Planar Point Sets Determine Many Pairwise Crossing Segments

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ABSTRACT

We show that any set of n points in general position in the plane determines $n^{1-o(1)}$ pairwise crossing segments. The best previously known lower bound, $\Omega\left(\sqrt{n}\right)$, was proved more than 25 years ago by Aronov, Erdős, Goddard, Kleitman, Klugerman, Pach, and Schulman. Our proof is fully constructive, and extends to dense geometric graphs.

CCS CONCEPTS

• **Mathematics of computing → Combinatoric problems**; *Extremal graph theory*; • **Theory of computation → Computational geometry**.

KEYWORDS

computational geometry, geometric graphs, intersection graphs, crossing edges, avoiding edges, partial orders, comparability graphs, extremal combinatorics

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1 INTRODUCTION

Let V be a set of n points in general position in the plane, that is, assume that no 3 points of V are collinear. A *geometric graph* is a graph $G = (V, E)$ whose vertex set is V and whose edges are represented by possibly crossing straight-line segments connecting certain pairs of points in V. If every pair of points in V is connected by a segment, we have $E = \binom{V}{2}$, and G is called a *complete geometric graph*. Two edges $pq, p'q' \in E$ are said to *cross* if the corresponding segments share an interior point. *Topological graphs* are defined similarly, except that their edges can be represented by any Jordan curves that have no interior points that belong to V.

Crossing patterns and intersection graphs. Finding maximum cliques or independent sets in intersection graphs of segments, rays, and other convex sets in the plane is a computationally hard problem and a classic topic in computational and combinatorial geometry [4, 10, 21, 30–32]. There are many interesting Ramsey-type problems and results about the existence of large cliques *or* large independent sets in intersection graphs of segments [7, 11, 34, 35, 42] and, more generally, of Jordan curves ("strings") [20, 22, 24]. Some of these questions are intimately related to counting incidences between points and lines [41, 45, 46], and to bounding the complexity of k-levels in arrangements of lines in $\mathbb{R}^2$ [16].

It appears to be a somewhat simpler task to understand the *combinatorial structure* of crossings between the edges of a geometric or topological graph. Despite decades of steady progress, we have very few asymptotically tight results in this direction. Perhaps the best known and most applicable theorem of this kind is the so-called *Crossing Lemma* of Ajtai, Chvátal, Newborn, Szemerédi [6] and Leighton [36], which states that any topological graph $G = (V, E)$ with $|E| > 4|V|$ determines at least $\Omega\left(|E|^3/|V|^2\right)$ crossing pairs of edges. Recently, a similar result has been established by the authors for contact graphs of families of Jordan curves [40].

According to another asymptotically tight result, for $t > 1$, every geometric graph G with $|E| \geq nt$ edges has two disjoint sets of edges, $E_1, E_2 \subset E$, each of size $\Omega(t)$, such that every edge in E_1 crosses all edges in E_2; see, e.g., [20, Theorem 6]. A similar theorem holds for topological graphs, with the difference that then $|E_1|, |E_2| = \Omega\left(t/\log t\right)$ [22]. It is a major unsolved question to decide whether under these circumstances G must also contain a family of *pairwise crossing* edges, whose size tends to infinity as $t \to \infty$. It is conjectured that one can always choose such a family consisting of almost t edges. If this stronger conjecture is true for $t^{1-o_t(1)}$ edges, then for $t \approx n/2$, it would imply that every *complete* geometric or topological graph on n vertices has $n^{1-o(1)}$ pairwise crossing edges.

A topological graph is called *t-quasi-planar* if it contains no t pairwise crossing edges. Let $f_t(n)$ (and $f'_t(n)$) denote the maximum number of edges that a t-quasi-planar geometric (resp., topological) graph of n vertices can have. Clearly, we have $f_t(n) \leq f'_t(n)$, for every t and n. For geometric graphs, Valtr [50] proved that $f_t(n) = O_t(n \log n)$, but in general the best known upper bound is only $f'_t(n) = O_t\left(n(\log n)^{O(\log t)}\right)$ [20]. It is conjectured that $f_t(n) \leq f'_t(n) \leq t^{1+o_t(1)}n$, which is known to be true only for $t \leq 4$; see [1–3].

Our results. The aim of the present paper is to find many pairwise crossing edges in dense geometric graphs. For every $n \geq 2$, let $T(n)$ denote the largest positive integer T with the property that any complete geometric graph with n vertices has at least T pairwise crossing edges. Equivalently, $T(n)$ is the largest number such that any set V of n points in general position in the plane determines at

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least T pairwise crossing segments. (A segment is *determined* by V if both of its endpoints belong to V.)

It was proved by Aronov, Erdős, Goddard, Kleitman, Klugerman, Pach, and Schulman [8] in 1991 that $T(n) = \Omega(\sqrt{n})$, cf. [49]. Since then no one has been able to improve this bound. The prevailing conjecture is that $T(n) = \Theta(n)$ [9]. A slightly sublinear lower bound holds for "uniformly distributed" point sets, in which the ratio of the largest distance and the smallest distance between two points is $O(\sqrt{n})$ [48]. Note that the best known construction found by a group of Austrian and Czech researchers gives $T(n) \leq \lceil n/5 \rceil$; see [5].

Our main theorem comes close to settling the above conjecture in the affirmative, and applies in a more general setting.

Theorem 1.1. *(i) Any set V of n points in general position in the plane determines at least $n/2^{O(\sqrt{\log n})}$ pairwise crossing segments.*

(ii) There exists an absolute constant c such that any geometric graph with n vertices and at least $n^{2-\varepsilon}$ edges with $\varepsilon > (\log n)^{-2/3}$ has at least $n^{1-c\sqrt{\varepsilon}}$ pairwise crossing edges.

Ignoring the specific order of the error term part (i) of the theorem implies the existence of $n^{1-o(1)}$ pairwise crossing edges in a complete geometric graph with n vertices, while part (ii) implies the same in *all* geometric graphs with n vertices and $n^{2-o(1)}$ edges.

Our proof of Theorem 1.1 is fully algorithmic. The crossing segments can be found by an efficient algorithm whose running time is near-quadratic in n for complete or dense geometric graphs. Our construction heavily relies on the assumption that the segments are straight, but otherwise it is fairly robust. Below we briefly sketch the key underlying ideas of our proof of Theorem 1.1.

The main observation of Aronov et al. [8] was that any n-point set V in general position in the plane contains a pair of subsets A and B, each of cardinality $\Omega\left(\sqrt{n}\right)$ that can be separated by a line, and so that all the points in B see the points in A in the same order, and vice versa. Such a pair $A, B \subset V$ is called *avoiding*, and its properties are reviewed in Section 3. Connecting each point of A to the "opposite" point of B, we obtain a family of $\Omega\left(\sqrt{n}\right)$ pairwise crossing segments. Valtr [49] showed that, for certain instances of V, one cannot find an avoiding pair with $|A| = |B|$ greater than some constant times $\sqrt{n}$.

To get around this barrier, we apply a divide-and-conquer approach. We first use a decomposition machinery from computational geometry [38] in order to obtain a pair of sets A and B whose size is close to n, which satisfies the following property: The points of A are separated from the points of B by a line, and all points of B see the points in A in the same *approximate* order (and vice versa). This order is described by a poset with $o\left(|A|^2\right)$ incomparable pairs. As a result, we are able to subdivide most points in A (resp., B) into smaller subfamilies $\{A_i\}_{i=1}^k$ (resp., $\{B_i\}_{i=1}^k$) so that any pair of vertices of $\bigcup_{i=1}^k A_i$ that are incomparable lie in the same set A_i, and a symmetric property holds for $\bigcup_{i=1}^k B_i$. It is easy to check that this decomposition is regular in the sense that for any four distinct sets $A_a, A_{a'}, B_b$ and $B_{b'}$, the property that the edge xy in the subinstance $A_a \times B_b$ crosses the edge $zw \in A_{a'} \times B_{b'}$ is invariant to the choice of the representatives x, y, z, w. Finally, we use the dual of Dilworth's Theorem [39] to obtain a large family of pairwise crossing such "super-edges" $A_a \times B_b$ while also maintaining the "almost avoiding" property of these pairs.

The rest of the paper is organized as follows. In Section 2, we prove three preliminary lemmas that are important for our analysis. The proof of Theorem 1.1 is given in Section 3. Section 4 contains the concluding remarks along with a brief discussion of the constructive aspects of Theorem 1.1. We also provide an analogue of Theorem 1.1 which yields many pairwise avoiding edges in dense geometric graphs. Here two segments are said to be *avoiding* if (the closure of) neither of them is intersected by the supporting line of the other segment [8, 43, 50]. (In particular, the segments must be disjoint. In much of the earlier literature, such pairs of segments are called "parallel".)

2 THREE PRELIMINARY LEMMAS

We use the term *poset* for finite partial ordered sets $(P, <)$. If the ordering is clear from the context, we write $x||y$ to indicate that the elements $x, y \in P$ are *incomparable*. Let $\iota(P, <)$ denote the number of incomparable pairs of elements in P. For subsets A and B of P we write $A < B$ if $a < b$ for all $a \in A$ and $b \in B$.

We start with the following lemma which is close to Theorem 4 (ii) in [47]. However, Tomon's bound is not sufficient for our purposes.

Lemma 2.1. *Let n and k be positive integers, and $(P, <)$ be a poset with $|P| > nk$ and $\iota(P, <) \leq \frac{(|P|-nk)^2}{16k}$.*

Then one can choose suitable n-element subsets $A_1, A_2, \ldots, A_k$ of P with $A_i < A_j$ for all $i < j$.

Proof. Let $I_x = \{y \in P \mid y||x\}$ be the set of elements incomparable to $x \in P$. Let $T = (|P| - nk)/(4k)$ and $Q = \{x \in P \mid |I_x| < T\}$. Since $\sum_{x \in P} |I_x| = 2\iota(P, <)$, we have

$$|P \setminus Q| \leq \frac{2\iota(P, <)}{T} \leq \frac{|P| - nk}{2}.$$

Let $x_1, x_2, \ldots, x_m$ be a linear ordering of the elements of Q compatible with the partial order $<$. That is, $m = |Q|$, $Q = \{x_1, \ldots, x_m\}$, and whenever $x_i < x_j$, we have $i < j$. If $x_i||x_j$ for some $1 \leq i < j \leq m$, then each element x_l with $i \leq l \leq j$ must be incomparable to either x_i or x_j. As both x_i and x_j are incomparable with less than T elements, this implies $j - i < 2T$. Therefore $j - i \geq 2T$ implies $x_i < x_j$.

We choose the sets A_i to be intervals of the linear order on Q with enough buffer between them to ensure the comparability. Namely, we set $A_i = \{x_j \mid (i-1)(n + \lfloor 2T \rfloor) < j \leq (i-1)(n + \lfloor 2T \rfloor) + n\}$ for every $1 \leq i \leq k$. We only need to check that Q is large enough for all these intervals to fit. This holds, because

$$m = |Q| = |P| - |P \setminus Q| \geq |P| - \frac{|P| - nk}{2} \geq k(n + 2T).$$

□

Let x and y be distinct points in the plane. Let us orient the line xy from x toward y, and denote the open half-plane to the left of this oriented line by $\ell(xy)$. Let B be a nonempty set in the plane. For any pair of distinct points in the plane, x and y, write $x <_B y$ if B is contained in $\ell(xy)$; see Figure 1 (top). The convex hull of a set B is denoted by $\mathrm{conv}(B)$.

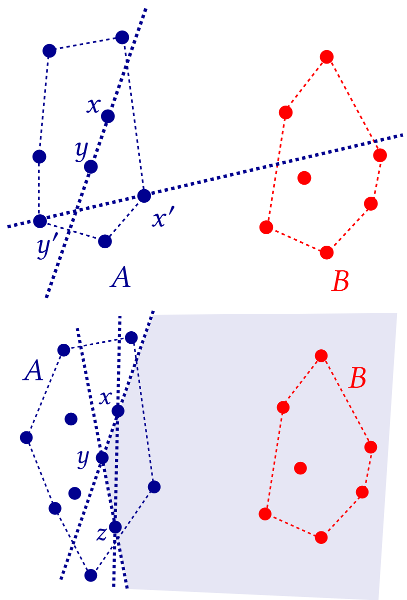

Figure 1: Top: The partial order $(A, <_B)$. We have $x <_B y$ if and only if B lies in the half-plane $\ell(xy)$ to the left of the directed line from x to y. (The points x' and y' are incomparable, because the line xy crosses the convex hull of B.) Bottom: Lemma 2.2 – proving that the relation $(A, <_B)$ is transitive.

Lemma 2.2. *Let A and B be nonempty sets in the plane such that their convex hulls are disjoint. Then $<_B$ defines a partial order on A, in which two points x and y are incomparable if and only if the line xy intersects* conv(B). *Further, if $x <_B y$ for $x, y \in A$ and $z <_A t$ for $z, t \in B$, then the segments xz and yt cross.*

Proof. To check transitivity, let us assume that $x <_B y$ and $y <_B z$ for a triple of points $x, y, z \in A$; see Figure 1 (bottom). This means that both $\ell(xy)$ and $\ell(yz)$ contain B. We have

$$\ell(xz) \supseteq (\ell(xy) \cap \ell(yz)) \setminus \mathrm{conv}(xyz),$$

and conv(xyz) is disjoint from B. Therefore, $B \subseteq \ell(xz)$ must hold, which means that $x <_B z$, as required.

The distinct points x and y are incomparable in $<_B$ if and only if B is not contained in either of the open half planes bounded by the line xy, which happens if and only if the line intersects the convex hull of B.

Finally, for any $x, y \in A$ and $z, t \in B$ that satisfy the inequalities $x <_B y$ and $z <_A t$, the quadrilateral $xyzt$ must be strictly convex, so the diagonals xz and yt must cross. □

Any finite family L of lines in $\mathbb{R}^2$ induces the *arrangement* $\mathcal{A}(L)$ – the partition of $\mathbb{R}^2$ into 2-dimensional *cells*, or 2-*faces*. Each of these cells is a maximal connected region of $\mathbb{R}^2 \setminus (\bigcup L)$; it is a (possibly unbounded) convex polygon whose boundary is composed of *edges* – portions of the lines of L, which connect vertices – crossings amongst the lines of L.

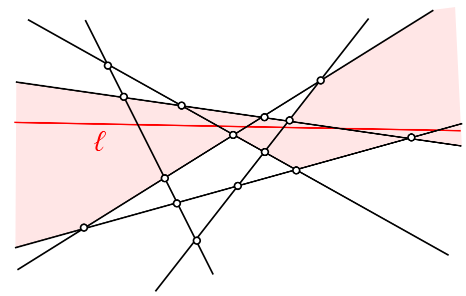

Figure 2: An arrangement of 6 lines. The cells in the zone of a (seventh) line ℓ are shaded.

The *zone* of a line ℓ within the arrangement $\mathcal{A}(L)$ is the union of all the cells that ℓ intersects; see Figure 2. The following result was implicitly established by Matoušek [38]; it comprises most of the proof of the so called Test Set Lemma.

Lemma 2.3 (Matoušek, [38]). *For any n-element point set V in general position in the plane and for any $\varepsilon > 0$, we can find $O(1/\varepsilon^2)$ lines such that the zone of any other line in their arrangement contains at most εn points of V.*

For his proof, Matoušek [38] used a result of Chazelle and Friedman [14], according to which any set of n lines admits an *optimal ε-cutting*. In other words, one can partition the plane into $O\left(\varepsilon^{-2}\right)$ simply shaped cells (triangles or trapezoids, some of which are unbounded) with the property that every cell is crossed by at most εn lines. He applied this result to the dual line set V^* of V [37, Section 5], and argued that the duals of the vertices of this cutting (that correspond to lines in the primal plane) satisfy the requirements of Lemma 2.3. Before giving a formal proof of the lemma, we outline a more explicit argument which yields the same result with the slightly weaker bound $O(\log^2(1/\varepsilon)/\varepsilon^2)$. This bound also suffices for our calculations.

We consider all *angular sectors* in $\mathbb{R}^2$; each of them can be obtained as the intersection of two open half-planes. We construct a set of points $Q \subset V$ which pierces every angular sector that contains at least $\varepsilon n/4$ points of V; such a set is known as an $(\varepsilon/4)$-*net.* In accordance with the standard theory of (strong) ε-nets [27], angular sectors have a *bounded VC-dimension* (namely, 5), so we can find such a net Q with $|Q| = O(\log(1/\varepsilon)/\varepsilon)$. Note that the family $\mathcal{L}$ of all lines determined by the point set Q is of size $O(\log^2(1/\varepsilon)/\varepsilon^2)$. To see that this family satisfies the requirements of the lemma, it suffices to check that the zone of any line ℓ is contained in the union of at most *four* angular sectors, each disjoint from Q. We show that the part of the zone in a half-plane F bounded by ℓ can be covered by two such angular sectors. For simplicity, we assume ℓ does not pass through any point in Q and is not parallel to a line determined by two points in Q. These restrictions are not essential but allow us to avoid a case analysis. If $Q \cap F$ is empty or consists of collinear points, then the statement is trivial. Otherwise, let x be the closest point to ℓ in $Q \cap F$ and let A be the smallest angular sector with apex x, whose closure contains $Q \cap F$; see Figure 3. Notice that the two delimiting lines of A belong to the family $\mathcal{L}$. It readily follows that $F \setminus \overline{A}$ is disjoint from Q, it is the union of two angular sectors, and it contains the part of the zone of ℓ inside F.

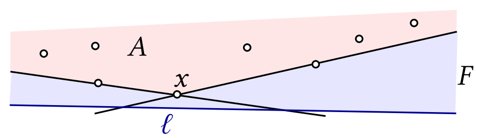

Figure 3: Proof of the relaxed variant of Lemma 2.3. The point $x \in Q$ is the closest such point to ℓ in the half-plane F (above ℓ). A is the smallest angular sector with apex x, and whose closure contains $Q \cap F$. The complement of the closure of A within F is covered by two angular sectors which together cover the part of the zone of ℓ within F.

Proof of Lemma 2.3. Let V^* denote the set of lines dual to the points of V. Since the points of V are in a general position, an analogous condition must hold for the dual set V^*: no three lines in V^* can meet at the same point, and none of these lines can be horizontal or vertical. We construct an optimal ε-cutting $\Xi(V^*)$ of the dual plane with respect to V^* [14]. Namely, $\Xi(V^*)$ is a decomposition of the dual plane into $O\left(1/\varepsilon^2\right)$ interior-disjoint triangles with the property that each of these triangles is crossed by at most $\lfloor \varepsilon|V^*|\rfloor = \lfloor \varepsilon n\rfloor$ dual lines belonging to V^*.[1]

Our set, which we denote by $\mathcal{L}$, consists of the lines dual to the vertices of $\Xi(V^*)$. To see that $\mathcal{L}$ satisfies the asserted property, let ℓ be an arbitrary non-vertical line in the primary plane. Assume first that the dual point ℓ^* is contained in the relative interior of some triangle $\Delta \in \Xi(V^*)$. Since the lines in V^* are in a general position, the argument readily extends to the points that lie on boundaries of the triangles of $\Delta \in \Xi(V^*)$. The crucial observation is that for any point of V that lies in the zone of ℓ within the arrangement of $\mathcal{L}$, its dual line $p^* \in V^*$ must cross the boundary of the triangle Δ (or, else, p will be "separated" from ℓ by the lines dual to the vertices of Δ); see [12] for the precise details. Hence, the number of such points cannot exceed εn. □

3 PROOF OF THEOREM 1.1

We begin by introducing some terminology.

Definition 3.1. *Two point sets $A, B \subset \mathbb{R}^2$ are called* separated *if* $\mathrm{conv}(A) \cap \mathrm{conv}(B) = \emptyset$. *Two separated m-element sets, A and B, are said to form an* ε-avoiding pair *for some $\varepsilon \geq 0$ if*

$$\iota(A, <_B) + \iota(B, <_A) \leq \varepsilon m^2.$$

For simplicity, a 0-avoiding pair is called avoiding. *See Figure 4 (top).*

The special case $\varepsilon = 0$ was studied in [8], where it was shown that any point set in general position in the plane contains an avoiding pair of $\Omega(\sqrt{n})$-element subsets. Then it was shown that there is a one-to-one correspondence between the elements of any pair of avoiding sets of the same size such that any two segments formed by the corresponding points cross each other (as depicted in Figure 4 (bottom) and hence every set of n points in general position in the plane determines $\Omega(\sqrt{n})$ pairwise crossing segments. Here the second statement follows from Lemma 2.2. Indeed, if the t-element point sets A and B form an avoiding pair, then $<_B$ linearly orders A, so $A = \{x_1, x_2, \ldots, x_t\}$ with $x_1 <_B x_2 <_B \cdots <_B x_t$ and similarly $B = \{y_1, y_2, \ldots, y_t\}$ with $y_1 <_A y_2 <_A \cdots <_A y_t$. So by Lemma 2.2 the segments x_iy_i are pairwise crossing for $i = 1, \ldots, t$.

This approach cannot yield any better result, because Valtr [49] showed that there are many n-element point sets with no avoiding pair of size larger than constant times $\sqrt{n}$. One of the key ideas of our proof is the relaxation of the notion of avoiding pairs, described in Definition 3.1.

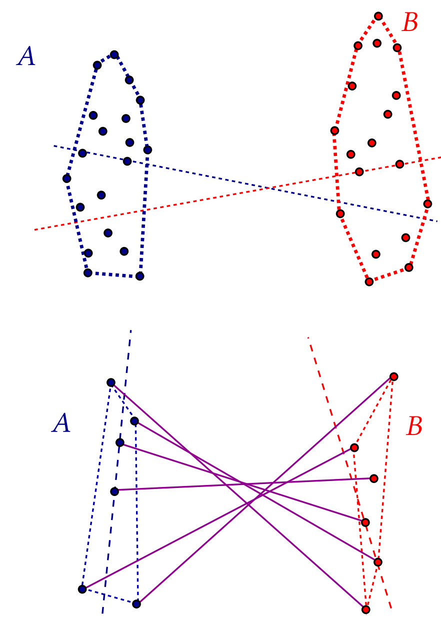

Figure 4: Top: Two separated sets A and B form an *ε-avoiding pair* if there exist at most εm^2 incomparable pairs in $\binom{A}{2} \cup \binom{B}{2}$, whose connecting lines cross the convex hull of the opposite set. Bottom: A 0-avoiding pair $\{A, B\}$ with $|A| = |B| = 6$, and its induced family of 6 pairwise crossing edges.

To be able to deal with geometric graphs that are not complete, we introduce a further definition. Let $G = (V, E)$ be a geometric graph. For any subsets $A, B \subseteq V$, let $E(A, B) \subseteq E$ denote the set of edges connecting a vertex in A with a vertex in B.

Definition 3.2. *Given a geometric graph $G = (V, E)$ and a number $\delta \geq 0$, a pair $\{A, B\}$ of disjoint m-element subsets of V is* δ-dense *with respect to G if $|E(A, B)| \geq \delta m^2$.*

If it is clear what the underlying geometric graph is, with no danger of confusion we simply say that the pair is *δ-dense*. Our next result states that in every sufficiently large and dense geometric graph, one can find two separated m-element sets that form an ε-avoiding, δ-dense pair.

Lemma 3.3. *There exists an absolute constant $c > 0$ with the following property. For any integer $m > 0$ and any reals $\varepsilon, \delta > 0$,*

[1] Some of the vertices of these triangles may lie at infinity on one of the imaginary lines $x = \infty$, $x = -\infty$, $y = \infty$, $y = -\infty$, which are incorporated in V^*.

every geometric graph $G = (V, E)$ with $|V| \geq \frac{cm}{\varepsilon^4\delta^5}$ and $|E| \geq \delta|V|^2$ has two separated m-element sets of vertices that form an ε-avoiding, δ-dense pair with respect to G.

Proof. Let us set $n = |V|$. We use Lemma 2.3 to obtain an arrangement $\mathcal{A}(L)$ of a set L of $r = O\left(1/(\varepsilon\delta)^2\right)$ lines such that the zone of any line contains at most $\varepsilon\delta n/2$ points of V. We assume that $r \geq 3$.

Using parallel segments or half-lines that do not pass through any point of V, split each cell of $\mathcal{A}(L)$ into smaller cells such that all but at most one of them contain precisely m elements of V, and if there is an exceptional cell, it contains fewer than m points. Denote the resulting cell decomposition of $\mathbb{R}^2$ by Π. Obviously, every cell in Π is convex. The set of m points of V inside a non-exceptional cell of Π is called a *cluster*; see Figure 5. Denoting the number of clusters by D, we have $D \leq n/m$.

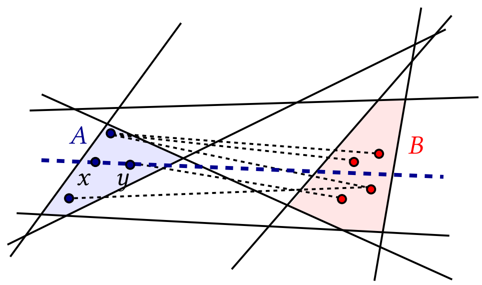

Figure 5: A pair of cells in the decomposition Π, which determine clusters A and B. The edges of $E(A, B)$ are depicted. If the pair $x, y \in A$ is incomparable with respect to $<_B$, then B must lie in the zone of the line xy.

Let H denote the set of points in V that do not belong to any cluster. Such a point must lie either in an exceptional cell of Π or on a line of L. There are at most $r^2 - 1$ exceptional cells, each containing fewer than m points in V, and Π has r lines, each passing through at most 2 points in V. Thus, we have $|H| \leq (r^2 - 1)m$. Every edge of G belongs to one of the following categories:

(i) it has a vertex belonging to H, or
(ii) it connects a pair of vertices in the same cluster, or
(iii) it connects a pair of vertices in distinct clusters that do *not* form a δ-dense pair, or
(iv) it connects a pair of vertices in distinct clusters that form a δ-dense pair.

The number of edges in category (i) is at most $|H|n \leq (r^2 - 1)mn$. The number of edges in category (ii) is at most $D\binom{m}{2} < mn$. The number of edges in category (iii) is less than $\binom{D}{2}\delta m^2 < \delta n^2/2$. By assumption, G has at least δn^2 edges, so more than $\delta n^2/2 - r^2 mn$ of them must belong to category (iv). The number of edges between any two distinct clusters is at most m^2. Therefore, the number of unordered δ-dense pairs of clusters is at least

$$\frac{\delta n^2/2 - r^2 mn}{m^2} = \frac{\delta n^2}{2m^2} - \frac{r^2 n}{m}. \tag{1}$$

Let

$$X = \sum_{\{A,B\}} (\iota(A, <_B) + \iota(B, <_A)),$$

where the sum is taken over all unordered pairs of distinct clusters $\{A, B\}$.

We can estimate X according to the point pairs incomparable with respect to a cluster. An unordered pair of distinct vertices $\{x, y\}$ will contribute to the sum X only if x and y come from the same cluster, and in this case its contribution will be the number of other clusters whose convex hulls are crossed by the line xy. All of these clusters belong to the zone of the line xy in the arrangement $\mathcal{A}(L)$. By the choice of L, the zone of any line contains at most $\varepsilon\delta n/2$ vertices. Thus, the contribution of any pair of points is at most $\varepsilon\delta n/(2m)$, so we have

$$X \leq D\binom{m}{2}\frac{\varepsilon\delta n}{2m} < \varepsilon\delta n^2/4.$$

On the other hand, each pair of clusters which is *not* ε-avoiding contributes more than εm^2 to X, so we have fewer than $X/(\varepsilon m^2)$ such pairs. This implies that the number of not ε-avoiding pairs of clusters is less than

$$\frac{\varepsilon\delta n^2/4}{\varepsilon m^2} = \frac{\delta n^2}{4m^2}. \tag{2}$$

Clearly, each cluster has m elements and any two of them are separated. We are done if we find an ε-avoiding δ-dense pair formed by two clusters. For this, it is sufficient to show that there are more δ-dense pairs of clusters than pairs that are not ε-avoiding. Comparing (1) and (2), this is true as long as

$$\frac{\delta n^2}{2m^2} - \frac{r^2 n}{m} \geq \frac{\delta n^2}{4m^2},$$

that is, if $n \geq 4mr^2/\delta$. This can be ensured by choosing the constant c large enough as a function of the constant hidden in $r = O(1/(\varepsilon\delta)^2)$, which comes from Lemma 2.3. □

The heart of the proof is the following partition result.

Lemma 3.4. *Let k, m, and $t \geq 3$ be positive integers, and set $\delta = 1/t$, $\varepsilon = 1/(32t^2k)$. Let A and B be two separated sets of vertices in a geometric graph G with $|A| = |B| = (t + 1)km$. Suppose that $\{A, B\}$ is a 8δ-dense, $\varepsilon\delta$-avoiding pair.*

Then we can find pairwise disjoint m-element subsets $A_1, A_2, \ldots, A_k \subset A$, $B_1, B_2, \ldots, B_k \subset B$ such that for every $1 \leq i \leq k$, $\{A_i, B_i\}$ is a δ-dense, ε-avoiding pair and, for $1 \leq i < j \leq k$, all edges in $E(A_i, B_i)$ cross every edge in $E(A_j, B_j)$.

Proof. It follows from the assumption that $\{A, B\}$ is an $\varepsilon\delta$-avoiding pair that the posets $(A, <_B)$ and $(B, <_A)$ both satisfy the conditions of Lemma 2.1 with $n = m$ and with tk in place of k. Indeed, by Definition 3.1, we obtain

$$\iota(A, <_B) + \iota(B, <_A) \leq \varepsilon\delta(t + 1)^2k^2m^2 = \frac{(1 + 1/t)^2km^2}{32t} <$$

$$< \frac{(|A| - mtk)^2}{16tk}.$$

Hence, we can apply Lemma 2.1 to find suitable subsets $C_i \subseteq A$ and $D_i \subseteq B$ for $1 \leq i \leq tk$ such that $|C_i| = |D_i| = m$ for all i, and $C_i <_B C_j$, $D_i <_A D_j$ for all $i < j$; see Figure 6 (top).

Lemma 2.2 implies that all edges in $E(C_i, D_i)$ cross every edge in $E(C_j, D_j)$ as long as $i \neq j$, which suggests to select the pairs (A_i, B_i) from among the the pairs (C_i, D_i). Unfortunately, many of these pairs (C_i, D_i) might *not* be δ-dense or ε-avoiding. We must be careful when we select a suitable pair for a set C_a. Any collection of δ-dense, ε-avoiding pairs $\{(C_a, D_b)\}$ will work here (still by Lemma 2.2), as long as $b = b(a)$ is a monotone increasing function of a.

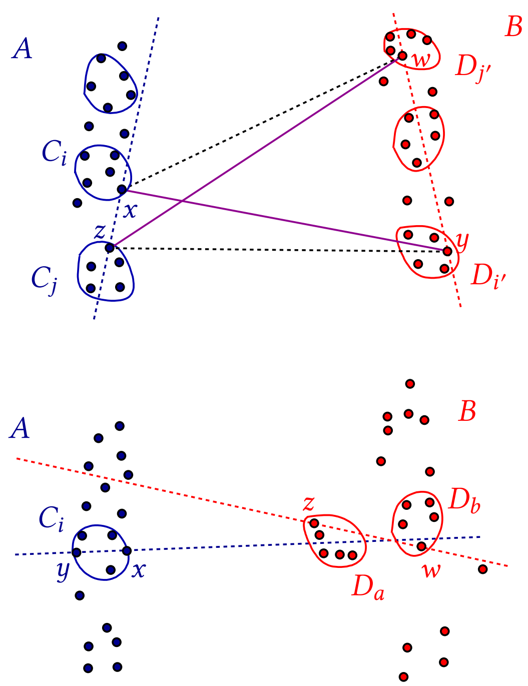

Figure 6: Top: The subsets $\{C_i\}$ and $\{D_i\}$. We have $C_i <_B C_j$ and $D_{i'} <_A D_{j'}$ whenever $i < j$ and $i' < j'$. Notice that in this case by Lemma 2.2, for any choice of $x \in C_i$, $z \in C_j$, $y \in D_{i'}$, and $w \in D_{j'}$, the segments xy and zw cross. Bottom: Proof of Claim 3.5. If x and y are incomparable with respect to both sets D_a and D_b, so that $a < b$, then y lies to the right of the line from z to w. This is contrary to the assumption that $D_a <_A D_b$.

Since $\{A, B\}$ is an 8δ-dense pair and $|A| = |B| = (t+1)km$, by definition we have

$$|E(A,B)| \geq 8\delta((t+1)km)^2.$$

Every edge in $E(A, B)$
(i) belongs to $E(C_a, D_b)$ for some δ-dense pair (C_a, D_b), or
(ii) belongs to $E(C_a, D_b)$ for a pair (C_a, D_b) that is not δ-dense, or
(iii) has at least one endpoint outside $X = \bigcup_{a=1}^{tk}(C_a \cup D_a)$.

Note that both A and B have exactly km elements outside X, so the number of edges that belong to the last category is less than $2km|A| = 2(t+1)k^2m^2$. If (C_a, D_b) is not δ-dense, then $|E(C_a, D_b)| < \delta m^2$. Hence, the total number of edges of category (ii) is smaller than $(tk)^2\delta m^2$. Thus, the number of edges of category (i) is larger than

$$|E(A,B)| - 2(t+1)k^2m^2 - (tk)^2\delta m^2 > 4tk^2m^2.$$

Each δ-dense pair (C_a, D_b) contributes $|E(C_a, D_b)| \leq m^2$ edges to this category. Therefore, the number of δ-dense pairs (C_a, D_b) is larger than $4tk^2$.

Consider now a pair of points $x, y \in A$. Clearly, if $x <_B y$, then $x <_{D_b} y$ for all $1 \leq b \leq tk$.

Claim 3.5. *If x and y are incomparable in $<_B$, they are still comparable with respect to all but at most one ordering $<_{D_b}$, $1 \leq b \leq tk$.*

Proof of Claim 3.5. To verify this claim, suppose that x and y are incomparable in $<_B$, so the line xy intersects $\mathrm{conv}(B)$. Assume by symmetry that x is closer to this intersection than y is. Suppose for a contradiction that x and y are incomparable in both $<_{D_a}$ and $<_{D_b}$ for some $1 \leq a < b \leq tk$; see Figure 6 (bottom). As $x <_{D_a} y$ does not hold, there is a point $z \in D_a$ that does not belong to $\ell(xy)$. Similarly, as $y <_{D_b} x$ does not hold, there is another point $w \in D_b$ that does not belong to $\ell(yx)$. Thus, the segment zw must intersect the line xy and, by our assumption, this intersection point is closer to x than to y. This implies that $y \notin \ell(zw)$ and, hence, $z <_A w$ does not hold. Therefore, the relation $D_a <_A D_b$ cannot hold either, contradicting our assumption that $a < b$. This contradiction proves the claim. □

It follows from Claim 3.5 that

$$\sum_{1 \leq a, b \leq tk} \iota(C_a, <_{D_b}) \leq \iota(A, <_B).$$

By symmetry, we also have

$$\sum_{1 \leq a, b \leq tk} \iota(D_b, <_{C_a}) \leq \iota(B, <_A).$$

Thus, we obtain that

$$\sum_{1 \leq a, b \leq tk} (\iota(C_a, <_{D_b}) + \iota(D_b, <_{C_a})) \leq \iota(A, <_B) + \iota(B, <_A) \leq$$

$$\leq \varepsilon\delta((t+1)km)^2,$$

where the last inequality follows from the assumption that $\{A, B\}$ is an $\varepsilon\delta$-avoiding pair. By definition, any pair (C_a, D_b) that is *not* ε-avoiding contributes more than εm^2 to this sum, so the number of such pairs is smaller than $\delta(t+1)^2k^2 < 2tk^2$.

All pairs $\{C_a, D_b\}$ are separated and consist of m-element sets. Call such a pair *eligible* if it is both δ-dense and ε-avoiding. As we saw above, the number of δ-dense pairs (C_a, D_b) is larger than $4tk^2$. We have just seen that fewer than $2tk^2$ of them are not ε-avoiding. Thus, the number of eligible pairs is larger than $2tk^2$.

Define a partial order on the set of eligible pairs, as follows. Let $\{C_a, D_b\} < \{C_{a'}, D_{b'}\}$ if $a < a'$ and $b < b'$. If there was no monotone chain of length k with respect to this partial order, then by the dual of Dilworth's theorem [39] all eligible pairs could be covered by fewer than k antichains (i.e., fewer than k sets of pairwise incomparable eligible pairs). The value $a - b$ is distinct for each pair $\{C_a, D_b\}$ in an antichain. Here we have $-kt < a - b < kt$, so antichains have fewer than $2kt$ eligible pairs. Hence, in this case the total number of eligible pairs would be smaller than $2tk^2$, contradicting our above estimate.

Thus, there exists a monotone chain of k eligible pairs, say $\{A_1, B_1\} < \{A_2, B_2\} < \cdots < \{A_k, B_k\}$. These pairs obviously satisfy the requirements of Lemma 3.4. □

By repeated application of Lemma 3.4, we obtain the following result.

Lemma 3.6. *Let s and u be positive integers such that u is a multiple of 8^s, and set $\delta = 8^s/u$, $\varepsilon = 1/(32u^{s+2})$, and $K = (512u)^{\binom{s}{2}}$.*

There exists a positive integer $M = M_s(u) \le u^s K$ such that for any two M-element sets of vertices, A and B that form a δ-dense ε-avoiding pair in a geometric graph G, the set of edges $E(A,B)$ connecting them contains K pairwise crossing elements.

Proof. We prove the lemma by induction on s. The statement is trivial for $s = 1$, as in this case we have $K = 1$, so we can choose $M = 1$ and select any one edge from $E(A,B)$ (which is nonempty, because $\delta > 0$).

Let $s > 1$ and assume that the statement holds for $s-1$ in place of s. We apply Lemma 3.4 with $t = u/8^{s-1}$, $k = (512u)^{s-1}$, and $m = M_{s-1}(u)$. Setting $M = M_s(u) = (t+1)km$, the lemma states that given a δ-dense ε-avoiding pair $\{A,B\}$ with $|A| = |B| = M$, we can find pairwise disjoint m-element subsets $A_1,\dots,A_k \subset A$ and $B_1,\dots,B_k \subset B$ such that (A_i,B_i) is a $1/t$-dense $1/(32t^2k)$-avoiding pair for every i, and every edge in $E(A_i,B_i)$ crosses all edges in $E(A_j,B_j)$, provided that $i \ne j$. For each i, we use the inductive hypothesis with the same u and with $s-1$ in place of s to find a family Z_i of $K_0 = (512u)^{\binom{s-1}{2}}$ pairwise crossing edges in $E(A_i,B_i)$. Note that (A_i,B_i) is $1/(32t^2k)$-avoiding. To apply the induction hypothesis, we need that (A_i,B_i) is $1/(32u^{s+1})$-avoiding, but this holds, as

$$1/(32t^2k) = 1/(32\cdot 8^{s-1}u^{s+1}) \le 1/(32u^{s+1}).$$

The union of the sets Z_i contains $kK_0 = K$ edges of $E(A,B)$, any two of which cross. Since $m = M_{s-1}(u) \le u^{s-1}K_0$, we obtain that

$$M = M_s(u) = (t+1)km \le uku^{s-1}K_0 = u^sK.$$

□

Part (ii) of Theorem 1.1 follows by combining Lemmas 3.3 and 3.6.

Proof of Theorem 1.1(ii). Let G have n vertices and at least n^{2-x} edges with $x > (\log n)^{-2/3}$. We set the value of s later and set $u = 8^s\lceil n^x\rceil$. Apply Lemma 3.3 with $\delta = 8^s/u$, $\varepsilon = 1/(32u^{s+2})$ and $m = M_s(u)$ from Lemma 3.6. The density condition of Lemma 3.3 is satisfied, so if n is large enough, we obtain separated vertex sets A and B of size m forming a δ-dense, ε-avoiding pair. If this happens, we apply Lemma 3.6 with the parameters s and u to obtain $K = (512u)^{\binom{s}{2}}$ pairwise crossing edges in $E(A,B)$.

It remains to do the calculation to check what value we can choose for s and how many pairwise crossing edges we find this way. We can apply Lemma 3.3 if $n \ge cm\delta^{-5}\varepsilon^{-4}$ for the absolute constant c in that lemma. Using $m = M_s(u) \le u^sK$ and the formulas defining δ and ε, we see that $n \ge 32^4cu^{5s+13}K$ suffices. In case $n = O(u^{5s+13}K)$, we have $n/K = O(u^{5s+13})$. Here $n \ge K \ge u^{\binom{s}{2}} \ge n^{x\binom{s}{2}}$, so $x\binom{s}{2} \le 1$ and $s = O(1/\sqrt{x})$. We also have $u = 2^{O(s)}n^x = n^{x+O(s/\log n)} = n^{O(x)}$, since $s/\log n = O(x^{-1/2}/\log n) = O(x)$ by our lower bound on x. We have $n/K = u^{O(s)} = n^{O(sx)} = n^{O(\sqrt{x})}$ as claimed.

In the above line of reasoning we assumed that the size of G is just suitable, namely $n \ge 32^4cu^{5s+13}K$, but barely. In the general case, we simply set the value of the parameter s to be the largest possible value. That is, we still have $n \ge 32^4cu^{5s+13}K$, but the same formula is violated if s is increased by one. Note that increasing s to $s^* = s+1$ changes the values of u to $u^* = 8u$ and K to $K^* = 512^s\cdot 8^{\binom{s+1}{2}}u^sK$. We have $n < 32^4cu^{*5s^*+13}K^* = 2^{O(s^2)}u^{O(s)}K$. We still have $n/K = 2^{O(s^2)}u^{O(s)} = n^{O(xs+s^2/\log n)} = n^{O(\sqrt{x})}$ as needed. Finally, if even the choice of $s = 1$ is not feasible, then finding any one edge of the graph suffices. □

Note that a complete graph on n vertices has n^{2-x} edges for $x = \Theta(1/\log n)$. Thus part (i) of Theorem 1.1 would follow as a special case of part (ii) if not for the $\varepsilon > (\log n)^{-2/3}$ requirement there. Because of this, part (ii) directly implies the existence of only $n/2^{O((\log n)^{2/3})}$ pairwise crossing segments determined by n points in the plane in general position. To obtain the slightly stronger estimate claimed in part (i) of Theorem 1.1 we have to do the calculations again for the special case of the complete geometric graph. Here we do not have to worry about maintaining the density condition. We can apply Lemmas 3.3 and 3.4 as stated but have to prove an analogue of Lemma 3.6:

Lemma 3.7. *Let s be a positive integer and set $K = 8^{\binom{s}{2}}$, $M = 9^sK$, $\varepsilon = 2^{-3s-11}$. Suppose the M-element point sets A and B form a ε-avoiding pair and $A\cup B$ is in general position. Then we can find K pairwise crossing segments, each connecting a point of A to a point of B.*

Proof. We prove the lemma by induction on s. For $s = 1$ we have $K = 1$, so the statement is trivial.

If $s > 1$ we apply Lemma 3.4 with $t = 8$, $k = 8^{s-1}$ and $m = M/(9k)$ to the complete geometric graph on $A\cup B$. As A and B form a $(2^{-14}/k)$-avoiding, 1-dense pair, the lemma finds us pairwise disjoint m-element subsets $A_i \subset A$ and $B_i \subset B$ such that $\{A_i,B_i\}$ is $2^{-11}/k$-avoiding for all i and each edge in $E(A_i,B_i)$ crosses all edges in $E(A_j,B_j)$ whenever $i \ne j$. We apply the inductive hypothesis for $s-1$ in place of s separately for each pair $\{A_i,B_i\}$. This results in a subset Z_i of $E(A_i,B_i)$ consisting of $K^* = 8^{\binom{s-1}{2}}$ pairwise crossing edges. Their union, $\cup_{i=1}^k Z_i$ is a subset of $E(A,B)$ consisting of $kK^* = K$ pairwise crossing edges as claimed. □

Proof of Theorem 1.1(i). We assume $n \ge 3$. Let us choose s to be the smallest positive integer such that V *does not* determine $K = 8^{\binom{s}{2}}$ pairwise crossing segments. Note that $s > 1$ as V determines at least $8^{\binom{1}{2}} = 1$ pairwise crossing segments. By Lemma 3.7, we do not have size $M = 9^sK$ subsets A and B of V forming an ε-avoiding pair with $\varepsilon = 2^{-3s-11}$. The complete geometric graph on the vertex set V has n vertices and δn^2 edges with $\delta = (n-1)/(2n) \ge 1/3$. Applying Lemma 3.3 to this graph yields $n = O(M\varepsilon^{-4}\delta^{-5}) = 2^{O(s)}K$.

By the choice of s, V determines at least $K^* = 8^{\binom{s-1}{2}} = K/2^{O(s)}$ pairwise crossing segments. Therefore, $n > K^* = 2^{\binom{s-1}{2}}$, so we must have $s = O(\sqrt{\log n})$. But we also have $K^* = K/2^{O(s)} = n/2^{O(s)} = n/2^{O(\sqrt{\log n})}$ as claimed. □

4 DISCUSSION

- Theorem 1.1 represents a substantial step towards characterizing the intersection structure of the edges in a geometric graph. We hope that further progress in this direction would facilitate the solution several related decades-old unsolved problems in combinatorial and computational geometry. The two most notorious questions of this kind are the following. Determine (1) the maximum number of halving lines of a set of n points in $\mathbb{R}^2$, and (2) the maximum number of incidences that can occur between n points and m pseudo-algebraic[2] curves in $\mathbb{R}^2$. Note that the best known general upper bounds for these problems [16, 41] are obtained by applying the Crossing Lemma to the edges in a suitable geometric or topological graph. Following the pioneering work of Dvir [17], Guth and Katz [25, 26], several of these questions have been revisited from an algebraic perspective [18, 28, 44].

- A closely related line of work [29, 33, 43, 50] concerns *pairwise avoiding edges* in geometric graphs. We say that a pair of segments in the plane are *avoiding* if (the closure of) neither of them is crossed by the supporting line of the other segment. Note that the results of Aronov et al. [8] extended to pairwise avoiding segments as well: they proved that any n points in general position in the plane determine $\Omega\left(\sqrt{n}\right)$ pairwise avoiding segments. Our bounds in Theorem 1.1 similarly apply to families of pairwise avoiding edges.

Theorem 4.1. *(i) Any set V of n points in general position in the plane determines at least $n/2^{O(\sqrt{\log n})}$ pairwise avoiding segments.*
(ii) There exists an absolute constant c such that any geometric graph with n vertices (in general position in the plane) and at least $n^{2-\varepsilon}$ edges with $\varepsilon > (\log n)^{-2/3}$ has at least $n^{1-c\sqrt{\varepsilon}}$ pairwise avoiding edges.

Part (i) of Theorem 7 easily follows from Theorem 1.1 via a reduction in [8, Theorem 2], while both parts can be derived by a slight modification of Lemma 3.4. To that end, it is sufficient to establish the following analogue of Lemma 3.4: For any separated 8δ-dense, $\varepsilon\delta$-avoiding pair (A, B) with $|A| = |B| = (t+1)km$, as in Lemma 3.4, there exist m-element subsets $A_1, \ldots, A_k \subset A$, $B_1, \ldots, B_k \subset B$ so that for every $1 \le i \le k$, the pair $\{A_i, B_i\}$ is δ-dense and ε-avoiding, and, for $1 \le i < j \le k$, all edges in $E(A_i, B_i)$ are parallel to every edge in $E(A_j, B_j)$. Our argument closely follows the proof of Lemma 3.4. Specifically, we consider the partial orders $(A, <_B)$ and $(B, <_A)$ and use the following geometric property: if $x <_A y$ for $x, y \in A$ and $z <_A w$ for $z, w \in B$ then the segments xw and yz are avoiding.[3] Recall that Lemma 3 yields m-size subsets $C_1 <_B C_2 <_B \ldots <_B C_{tk}$ and $D_1 <_A D_2 <_A \ldots <_A D_{tk}$ of, respectively, A and B. Note that whenever $1 \le i < j \le tk$ and $1 \le i' < j' \le tk$, all edges in $E(C_i, D_{j'})$ are avoiding with respect to every edge in $E(C_j, D_{i'})$; see Figure 7. Thus, the dual of the Dilworth's Theorem must now be invoked with respect to the *modified* partial order in which $(C_a, D_b) < (C_{a'}, D_{b'})$ whenever $a < a'$ and $b > b'$.

[2] We say that a family of curves in $\mathbb{R}^2$ is pseudo-algebraic if any two of the curves intersect at most a fixed number of times. In particular, this includes families of bounded-degree algebraic curves.
[3] Similar to Lemma 2.2, this follows because x, y, w, z are vertices of a convex quadrilateral.

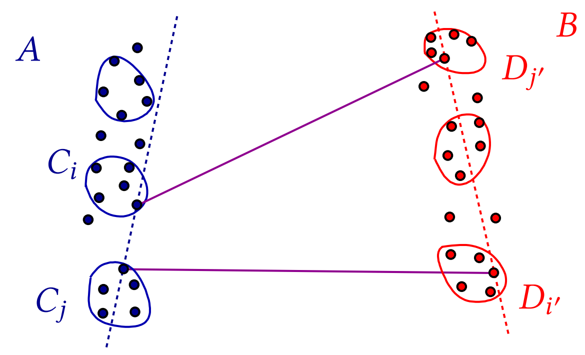

Figure 7: Proof of Theorem 4.1. All edges in $E(C_i, D_{j'})$ are avoiding with respect to every edge in $E(C_j, D_{i'})$ whenever $C_i <_B C_j$ and $D_{i'} <_A D_{j'}$.

- Our proof of Theorem 1.1, just like the arguments of Aronov et al. [8], crucially relies on the assumption that the edges of our graph are straight-line segments. In particular, the decomposition provided by Lemma 2.3 is based on the machinery that was developed in computational geometry to support efficient searching in arrangements of lines and hyperplanes. Extending Theorem 1.1 to more general families of topological graphs (e.g., whose edges are contained in fixed-degree algebraic curves, or any two of whose edges cross a bounded number of times) would require new ideas. Most of the previous lower bounds of this kind are based on Ramsey-type results applied to the intersection graph of the edge set [7, 19, 20, 22, 24]. Hence, it is likely that they can be improved by a careful divide-and-conquer scheme applied to the vertex set V, in the spirit of our proof of Theorem 1.1.
- Our proof of Theorem 1.1 is *fully constructive*. As the primary focus of this study is on the combinatorial aspects of geometric graphs, we did not seek to optimize the construction cost of our family of pairwise crossing edges. Nevertheless, our argument yields an algorithm whose running time is $O\left(n^{2+O(\sqrt{x})}\right)$ if the input graph (V, E) has at least n^{2-x} edges (for $x \ge (\log n)^{-2/3}$).
Specifically, the $O\left(\frac{1}{\varepsilon^2}\right)$-size line set L of Lemma 2.3 can be computed in time $O\left(\frac{1}{\varepsilon^2} + \frac{n}{\varepsilon}\right)$ using the deterministic algorithm of Chazelle [13] for finding an optimal cutting of the dual plane. The ε-avoiding, δ-dense pair $\{A, B\}$ (together with the posets $(A, <_B)$ and $(B, <_A)$) in Lemma 3.3 can be computed, using a naive implementation, in $O\left(\frac{n^2}{\varepsilon^4\delta^4}\right)$ time. The initial decomposition $\{C_a\}$ and $\{D_b\}$ of the sets A and B in Lemma 3.4 can be obtained (as designated in Lemma 2.1) via a simple topological sorting of the posets $(A, <_B)$ and $(B, <_A)$, and in time that is at most quadratic in $|A| = |B|$. The induced incomparability graphs $\iota(C_a, <_{D_b})$ (and, therefore, the eligible pairs (C_a, D_b)) can be determined in time $O\left((tk) \cdot (\varepsilon\delta|A|^2)\right) = O(|A|^2)$. The desired longest chain of eligible pairs can be obtained in $O\left(t^4k^4\right)$ time (again, via the standard topological sorting of their poset [15]).

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