega(G)}{2} \right\rfloor + 1 \le \psi_{\mathrm{d,min}}(G) \le \left\lfloor \frac{\chi(G)}{2} \right\rfloor + 1.$$

In particular, if G has equal clique number and chromatic number, then equality holds in both inequalities.

Proof. Let G be a graph with chromatic number r, which means that there is a homomorphism from G to K_r. Orient the edges of K_r so that the maximum outdegree become as small as possible. Clearly, this minimal maximum outdegree is $\lfloor r/2 \rfloor$.

Let $c : V(G) \to V(K_r)$ be an optimal coloring of G. For each edge $\{u, v\}$ of G orient it from u to v if and only if the edge $\{c(u), c(v)\}$ of K_r is oriented from $c(u)$ to $c(v)$ above. The set of colors in the outneighborhood of a vertex v of G will be the set of outneighbors of $c(v)$ in K_r. This proves $\psi_{\mathrm{d,min}}(G) \le \lfloor \frac{r}{2} \rfloor + 1$.

If the clique number of G is q then some vertex of a q-clique of G must have at least $\lfloor \frac{q}{2} \rfloor$ other vertices of this clique in its outneighborhood. Since all these must have different colors, it implies $\psi_{\mathrm{d,min}}(G) \ge \lfloor \frac{q}{2} \rfloor + 1$.

Clearly, if $\chi(G) = \omega(G)$, then the two bounds coincide. $\square$

3 Shift graphs

Shift graphs were introduced by Erdős and Hajnal [7].

Definition 5 ([7]) *The shift graph H_m is defined on the ordered pairs (i, j) satisfying $1 \le i < j \le m$ as vertices and two pairs (i, j) and (k, ℓ) form an edge if and only if $j = k$ or $\ell = i$.*

Note that H_m is isomorphic to the line graph of the transitive tournament on m vertices. It is well-known (see, e.g., [16], Problem 9.26) that $\chi(H_m) = \lceil \log_2 m \rceil$.

Shift graphs are relevant for us for two different reasons. One is what we already mentioned in the Introduction that their minimum directed local chromatic number is much below their local chromatic number. The other reason is explained below.

While the local chromatic number is obviously bounded from above by the chromatic number, in [13] it was shown to be bounded from below by the fractional chromatic number. This motivated the study of the local chromatic number for graphs with a large difference between the latter two bounds (see [23]). Determining the chromatic number of such graphs often requires special tricks as one needs some lower bound that is not a lower bound for the fractional chromatic number. In case of Kneser graphs this difficulty was overcome by Lovász [15] thereby introducing his topological method that was later successfully applied also for other graph families with the above property. Examples include Schrijver graphs ([22]) and generalized Mycielski graphs ([25, 11]). See also [18] for an excellent introduction to this method.

In [23] (see also [24]) we investigated the local chromatic number of graphs for which the chromatic number is far from the fractional chromatic number and can be determined by a particular implementation of the topological method. If this implementation gave t as a lower bound of the chromatic number, we called a graph *topologically t-chromatic*, and showed that if a graph is topologically t-chromatic, then $\lceil t/2 \rceil + 1$ is an often tight lower bound for its local chromatic number.

For shift graphs this topological lower bound for the chromatic number is not tight (except for some very small initial cases), in other words they are not topologically t-chromatic for t being the actual chromatic number, see Proposition 6 below. On the other hand, shift graphs do have the property that there is a large gap between their fractional and ordinary chromatic numbers. Thus the above mentioned result of [13] equally motivates the investigation of their local chromatic number while the methods of [23, 24] cannot give good bounds for it.

To see that the fractional chromatic number $\chi_f(H_m)$ is small it is worth defining the *symmetric shift graph S_m* (which later will also be considered for its own sake) that contains all ordered pairs (i, j) where $1 \leq i, j \leq m$, $i \neq j$, as vertices (i.e., (i, j) is a vertex even if $i > j$) and (i, j) and (k, ℓ) are adjacent again if $j = k$ or $\ell = i$. (Note that S_m is the line graph of the complete directed graph on m vertices.) It is obvious that S_m is vertex-transitive, thus $\chi_f(S_m) = \frac{|V(S_m)|}{\alpha(S_m)}$ (cf., e.g. [21]), where $\alpha(G)$ stands for the independence number of graph G. Since $\alpha(S_m) = \lceil \frac{m}{2} \rceil \lfloor \frac{m}{2} \rfloor$ (vertices (i, j) with $i \leq \lceil \frac{m}{2} \rceil < j$ form an independent set of this size and one easily sees that no larger one can be formed), we get $\chi_f(H_m) \leq \chi_f(S_m) = \frac{m(m-1)}{\lceil \frac{m}{2} \rceil \lfloor \frac{m}{2} \rfloor} < 4$, where the first inequality follows from H_m being a subgraph of S_m.

Thus by the inequalities $\chi_f(H_m) \leq \psi(H_m) \leq \chi(H_m)$ the value of $\psi(H_m)$ could be anywhere between 4 and $\lceil \log_2 m \rceil$. Now we show that the lower bound cannot be improved by the methods used in [23].

The lower bound on $\psi(G)$ in [23] mentioned above is proven by showing (cf. also [9] for a special case), that if G is a topologically t-chromatic graph, then whatever way we color its vertices properly (with any number of colors, thus the coloring need not be optimal) there always appears a complete bipartite subgraph $K_{\lceil t/2 \rceil, \lfloor t/2 \rfloor}$, all t vertices of which get a different color. Though we do not give here the exact definition of topological t-chromaticity, it makes sense to state the following proposition that can be proven using the result just described. We remark that topological t-chromaticity is a monotone property, that is, it implies topological $(t-1)$-chromaticity.

Proposition 6 *The graph H_m is not topologically 4-chromatic and S_m is not topologically 5-chromatic.*

Proof. Let us color the vertex (i,j) with color i. This gives a proper coloring of H_m.

One can easily check that if two vertices (i_1,j_1) and (i_2,j_2) of H_m have two common neighbors (k_1,ℓ_1), (k_2,ℓ_2), then either $j_1 = j_2 = k_1 = k_2$ or $i_1 = i_2 = \ell_1 = \ell_2$. Thus H_m can be properly colored in such a way it has no $K_{2,2}$ subgraph with all four vertices receiving a different color. By the above described result in [23], this implies that H_m is not topologically 4-chromatic.

The same coloring (assigning color i to the vertex (i,j)) is also a proper coloring of S_m but here for $m \geq 4$ some $K_{2,2}$ subgraphs (like the one consisting of the vertices $(1,2)$, $(2,3)$, $(3,4)$, $(4,1)$) receive four distinct colors. However no $K_{2,3}$ subgraphs receive five distinct colors, so by the same quoted result S_m is not topologically 5-chromatic. □

We remark that S_m is not even topologically 4-chromatic, but to see this is beyond the scope of the present paper because every proper coloring of S_4 makes a $K_{2,2}$ subgraph (a 4-cycle) receive four distinct colors. Every non-bipartite graph is topologically 3-chromatic, so the graphs H_m for $m \geq 5$ and S_m for $m \geq 3$ are topologically 3-chromatic.

Although the local chromatic number of shift graphs could be as low as 3 if considering only the topological lower bound of the local chromatic number given in [23], the main result of this section below states that it is much higher.

Theorem 7 *We have*

$$\psi(H_m) = \chi(H_m)$$

whenever $2^k + 2^{k-1} < m \leq 2^{k+1}$ *for some positive integer* k. *If* $2^k < m \leq 2^k + 2^{k-1}$ *holds for some* k *instead, then we have*

$$\chi(H_m) - 1 \leq \psi(H_m) \leq \chi(H_m).$$

We prove this theorem in Subsection 3.2. It shows not only that the local chromatic number of shift graphs is close to their chromatic number but also that the gap between the directed local chromatic number and the local chromatic number of the underlying undirected graph can be arbitrarily large. This statement follows when comparing Theorem 7 to the following simple observation. (For the appearance of more general shift graphs in a similar context, see the starting example in [6].)

Proposition 8 *For* $m \geq 3$ *we have*

$$\psi_{\mathrm{d,min}}(S_m) = \psi_{\mathrm{d,min}}(H_m) = 2.$$

Proof. As H_m is a subgraph of S_m and $\psi_{\mathrm{d,min}}(H_m) \geq 2$ is obvious for $m \geq 3$, it is enough to prove $\psi_{\mathrm{d,min}}(S_m) \leq 2$. Let $\tilde{S}_m$ be the oriented version of S_m in which edge $\{(a,b),(b,c)\}$ is oriented from vertex (a,b) to vertex (b,c) whenever a, b and c are distinct while we choose arbitrarily when orienting the edge between the vertices (a,b) and (b,a) for $a \neq b$. Color each vertex (x,y) by its first element x. Let (a,b) be an arbitrary vertex and observe that every element of its outneighborhood is given color b. This shows $\psi_{\mathrm{d}}(\tilde{S}_m) \leq 2$ thereby proving the statement. □

Note the easy fact, that if we modify the directed graph $\tilde{S}_m$ in the above proof so that for edges $\{(a,b),(b,a)\}$ we include both orientations then the so obtained graph $\hat{S}_m$ is a homomorphism universal graph: it has the property that a digraph F admits a coloring with m colors attaining $\psi_{\mathrm{d}}(F) \leq 2$ if and only if there exists a homomorphism from F to $\hat{S}_m$. (With the notation of [13] $\hat{S}_m$ is just the graph $U_d(m,2)$.) We will refer to the graphs $\hat{S}_m$ as the *symmetric directed shift graphs*.

3.1 Bollobás-type inequalities

A key observation in proving Theorem 7 will be the close connection between local colorings of shift graphs and cross-intersecting set systems. Here we state two classical results about the latter that will be relevant for us. The first of these is due to Bollobás.

Theorem 9 ([4]) *Let $A_1, \dots, A_m$ and $B_1, \dots, B_m$ be finite sets satisfying the property that $A_i \cap B_j \neq \emptyset$ for all $1 \leq i, j \leq m$ with $i \neq j$, while $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq m$. Then*

$$\sum_{i=1}^{m} \binom{|A_i| + |B_i|}{|A_i|}^{-1} \leq 1.$$

Note that if $|A_i| = r$ and $|B_i| = s$ holds for all i then the above statement implies $m \leq \binom{r+s}{r}$. This consequence is generalized by Frankl as follows.

Theorem 10 ([10]) *Let $A_1, \dots, A_m$ and $B_1, \dots, B_m$ be sets satisfying $|A_i| = r, |B_i| = s, A_i \cap B_i = \emptyset$ for all $1 \leq i \leq m$, and the additional property that $A_i \cap B_j \neq \emptyset$ whenever $1 \leq i < j \leq m$. Then*

$$m \leq \binom{r+s}{r}.$$

We remark that further relaxing the condition $A_i \cap B_j \neq \emptyset$ whenever $1 \leq i < j \leq m$ to $1 \leq i < j \leq m \Rightarrow (A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset)$, we arrive to a problem that, by our current knowledge, is not completely solved for $r, s \geq 2$, cf. [27].

The following lemma shows the connection between our problem and cross-intersecting set systems.

Lemma 11 *The inequality $\psi(H_m) \leq k$ is equivalent to the following statement. There exist finite sets, $A_1, \dots, A_m$ and $B_1, \dots, B_m$ such that $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq m$ and for all $1 \leq i < j \leq m$ we have $A_i \cap B_j \neq \emptyset$ and $|A_j \cup B_i| \leq k - 1$.*

Proof. Assume first that $\psi(H_m) \leq k$ and let $c : V(H_m) \to \mathbb{N}$ be a proper coloring that attains the local chromatic number. For each $1 \leq i \leq m$ form the sets A_i, B_i by $A_i := \{c(i,\ell) : i < \ell \leq m\}, B_i := \{c(\ell,i) : 1 \leq \ell < i\}$. Since the coloring is proper we must have $A_i \cap B_i = \emptyset$ for all i. For $1 \leq i < j \leq m$ we have $c(i,j) \in A_i \cap B_j$, thus we have $A_i \cap B_j \neq \emptyset$ for all $i < j$. A given vertex (i,j) of H_m is adjacent to the vertices (ℓ,i) and

(j,q) where $\ell < i < j < q$. By our condition on the local chromatic number this implies $|B_i \cup A_j| \le k-1$ for all $i < j$.

On the other hand, if $A_1, \dots, A_m, B_1, \dots, B_m$ with the above properties exist, then we can define the coloring c of the vertices of H_m as follows. For each vertex $(i,j) \in V(H_m)$ let $c(i,j)$ be an arbitrary element of the nonempty set $A_i \cap B_j$. As $A_i \cap B_i = \emptyset$ for all i this coloring is proper. By $|A_j \cup B_i| \le k-1$ the local chromatic number attained by this coloring is at most k. □

3.2 Proof of Theorem 7

We will show that if the sets $A_1, \dots, A_m$ and $B_1, \dots, B_m$ satisfy the conditions in Lemma 11, then $m \le 2^k + 2^{k-1}$. By Lemma 11 and $\chi(H_m) = \lceil \log_2 m \rceil$, this implies the statement of Theorem 7.

For obtaining the above upper bound on m we partition the pairs (A_i, B_i) according to the sizes of the sets A_i, B_i. For every $0 \le r$ set

$$\mathcal{D}_1^{(r)} = \{i : 1 \le i \le m, |A_i| = r, |A_i| + |B_i| < k\}$$

and

$$\mathcal{D}_2^{(r)} = \{i : 1 \le i \le m, |A_i| = r, |A_i| + |B_i| \ge k\}.$$

Note that by its definition $\mathcal{D}_1^{(r)} = \emptyset$ for $r \ge k$ and $|A_j \cup B_1| \le k-1$ for $1 < j \le m$ implies $\cup_{r \ge k} \mathcal{D}_2^{(r)} \subseteq \{1\}$.

Fix some $r \ge 0$. Notice that for each $i \in \mathcal{D}_1^{(r)}$ we have $|B_i| \le k-1-r$ and add $k-1-r-|B_i|$ new elements to the set B_i that do not appear elsewhere. Denote the resulting set by B_i'. Note that the pairs (A_i, B_i') for $i \in \mathcal{D}_1^{(r)}$ satisfy the conditions in Frankl's Theorem 10 (with $s = k-1-r$), implying $|\mathcal{D}_1^{(r)}| \le \binom{k-1}{r}$. This further implies

$$\sum_{r \ge 0} |\mathcal{D}_1^{(r)}| \le 2^{k-1}.$$

For bounding the size of sets $\mathcal{D}_2^{(r)}$ observe that the condition $|A_j \cup B_i| \le k-1$ satisfied for all $i < j$ is equivalent to $|A_j \cap B_i| \ge |A_j| + |B_i| - k + 1$. Fix some $0 \le r < k$ and notice that for $i \in \mathcal{D}_2^{(r)}$ we have $|B_i| \ge k - r$. Let B_i' be an arbitrary subset of B_i of size $k-r$. The pairs (A_i, B_i') for $i \in \mathcal{D}_2^{(r)}$ still satisfy that $A_j \cap B_i' \ne \emptyset$ whenever $j > i$, while $A_i \cap B_i' = \emptyset$ is also true. Thus the conditions of Theorem 10 hold again (now with $s = k-r$ and by reversing the order of indices) implying $|\mathcal{D}_2^{(r)}| \le \binom{k}{r}$. This further implies

$$\sum_{r \ge 0} |\mathcal{D}_2^{(r)}| \le \sum_{r=0}^{k-1} |\mathcal{D}_2^{(r)}| + 1 \le 2^k.$$

Thus we obtained $m = \sum_{r \ge 0} |\mathcal{D}_1^{(r)}| + \sum_{r \ge 0} |\mathcal{D}_2^{(r)}| \le 2^k + 2^{k-1}$ completing the proof. □

3.3 Symmetric shift graphs

In view of the above it is natural to ask what is the local chromatic number of the symmetric shift graph S_m. We trivially have $\psi(S_m) \geq \psi(H_m)$. In view of Theorem 7 this shows that $\psi(S_m)$ is close to $\chi(S_m) = \min\left\{k : \binom{k}{\lceil k/2 \rceil} \geq m\right\}$ (see, e.g. [16], Problem 9.26.), but this trivial observation allows for an unbounded difference of the order $\log(\chi(S_m))$ or $\log\log m$. In view of Theorem 7 it seems very unlikely that there could be such a large gap between $\psi(S_m)$ and $\chi(S_m)$. In fact, we are inclined to believe that both $\psi(S_m)$ and $\psi(H_m)$ coincides with the corresponding chromatic numbers, $\chi(S_m)$ and $\chi(H_m)$, respectively.

In this subsection we apply the method of the preceding section to improve the above trivial lower bound on $\psi(S_m)$. The improvement we obtain is rather modest: we increase the lower bound by 1 for some m.

The analogue of Lemma 11 is the following.

Lemma 12 *The inequality $\psi(S_m) \leq k$ is equivalent to the following statement. There exist finite sets $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ such that $A_i \cap B_i = \emptyset$ for all $1 \leq i \leq m$ and for all $1 \leq i, j \leq m$ with $i \neq j$ we have $A_i \cap B_j \neq \emptyset$ and $|A_i \cup B_j| \leq k - 1$.*

The proof is essentially identical to that of Lemma 11, therefore we omit it.

Theorem 13 *The local chromatic number of the symmetric shift graph S_m satisfies*

$$\psi(S_m) \geq \lceil \log_2(m+2) \rceil.$$

Proof. We do the same as in the proof of Theorem 7. By Lemma 12 it is enough to show that if $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ are two families of finite sets satisfying the conditions there, then $m \leq 2^k - 2$.

To this end we define $\mathcal{D}^{(r)} = \{i : 1 \leq i \leq m, |A_i| = r\}$.

Note that for $r \geq k$ $\mathcal{D}^{(r)} = \emptyset$ follows from the condition $|A_i \cup B_j| \leq k - 1$ for $i \neq j$. Similarly, $\mathcal{D}^{(0)} = \emptyset$ follows from $A_i \cap B_j \neq \emptyset$ for $i \neq j$.

Fix some $0 < r < k$ and consider $i \in \mathcal{D}^{(r)}$. If $|B_i| > k - r$ let B'_i be an arbitrary subset of B_i of size $k - r$, otherwise let $B'_i = B_i$. The conditions imply that the pairs (A_i, B'_i) for $i \in \mathcal{D}^{(r)}$ satisfy the conditions of Theorem 9. Since we have $|A_i| = r$, $|B'_i| \leq k - r$ for all $i \in \mathcal{D}^{(r)}$, this further implies $|\mathcal{D}^{(r)}| \leq \binom{k}{r}$. Summing for all r we obtain

$$m = \sum_{r=1}^{k-1} |\mathcal{D}^{(r)}| \leq 2^k - 2$$

completing the proof. $\square$

3.4 A homomorphism duality result

In this subsection we prove that the following homomorphism duality statement (see [12] for more on this term) holds for symmetric directed shift graphs $\hat{S}_m$ (see their definition after Proposition 8). We need the notion of an alternating odd cycle, which is an oriented odd cycle with exactly one vertex of outdegree one. It was observed in [13] that a directed odd cycle has directed local chromatic number 3 if and only if it contains an alternating odd cycle as a subgraph. (The two may not be equal as a directed odd cycle, unlike oriented ones, can contain multiple edges.) The following is a straightforward extension of this observation.

Proposition 14 *A directed graph $\hat{G}$ admits a homomorphism into $\hat{S}_m$ for some m if and only if no alternating odd cycle admits a homomorphism to $\hat{G}$.*

Proof. It is clear (and also contained in [13]) that alternating odd cycles have directed local chromatic number 3. By the remark following the proof of Proposition 8 this implies that there is no homomorphism from any alternating odd cycle to $\hat{S}_m$ for any m, or to any graph that admits a homomorphism to a symmetric directed shift graph $\hat{S}_m$ for some m.

On the other hand, we claim that if $\psi_{\mathrm{d}}(\hat{G}) > 2$ (which is equivalent to $\hat{G}$ not having a homomorphism to any $\hat{S}_m$), then an alternating odd cycle has a homomorphism to $\hat{G}$. (We remark that this also implies that $\hat{G}$ contains an alternating odd cycle as a subgraph.) Indeed, call two vertices u and v *related* if they both belong to the outneighborhood of the same vertex w. The transitive closure of this relation defines equivalence classes of the vertices. Let us color the vertices according to the equivalence class they belong to. Clearly, the outneighborhood of any vertex is monochromatic, so $\psi_{\mathrm{d}}(\hat{G}) > 2$ implies that this is not a proper coloring of $\hat{G}$. Let a and b be adjacent vertices in an equivalence class. There must be a sequence $a = u_0, u_1, \ldots, u_h = b$ of vertices such that u_i is related to u_{i+1} for $0 \leq i < h$. Let w_i be the vertex having both u_i and u_{i+1} in its outneighborhood. The vertices of an alternating odd cycle of length $2h+1$ can be homomorphically mapped to $u_0, w_0, u_1, w_1, \ldots, u_h$ in this order. $\Box$

4 Borsuk-like graphs

Borsuk-graphs were also introduced by Erdős and Hajnal [8].

Definition 15 ([8]) *The Borsuk graph $B(n, \alpha)$ is defined for every positive integer n and $0 < \alpha < 2$ on the unit sphere $\mathbb{S}^{n-1}$ of the n-dimensional Euclidean space as vertex set. Two vertices form an edge if their Euclidean distance is larger than α.*

It is easy to see that the statement $\chi(B(n, \alpha)) \geq n+1$ is equivalent with the celebrated Borsuk-Ulam theorem, see [8, 17]. It is also well-known and easy to see, that if α is larger than a certain threshold, than $n+1$ colors suffice: inscribe a regular simplex into $\mathbb{S}^{n-1}$

and color each point of the sphere with the side of the simplex intersected by the line segment joining this point to the center of the sphere. Note that besides being proper this coloring has a further remarkable property: for every $s \in \mathbb{N}$ there exists $\alpha_{n,s} < 2$ such that if $\alpha > \alpha_{n,s}$ then there is no walk of length $2s-1$ in $B(n,\alpha)$ between any pair of vertices that have the same color. Several other interesting graphs also have optimal colorings with this property, see [23].

Definition 16 ([23], cf. also [1]) *Let s be a positive integer. A coloring c of a graph G is called* s-wide *if there is no walk of length $2s-1$ in G between any two vertices u and v with $c(u) = c(v)$.*

Observe that 1-wide colorings are exactly the proper colorings, while being 2-wide means that the neighborhood of each color class is independent. Graphs with colorings of the latter property were investigated in [11]. 3-wide colorings were simply called *wide* in [23] as they had a key role there in bounding the local chromatic number from above. Namely, we proved in [23] that if a graph G has a 3-wide coloring with t colors then $\psi(G) \leq \lfloor t/2 \rfloor + 2$. (To see that this bound is sharp for several graphs, cf. [23, 24].)

Recall that the Kneser graph $\mathrm{KG}(n,k)$ is defined for $n \geq 2k$ on all k-element subsets of the n element set $[n] = \{1, \ldots, n\}$ as vertex set and two such subsets form an edge if they are disjoint. Their chromatic number is $n-2k+2$ as conjectured by Kneser [14] and proved by Lovász [15]. Schrijver found a very nice family of induced subgraphs of Kneser graphs. They have the same chromatic number as the corresponding Kneser graphs but at the same time they are also vertex color-critical.

Definition 17 ([22]) *The* Schrijver graph $\mathrm{SG}(n,k)$ *is defined for $n \geq 2k$ as follows.*

$$\begin{aligned} V(\mathrm{SG}(n,k)) &= \{A \subseteq [n] : |A| = k, \forall i : \ \{i, i+1\} \nsubseteq A \ \text{ and } \ \{1, n\} \nsubseteq A\} \\ E(\mathrm{SG}(n,k)) &= \{\{A, B\} : A \cap B = \emptyset\} \end{aligned}$$

The following generalization of Mycielski's construction [19] appears in several papers, see, e.g., [11, 25, 26] for their chromatic properties.

Definition 18 *For a graph G and integer $r \geq 1$ the generalized Mycielskian $M_r(G)$ of G is the graph on vertex set*

$$V(M_r(G)) = \{(i, v) : v \in V(G), 0 \leq i \leq r-1\} \cup \{z\}$$

with edge set

$$E(M_r(G)) = \{\{(i, u), (j, v)\} : \{u, v\} \in E(G) \text{ and } i = j = 0 \text{ or } 0 \leq i = j-1 \leq r-2\} \cup$$
$$\{\{(r-1, u), z\} : u \in V(G)\}.$$

The Mycielskian $M(G)$ of a graph is identical to $M_2(G)$. The main property of this construction is that while it does not change the clique number for $r \geq 2$, the chromatic number of $M(G)$ is 1 more than that of G. We have $\chi(M_r(G)) \leq \chi(G)+1$ for an arbitrary r, but $\chi(M_r(G)) = \chi(G)$ can happen for $r \geq 3$ (an example is $G = \bar{C}_7$, see [26], or see [5] for another example with fewer edges). Stiebitz [25] proved, however, that Lovász's topological lower bound on the chromatic number is always 1 more for $M_r(G)$ than for G. Thus, if this bound is tight for G then the chromatic number of $M_r(G)$ is 1 larger than $\chi(G)$. Moreover, in this case this new bound is also tight for $M_r(G)$, so this argument can be used recursively.

The chromatic number of the above graphs were determined by using the topological method, in particular, the Borsuk-Ulam theorem, for getting the appropriate lower bound, see [15, 22, 25, 11] and also [18]. Another similarity between Schrijver graphs and generalized Mycielski graphs is that for any given chromatic number χ and parameter s one can find a member of either family with chromatic number χ having an s-wide χ-coloring. (We note that a topological similarity of Schrijver graphs and their iterated generalized Mycielskians that is not shared by Kneser graphs is that their so-called neighborhood complex, cf. [15, 18], is homotopy equivalent to a sphere, see [2, 25].)

We conclude the introductory part of this section by stating a result about the maximum directed local chromatic number of Mycielski graphs. It is a rather straightforward generalization of Proposition 10 in [23]. Though its proof is almost identical to that of this quoted result, we include it for the sake of completeness.

Proposition 19 *For any graph G we have*

$$\psi_{\mathrm{d,max}}(M(G)) \geq \psi_{\mathrm{d,max}}(G) + 1.$$

In particular, if $\psi_{\mathrm{d,max}}(G) = \chi(G)$, then $\psi_{\mathrm{d,max}}(M(G)) = \psi_{\mathrm{d,max}}(G) + 1 = \chi(M(G))$.

Proof. First we give the orientation. Fix an orientation of G that attains $\psi_{\mathrm{d,max}}(G)$ and orient the subgraph of $M(G)$ induced by the vertices $(0, v)$ accordingly. Orient each edge of the form $\{(1, u), (0, v)\}$ consistently with the corresponding edge $\{(0, u), (0, v)\}$, i.e., so that either both have its head or both have its tail at the vertex $(0, v)$. Finally, orient all edges $\{(1, u), z\}$ towards z.

Now consider an arbitrary proper coloring $c : V(M(G)) \to \mathbb{N}$. For a subset $U \subseteq V(M(G))$ let $c(U) := \{c(u) : u \in U\}$. Consider also the modified coloring c' of G defined by

$$c'(x) = \begin{cases} c(0, x) & \text{if } c(0, x) \neq c(z) \\ c(1, x) & \text{otherwise.} \end{cases}$$

It follows from the construction that c' is a proper coloring of G, which does not use the color $c(z)$.

By our orientation of G there is some vertex v of G for which $|c'(N_+(v))| \geq \psi_{\mathrm{d,max}}(G) - 1$. (Note that $N_+(.)$ and $N_+(.,.)$ here refer to outneighborhoods in the considered orientations of G and $M(G)$, respectively.) If there is no vertex $u \in N_+(v)$ for which $c(0,u) \neq c'(u)$, then the color $c(z)$ does not appear in the outneighborhood of $(0,v)$ in $M(G)$. In this case the set $c(N_+(1,v))$ contains all the colors in $c'(N_+(v))$ plus the additional color $c(z)$. If, however, there is some $u \in N_+(v)$ for which $c(0,u) \neq c'(u)$, then we have $c(0,u) = c(z)$. In this case the set $N_+(0,v)$ contains all the colors appearing in $c'(N_+(v))$ and also the additional color $c(z)$ as the color of $(0,u)$. In either case, some vertex has at least $\psi_{\mathrm{d,max}}(G)$ colors in its outneighborhood, proving $\psi_{\mathrm{d,max}}(M(G)) \geq \psi_{\mathrm{d,max}}(G) + 1$.

The second statement trivially follows from the first using the well-known fact $\chi(M(G)) = \chi(G) + 1$ and the obvious inequalities $\psi_{\mathrm{d,max}}(G) \leq \psi(G) \leq \chi(G)$. $\Box$

Note that Proposition 19 implies that $\psi_{\mathrm{d,max}}(G) = \psi(G)$ holds whenever G is a Mycielski graph, that is a graph obtained from a single edge by repeated use of the Mycielski construction. We also remark that unlike the analogous inequality for $\chi(G)$ or $\psi(G)$ the inequality $\psi_{\mathrm{d,max}}(M(G)) \leq \psi_{\mathrm{d,max}}(G) + 1$ does not seem to be obvious. Though we do not have a counterexample we are not completely convinced about its validity.

4.1 Lower bound by topological t-chromaticity

As we have already mentioned in Section 3 we called a graph topologically t-chromatic in [23] if a particular implementation of the topological method gave t as a lower bound for its chromatic number. We also mentioned there that a result in [23] shows (cf. also [9]) that in every proper coloring of a topologically t-chromatic graph a complete bipartite subgraph $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$ occurs, all t vertices of which get a different color. This result was used in [23] to bound ψ from below. In a similar manner it also gives a lower bound for $\psi_{\mathrm{d,min}}$.

Theorem 20 *If G is a topologically t-chromatic graph with $t \geq 2$, then*

$$\psi_{\mathrm{d,min}}(G) \geq \lceil t/4\rceil + 1.$$

Proof. Let G be a topologically t-chromatic graph, c its proper coloring and D its multicolored complete bipartite subgraph whose existence is guaranteed by the result mentioned above. The number of edges in D is $\lceil t/2\rceil\lfloor t/2\rfloor$ implying that for any orientation of D its average outdegree is $(1/t)\lceil t/2\rceil\lfloor t/2\rfloor$ the upper integer part of which is $\lceil t/4\rceil$. Since all vertices of D receive different colors, its maximum outdegree vertex have at least $\lceil t/4\rceil$ different colors in its outneighborhood in any orientation. This proves that $\psi_{\mathrm{d,min}} \geq \lceil t/4\rceil + 1$. $\Box$

4.2 Upper bound by wide colorability

4.2.1 Graphs with chromatic number at most six

If a graph G is at most 3-chromatic (but not edgeless), then Proposition 4 implies that its minimum directed local chromatic number $\psi_{\mathrm{d,min}}(G) = 2$. Below we will show that the same conclusion holds for 4-chromatic graphs with 2-wide 4-colorings. The same method will be used to prove the sharpness of our topological lower bound for certain graphs of chromatic number at most 6.

The following notations and lemmas will be useful. Given a coloring c of a graph G for each vertex $v \in V(G)$ let $S_c(v) = \{c(u) : \{u, v\} \in E(G)\}$ and $s_c(v) = |S_c(v)|$. That is, $s_c(v)$ is the number of colors given to the neighbors of v.

Lemma 21 *If c is a 2-wide coloring and u, v are adjacent vertices of a graph G then $S_c(u) \cap S_c(v) = \emptyset$. In particular, if c uses t colors, then $s_c(u) + s_c(v) \leq t$.*

Proof. Assume indirectly that $S_c(u) \cap S_c(v) \neq \emptyset$, i.e., u has a neighbor x and v has a neighbor y with $c(x) = c(y)$. But then the walk $xuvy$ connects vertices of the same color and contradicts the assumption that c is 2-wide. This proves the first statement of the lemma, that obviously implies the second one completing the proof. $\Box$

Lemma 22 *If a graph G has a 2-wide coloring using $2h$ colors with $h \geq 2$, then $\psi_{\mathrm{d,min}}(G) \leq h$.*

Proof. Consider G as colored by a fixed 2-wide $2h$-coloring $c : V(G) \to H$ with $|H| = 2h$.

Let us consider the subgraph G' obtained from G by removing all vertices $u \in V(G)$ with $s_c(u) < h$. We claim that G' has an orientation $\hat{G}'$ such that the outneighborhood of any vertex receives at most $\lceil h/2 \rceil$ distinct colors by c.

Indeed, by Lemma 21 if $\{u, v\}$ is an edge of G', then $S_c(u)$ and $S_c(v)$ are complementary sets of colors, each of size h. So each nontrivial component of G' is a bipartite graph with one side containing vertices u with $S_c(u) = H_1$ for some fixed set H_1 of h colors and with the other side containing vertices v with $S_c(v) = H_2 = H \setminus H_1$. Clearly, the vertices in the former side receive colors in H_2, while vertices on the latter side have colors in H_1. To prove the claim it is enough to find a suitable orientation for each of the components separately, so let us fix H_1 and H_2. Consider the complete bipartite graph K_{H_1,H_2} on the vertex set H consisting of the edges connecting elements of H_1 and H_2. Orient the edges of this graph, so that every outdegree is at most $\lceil h/2 \rceil$. Now orient the edge $\{u, v\}$ in this connected component of G' according to the orientation of $\{c(u), c(v)\}$ in K_{H_1,H_2}. Clearly, this orientation satisfies the requirement of the claim.

Having found the orientation $\hat{G}'$, extend it to an orientation $\hat{G}$ of G by orienting each edge of G not in G' away from a vertex u with $s_c(u) < h$. The outneighborhood of a vertex in G' is the same in $\hat{G}$ and in $\hat{G}'$, so it receives at most $\lceil h/2 \rceil \leq h - 1$ colors at c. For the rest of the vertices of G their entire neighborhood receives at most $h - 1$ colors, so we have $\psi_{\mathrm{d}}(\hat{G}) \leq h$. This completes the proof of the lemma.

Notice that the coloring establishing the bound on the directed local chromatic number is the 2-wide coloring itself. $\Box$

Corollary 23 *If a non-edgeless graph G has a 2-wide 4-coloring, then $\psi_{\mathrm{d,min}}(G) = 2$.*

Proof. The statement immediately follows by applying Lemma 22 with $h = 2$. $\square$

Corollary 24 *If a topologically* 5*-chromatic graph* G *has a* 2*-wide coloring using at most* 6 *colors, then* $\psi_{\mathrm{d,min}}(G) = 3$.

Proof. Theorem 20 implies $\psi_{\mathrm{d,min}}(G) \geq 3$. Lemma 22 implies $\psi_{\mathrm{d,min}}(G) \leq 3$. $\square$

4.2.2 General upper bound

In this section we improve Lemma 22 so that the upper bound it gives will match the lower bound of Theorem 20 for several graphs of higher (local) chromatic number. For this we need to assume the existence of s-wide colorings for larger values of s. In [23] the minimal universal graphs for s-wide t-colorability were found. (Cf. [11] for some larger universal graphs for this property.) We will use them here.

Definition 25 *Let* $s \geq 1$ *and* $t \geq 2$ *be integers. The vertex set of the graph* $W(s,t)$ *consists of the functions* $f : \{1,\ldots,t\} \to \{0,1,\ldots,s\}$ *satisfying that* $f(i) = 0$ *holds for exactly one index* i *and* $f(i) = 1$ *holds for at least one index* i. *Two vertices* f *and* g *are connected in* $W(s,t)$ *if for every* i *one has* $|f(i) - g(i)| = 1$ *or* $f(i) = g(i) = s$.

The natural coloring of $W(s,t)$ *assigns the color* i *to the vertex* f *if* $f(i) = 0$.

Lemma 26 ([23], cf. also [1]) *For* $s \geq 1$ *and* $t \geq 2$ *the natural coloring of* $W(s,t)$ *is an* s*-wide* t*-coloring. A graph* G *admits an* s*-wide* t*-coloring if and only if there is a homomorphism from* G *to* $W(s,t)$.

Theorem 27 *For every* $t \in \mathbb{N}$ *there is an* $s = s_t$ *for which the following is true. If a graph* G *has an* s*-wide coloring with* t *colors then* $\psi_{\mathrm{d,min}}(G) \leq \lceil t/4 \rceil + 1$.

Proof. We will find an orientation $\hat{W}$ of $W(s,t)$ with directed local chromatic number bounded above by $\lceil t/4 \rceil + 1$. This is enough by Lemma 26 and the trivial observation that if there is a homomorphism from a graph G to another graph W, then we have $\psi_{\mathrm{d,min}}(G) \leq \psi_{\mathrm{d,min}}(W)$.

Let χ stand for the natural coloring of $W(s,t)$. This is the coloring establishing our bound on $\psi_{\mathrm{d}}(\hat{W})$. We write τ for $\lceil t/4 \rceil$. We will define a set $S(f)$ of colors for every vertex f of $W(s,t)$. We make sure that

1. $|S(f)| \leq \tau$ for every vertex f and
2. either $\chi(f) \in S(g)$ or $\chi(g) \in S(f)$ holds for every edge $\{f,g\}$ of $W(s,t)$.

We obtain the orientation $\hat{W}$ by orienting an edge from f to g only if $\chi(g) \in S(f)$. Property 2 ensures that all edges of $W(s,t)$ can be oriented this way. Property 1 makes sure that the natural coloring χ establishes $\psi_{\mathrm{d}}(\hat{G}) \leq \tau + 1$. So finding the sets $S(f)$ with these properties completes the proof of the theorem.

Let us fix a vertex f of $W(s,t)$. Let $c = \chi(f)$, $E = \{1 \leq i \leq t : f(i) \text{ is even}\}$ and $O = \{1 \leq i \leq t : f(i) \text{ is odd}\}$. For $1 \leq i \leq t$ let $p_i = \sum_{j\in E, j\leq i}(s - f(j))$ and $q_i = \sum_{j\in O, j\leq i}(s - f(j))$. Note that $f(c) = 0$, so $p_t \geq s$ and as there is an index i with $f(i) = 1$ we have $q_t \geq s - 1$.

The idea is to represent the colors in E and in O as points of the real interval $[0,1]$ and orient the edges from f towards those other vertices whose color in the natural coloring (that is those colors that we will put into $S(f)$ to get the said orientation) is represented by a point which is circularly (that is, when identifying 1 with 0) "somewhat to the right" from the point representing the color of f. To make this orientation consistent for the different vertices of $W(s,t)$ we apply appropriate weightings to determine the distances between the points representing different colors. These weights will depend on the actual values $f(i)$ for each color i that measure the length of the shortest walk in $W(s,t)$ from f to a vertex of color i in the natural coloring.

If $f(1)$ is even, we set $P_i = (p_i - (s - f(1))/2)/p_t$ and $Q_i = q_i/q_t$ for $1 \leq i \leq t$. If $f(1)$ is odd we set $P_i = p_i/p_t$ and $Q_i = (q_i - (s - f(1))/2)/q_t$. We have $0 \leq P_i, Q_i \leq 1$.

Note that $s - f(1)$ is a summand in one of p_i and q_i and the correction term of subtracting half of this summand is a technicality that we will need to be able to prove the theorem also in the case when t is divisible by 4.

Let $\varepsilon = t/(s-1)$. Note that $\varepsilon > 0$ can be made arbitrarily close to zero by choosing s large enough for a fixed t. We express this relationship simply by saying ε is *close* to zero and will use this term in similar meaning later in this proof.

In case there are at most τ indices i with $f(i) = 1$ we define $S(f)$ to be the set of these indices. Otherwise we compute $D_i = Q_i - P_c + 2\varepsilon$ for all indices i with $f(i) = 1$ and let $S(f)$ be formed by the τ indices that have the smallest fractional parts $X_i = D_i - \lfloor D_i \rfloor$.

Property 1 is clear from the definition. In the rest of this proof we establish property 2 if s is large enough.

Assume for a contradiction that the vertices f and f' are connected in $W(s,t)$ but property 2 fails for this edge. Let c, p_i, q_i, P_i, Q_i, D_i and X_i be the above defined values for the vertex f and let c', p'_i, q'_i, P'_i, Q'_i, D'_i and X'_i be the corresponding values for f'.

First observe that as f and f' are connected $|f(i) - f'(i)| \leq 1$ for all i while $f(i)$ and $f'(i)$ are of different parity unless $f(i) = f'(i) = s$. This shows that $|p_i - q'_i| \leq t$ and $|q_i - p'_i| \leq t$ for all i. Easy calculation shows that with our lower bound on p_t and q_t this implies $|P_i - Q'_i| \leq 2\varepsilon$ and similarly $|Q_i - P'_i| \leq 2\varepsilon$.

We have $f(c) = 0$, $f'(c) = 1$, $f'(c') = 0$ and $f(c') = 1$. By the formula defining D_i we have $0 \leq D_{c'} + D'_c \leq 8\varepsilon$. For the fractional parts this means $X_{c'} + X'_c \leq 1 + 8\varepsilon$. We assumed that property 2 is violated, so there are τ indices i with $f(i) = 1$ and $X_i < X_{c'}$ and similarly, for τ indices j we have $f'(j) = 1$ and $X'_j < X'_c$.

It is easy to see that the values X_i for indices satisfying $f(i) = 1$ are separated from each other by at least $(s-1)/q_t$, so we have $X_{c'} \geq \tau(s-1)/q_t$ and therefore $q_t \geq \tau(s-1)/X_{c'}$. Similarly we have $q'_t \geq \tau(s-1)/X'_c$. Using also the bound on $X_{c'} + X'_c$ we obtain $q_t + q'_t \geq 4\tau(s-1)/(1+8\varepsilon)$.

Notice that no index i can contribute to both q_t and q'_t. This is because either one of $f(i)$ or $f'(i)$ is even and thus does not contribute or if $f(i) = f'(i) = s$ is odd, then both contributions are zero. Those indices that do contribute to either q_t or q'_t contribute at most $s-1$, so we have $q_t + q'_t \le t(s-1)$. If $t < 4\tau$ and ε is small enough this contradicts our lower bound on $q_t + q'_t$ and thus completes the proof of property 2 in the $t < 4\tau$ case.

In the tight $t = 4\tau$ case we have to work more for the contradiction. We still have $t(s-1) \ge q_t + q'_t \ge 4\tau(s-1)/(1+8\varepsilon)$, but this inequality does not lead directly to a contradiction. Let $\alpha > 0$. If ε is small enough (the threshold depends on t and α), then it yields that $q_t + q'_t \ge (t-\alpha)(s-1)$ and therefore, since any index can contribute at most $(s-1)$ to one of q_t and q'_t, each index i must contribute at least $(1-\alpha)(s-1)$ to q_t or q'_t (in other words $f(i)$ must be small relative to s). Also, from $t(s-1) \ge q_t + q'_t \ge \tau(s-1)/X_{c'} + \tau(s-1)/X'_c$ one obtains $1/X_{c'} + 1/X'_c \le 4$, thus $X_{c'}$ must be close to $1/2$. (Recall that this means that fixing t and choosing s large enough $|X_{c'} - 1/2|$ can be made arbitrarily small.) Now from $q_t \ge \tau(s-1)/X_{c'}$ (and s large enough) it follows that at least 2τ indices contribute to q_t and similarly, at least 2τ indices contribute to q'_t, so by $4\tau = t$, exactly 2τ indices contribute to each. Thus exactly 2τ indices contribute to p_t, as well.

We can assume by symmetry that $f(1)$ is odd: otherwise switch the roles of f and f'. Now we can estimate P_c and $Q_{c'}$. We have $P_c = p_c/p_t$ and, by the above, this is close to $2k/t$, where $k = |\{1 \le i \le c : f(i) \text{ is even}\}|$. We have $Q_{c'} = (q_{c'} - (s - f(1))/2)/q_t$, and, similarly, this is close to $(2\ell - 1)/t$, where $\ell = |\{1 \le i \le c' : f(i) \text{ is odd}\}|$. This makes $D_{c'} = Q_{c'} - P_c + 2\varepsilon$ close to $(2\ell - 2k - 1)/t$. Here the numerator is odd, the denominator is the fixed value t divisible by 4, so the fractional part $X_{c'}$ of this number cannot be close to $1/2$. This provides the contradiction proving property 2 and completing the proof of the theorem. □

In the following corollaries $s = s_t$ always refers to the s_t of Theorem 27.

Corollary 28 *If G is a topologically t-chromatic graph that has an s-wide t-coloring for the value $s = s_t$, then $\psi_{\mathrm{d,min}}(G) = \lceil t/4 \rceil + 1$.*

Proof. Follows from Theorems 20 and 27. □

Finally, we specify two interesting special cases of Corollary 28. They rely on the topological and wide colorability properties of the relevant graphs established in [23].

Corollary 29 *If $t = n - 2k + 2$ is fixed and $n \ge (2s-2)t^2 - (4s-5)t$ for $s = s_t$, then*

$$\psi_{\mathrm{d,min}}(\mathrm{SG}(n,k)) = \left\lceil \frac{t}{4} \right\rceil + 1.$$

Proof. It is shown in Lemma 5.1 of [23] that if the conditions in the statement are satisfied, then $\mathrm{SG}(n,k)$ admits an s-wide t-coloring. Thus the statement is implied by Corollary 28 and the fact that $\mathrm{SG}(n,k)$ is topologically t-chromatic (cf. [18, 22] or Proposition 8 in [23]). □

Corollary 30 *If G is a topologically t-chromatic graph admitting an s-wide t-coloring for $s = s_t$ and $r \geq 3s - 2$, then*

$$\psi_{\mathrm{d,min}}(M_r(G)) = \left\lceil \frac{t+1}{4} \right\rceil + 1.$$

Proof. By a straightforward generalization of Lemma 4.3 in [23], which itself is a straightforward extension of (a special case of) Lemma 4.1 from [11], one can prove that if G has an s-wide t-coloring and $r \geq 3s - 2$, then $M_r(G)$ has an s-wide $(t+1)$-coloring. Thus the statement follows by Corollary 28 combined with the result of Stiebitz [25] stating that topological t-chromaticity of G implies topological $(t+1)$-chromaticity of $M_r(G)$, cf. also Csorba [5]. □

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