# Crossings between non-homotopic edges*

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**Abstract**

A *multigraph* drawn in the plane is called *non-homotopic* if no pair of its edges connecting the same pair of vertices can be continuously transformed into each other without passing through a vertex, and no loop can be shrunk to its end-vertex in the same way. Edges are allowed to intersect each other and themselves. It is easy to see that a non-homotopic multigraph on $n > 1$ vertices can have arbitrarily many edges. We prove that the number of crossings between the edges of a non-homotopic multigraph with $n$ vertices and $m > 4n$ edges is larger than $c\frac{m^2}{n}$ for some constant $c > 0$, and that this bound is tight up to a polylogarithmic factor. We also show that the lower bound is not asymptotically sharp as $n$ is fixed and $m$ tends to infinity.

## 1 Introduction

A standard parameter for measuring the non-planarity of a graph $G$ is its *crossing number*, which is defined as the smallest number $\mathrm{cr}(G)$ of crossing points in any drawing of $G$ in the plane. For many interesting variants of the crossing number, see [17, 20, 21, 25]. Computing $\mathrm{cr}(G)$ is an NP-complete problem [7].

Perhaps the most useful result on crossing numbers, is the so-called *crossing lemma*, proved independently by Ajtai, Chvátal, Newborn, Szemerédi [3] and Leighton [11], according to which the crossing number of any graph with $n$ vertices and $m > 4n$ edges is at least $c\frac{m^3}{n^2}$, for a suitable constant $c > 0$. For the best known value of the constant $c$, see [1, 14]. This result, which is tight up to the constant factor, has been successfully applied to a variety of problems in discrete and computational geometry, additive number theory, algebra, and elsewhere [5, 24]. In some applications, it was the bottleneck that one needed a lower bound on the

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crossing number of a *multigraph* rather than a graph, see [15, 23, 22]. Obviously, the crossing lemma does not hold in this case, as stated. Indeed, one can connect a pair of vertices ($n = 2$) with $m$ parallel edges without creating any crossing. However, for multigraphs $G$ with maximum edge *multiplicity* $k$ and $m > 4kn$ edges, Székely [24] established the lower bound $\mathrm{cr}(G) > c' \frac{m^3}{kn^2}$, where $c' > 0$ is another constant. This bound is also tight, up to the constant factor, and $c'$ can be chosen to be the same as the best known constant $c$ in the crossing lemma (presently, $\frac{1}{29}$) [1, 19].

As the multiplicity $k$ increases, Székely's bound gets weaker and weaker. Luckily, the term $k$ in the denominator can be eliminated in several special cases; see [18, 10]. That is, the result holds without putting any upper bound on the edge multiplicity. However, in all of these cases, we have to assume (among other things) that no two adjacent edges cross.

In this paper, we study the analogous question under the weakest possible assumption. Obviously, we need to assume that no pair of parallel edges or loops are *homotopic*, i.e., they cannot be continuously deformed into each other so that their interiors do not pass through any vertex. As we have noted above, without this assumption, a multigraph can have arbitrarily many non-crossing edges. For simplicity, we will also assume that there are no *trivial* loops, that is, no loop can be transformed into a point. Clearly, this latter assumption can be eliminated as the first condition already implies that there is at most a single trivial loop at any vertex.

To state our results, we need to agree about the definitions.

A *multigraph* is a graph in which parallel edges and loops are permitted. A *topological graph (or multigraph)* is a graph (multigraph) $G = (V, E)$ drawn in the plane with the property that every vertex is represented by a distinct point and every edge $e \in E$ is represented by a continuous curve, i.e., a continuous function $f_e\colon [0,1] \to \mathbb{R}^2$ with $f_e(0)$ and $f_e(1)$ being the endpoints of $e$. In terminology, we do not distinguish between the vertices and the points representing them. In the same spirit, if there is no danger of confusion, we often use the term "edge" instead of the "curve" $f_e$ representing it or the "image" of $f_e$. We assume that no edge passes through any vertex (i.e., $f_e(t) \notin V$ for $0 < t < 1$).

The *crossing number* of a *topological multigraph* $G$ is the number of crossings between its edges, i.e, the number of unordered pairs of distinct pairs $(e,t), (e',t') \in E \times (0,1)$ with $f_e(t) = f_{e'}(t')$. With a slight abuse of notation, this number will be denoted also by $\mathrm{cr}(G)$.

Two parallel edges, $e, e'$, connecting the same pair of vertices, $u, v \in V$ are *homotopic*, if there exists a continuous function (*homotopy*) $g\colon [0,1]^2 \to \mathbb{R}^2$ satisfying the following three conditions.

$$g(0,t) = f_e(t) \text{ and } g(1,t) = f_{e'}(t) \text{ for all } t \in [0,1],$$

$$g(s,0) = u \text{ and } g(s,1) = v \text{ for all } s \in [0,1],$$

$$g(s,t) \notin V \text{ for all } s,t \in (0,1).$$

As we deal with non-oriented multigraphs, we also call $e$ and $e'$ homotopic if $f_e(1-t)$ and $f_{e'}(t)$ are homotopic in the above sense. A loop at vertex $u$ is said to be *trivial* if it is homotopic to the constant function $f(t) = u$.

A topological multigraph $G = (V, E)$ is called *non-homotopic topological multigraph* or simply *non-homotopic multigraph* if it does not contain two homotopic edges, and does not contain any trivial loop.

Obviously, if $G$ is a simple topological graph (no parallel edges or loops), then it is non-homotopic. A non-homotopic multigraph with zero or one vertex has no edge. However, if the number of vertices $n$ is at least 2, the number of edges can be arbitrarily large, even infinite. Our first result provides a lower bound on the crossing number of non-homotopic topological multigraphs in terms of the number of their vertices and edges.

**Theorem 1** *The crossing number of a non-homotopic multigraph $G$ with $n > 1$ vertices and $m > 4n$ edges satisfies* $\mathrm{cr}(G) \geq \frac{1}{24}\frac{m^2}{n}$.

This bound is tight up to a polylogarithmic factor.

**Theorem 2** *For any $n \geq 2$, $m > 4n$, there exists a non-homotopic multigraph $G$ with $n$ vertices and $m$ edges such that its crossing number satisfies* $\mathrm{cr}(G) \leq 100\frac{m^2}{n}\log_2^2\frac{m}{n}$.

The constant 100 in the theorem was chosen for the proof to work for all $m > 4n$ and we made no attempt to optimize it. However, it can be replaced by $1 + o(1)$ if both $n$ and $m/n$ go to infinity.

Define the function $\mathrm{cr}(n, m)$ as the minimum crossing number of a non-homotopic multigraph with $n$ vertices and $m$ edges. Theorems 1 and 2 can be stated as

$$\frac{1}{24}\frac{m^2}{n} \leq \mathrm{cr}(n, m) \leq 100\frac{m^2}{n}\log_2^2\frac{m}{n},$$

for any $n \geq 2$ and $m > 4n$. We have been unable to close the gap between the lower and upper bounds. However, our next theorem shows that the lower bound is not tight.

**Theorem 3** *The minimum crossing number of a non-homotopic multigraph with $n \geq 2$ vertices and $m$ edges is super-quadratic in $m$. That is, for any fixed $n \geq 2$, we have*

$$\lim_{m\to\infty}\frac{\mathrm{cr}(n, m)}{m^2} = \infty.$$

*Explicitly, we obtain*

$$\frac{\mathrm{cr}(n, m)}{m^2} = \begin{cases} \Omega\left(\frac{(\log m/\log\log m)^{1/6}}{n^8}\right) & \text{for } m > 4n, \\ \Omega\left(\log^{2/3} m\right) & \text{for fixed } n. \end{cases} \tag{1}$$

Let $n, k$ be positive integers, and consider a set $S$ obtained from the Euclidean plane by removing $n$ distinct points. Fix a point $x \in S$. An oriented loop in $S$ that starts and ends at $x$ is called an *$x$-loop*. An $x$-loop may have self-intersections. Contrary to our convention for edges of a topological multigraph, we do distinguish between an $x$-loop and its reverse. We consider the homotopy type of $x$-loops in $S$, that is, we consider two loops *homotopic* if one can be continuously transformed to the other within $S$. When counting self-intersections of $x$-loops or intersections between two $x$-loops, we count points of multiple intersections with the appropriate multiplicity.

To establish Theorems 2 and 3, we study the following topological problem of independent interest.

**Problem 4** *Let $n, k \geq 1$ be integers, let $S$ denote the set obtained from $\mathbb{R}^2$ by removing $n$ distinct points, and let us fix $x \in S$. Determine or estimate the maximum number $f(n, k)$ of pairwise non-homotopic $x$-loops in $S$ such that none of them passes through $x$, each of them has fewer than $k$ self-intersections and every pair of them cross fewer than $k$ times.*

The finiteness of $f(n, k)$ is crucially important for our proof of Theorem 3. We provided a proof of this fact in the preliminary version of this paper for the proceedings of *Graph Drawing 2020* [16]. But this is a well studied subject: various upper bounds of $f(n, k)$ were established earlier in [4] and [8]. Juvan *et al.* [8] did not give an explicit bound, but a careful analysis of their argument implies the first part of the next theorem. The bound proved by Aougab and Sauto [4] (see the second part of the next theorem) is asymptotically stronger for any fixed $n$, but it seems to be hard to make this bound explicit in terms of its dependence on $n$. This is why we state both bounds here.

**Theorem 5** *1. [8] For any integers $n \geq 2$ and $k \geq 1$, we have*

$$f(n,k) < (nk)^{O(nk^2)}.$$

*2. [4] For $n \geq 2$ fixed we have*

$$f(n,k) \leq 2^{O(\sqrt{k})}.$$

For the proof of Theorem 2, we need a lower bound on $f(2,k)$. Juvan *et al.* [8] established a lower bound for $f(n,k)$. Their construction used non-selfintersecting curves. Therefore, they obtained a bound which, for a fixed $n$, was only polynomial in $k$.

Our following theorem provides an exponential lower bound. For fixed $n$ it matches the upper bound in Theorem 5/2 except for the hidden constant in the exponent.

**Theorem 6** *Let $k \geq n \geq 2$ be integers. We have*

$$f(n,k) \geq 2^{\sqrt{nk}/3}.$$

Our paper is organized as follows. In Section 2, we establish Theorem 1. In Section 3, we present some constructions proving Theorem 6, and apply them to deduce Theorem 2. In Section 4 we prove Theorem 3.

## 2 Loose multigraphs—Proof of Theorem 1

One can also define topological multigraphs and non-homotopic multigraphs on the sphere $S^2$. If we consider $S^2$ as the single point compactification of the plane with the *ideal point* $p^*$, then any topological multigraph $H$ drawn in the plane remains a topological multigraph on the sphere. However, even if $H$ is non-homotopic on the plane, it may lose this property on the sphere, as the addition of the ideal point $p^*$ may turn a loop trivial or two parallel edges homotopic. This can be avoided by adding $p^*$ as an isolated vertex to $H$: in this case, the resulting multigraph $H^*$ is non-homotopic on the sphere.

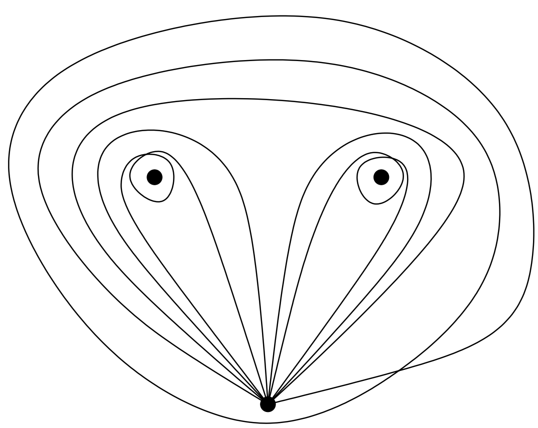

Figure 1: A non-homotopic loose multigraph with 3 vertices and 6 edges, all of which are loops.

We say that a topological multigraph is *loose* if no pair of distinct edges cross each other. An edge (in particular, a loop) is allowed to cross itself. We start by finding the maximum number of edges in a loose non-homotopic multigraph on the sphere or in the plane, for a given number of vertices. We will see that despite allowing parallel edges, loops, and self-intersections, loose non-homotopic multigraphs with $n > 2$ vertices on the sphere cannot have more than $3n-6$ edges, the maximum number of edges of a simple planar graph. However, there are many other nontrivial examples, for which this bound is tight. The interested reader can verify that, for all $n > 2$, there are extremal examples, all of whose edges are loops. See Fig. 1 for the case of three vertices in the plane.

**Lemma 7** *On the sphere, any loose non-homotopic multigraph with $n > 2$ vertices has $m \leq 3n - 6$ edges. For $n = 2$, the maximum number of edges is 1.*

**Proof.** Assume for contradiction that there is a loose non-homotopic multigraph $H$, which is a counterexample to the lemma. We may assume the $\mathrm{cr}(H)$ is finite, as this can be achieved by infinitesimal perturbation. We choose $H$ to be a counterexample with *minimum crossing number*. If this number is zero (that is, the edges of $H$ have no self-intersections), we further minimize the number of *connected components* in $H$. Let $n$ stand for the number of vertices of $H$, the minimal counterexample, and let $m$ stand for the number of its edges.

Assume first that there is no self-intersecting edge in $H$, so we deal with a *planar* drawing. In this case, we can also assume that $H$ is connected, otherwise two components could be joined by an extra edge without creating a crossing. (Note that this argument fails if we permit self-intersecting edges, as they may prevent the addition of such an edge between two components without creating a crossing, see Fig. 1.) Thus, the boundary of each face of $H$ can be visited by a single walk. These walks collectively cover every edge twice, so if each of them have at least three edges and the number of faces is $s$, then we have $3s \leq 2m$. Combining this inequality with Euler's formula $n + s = m + 2$ gives $m \leq 3n - 6$, which contradicts our assumption that $H$ was a counterexample. Therefore, $H$ must have a face bounded by a walk consisting of one or two edges. A boundary walk consisting of a single edge is a trivial loop, which is not permitted in a non-homotopic graph. A boundary walk of two edges is typically formed by two parallel edges that are homotopic, which is also disallowed in a non-homotopic graph. The only possibility is that the walk is back and forth along the same edge. In this case, we have $n = 2$ and $m = 1$, and $H$ is not a counterexample.

Therefore, our minimal counterexample $H$ must have at least one self-intersecting edge $e$. Find a minimal interval $\gamma$ of $e$ between two occurrences of the same intersection point $p$. This is a simple closed curve in the plane avoiding all vertices. It partitions the sphere $S^2$ into two connected components. We call them (arbitrarily) the *left* and *right sides* of $\gamma$. Obviously, $e$ is the only edge that may run between these sides. Let $H_1$ and $H_2$ be the subgraphs of $H \setminus \{e\}$ induced by the vertices in the left and right sides of $\gamma$, respectively. Both of them are loose topological multigraphs, but they may contain homotopic edges. By adding $p$ to both of them as an isolated vertex, they become non-homotopic. If an endpoint $u$ of $e$ lies in the left part, then by adding to $H_1$ a non-self-intersecting edge connecting $p$ and $u$ along $e$, we create no new intersection and do not violate the non-homotopic condition either. The resulting topological multigraph $H_1'$ is a loose non-homotopic multigraph on the sphere with $n_1$ vertices and $m_1$ edges. Analogously, we can construct the loose non-homotopic multigraph $H_2'$ from $H_2$. Denote its number of vertices and edges by $n_2$ and $m_2$, respectively. We have $n_1 + n_2 = n + 2$ and $m_1 + m_2 \geq m$. We eliminated a self-crossing (of $e$) and did not add any new crossings, so the crossing numbers of both $H_1'$ and $H_2'$ are smaller than $\mathrm{cr}(H)$.

If $n_1, n_2 > 2$, then we have $m_1 \leq 3n_1 - 6$ and $m_2 \leq 3n_2 - 6$, by the minimality of $H$. Summing up these inequalities, we get $m \leq 3n - 6$, contradicting our assumption that $H$ was a counterexample.

If $n_1 = 1$ or $n_2 = 1$, all vertices of $H$ lie on the same side of $\gamma$. In this case, by deleting $\gamma$ from $e$, the homotopy class of $e$ remains the same. Hence, the resulting topological multigraph is still a loose non-homotopic multigraph with $n$ vertices and $m$ edges, but its crossing number is smaller than that of $H$, contradicting the minimality of $H$.

Finally, consider the case $n_1 = 2$ or $n_2 = 2$. By symmetry, we can assume that $n_1 = 2$, $n_2 = n$, so we have a single vertex $u$ of $H$ on the left side of $\gamma$ and $n - 1$ vertices on the right side. Note that no edge of $H \setminus \{e\}$ can lie in the left side. Indeed, such an edge would be a trivial loop. If $e$ has at least one endpoint in the right part, then we have $m_2 = m$. This implies that $H'_2$ is another counterexample to the lemma with fewer crossings, contradicting the minimality of $H$.

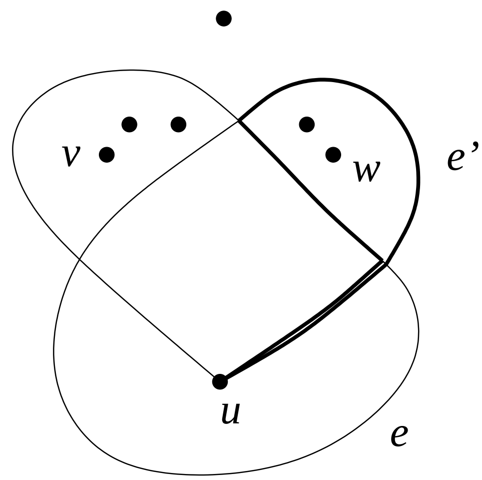


Figure 2: The replacement of $e$ by $e'$ in the proof of Lemma 7.

Therefore, $e$ must be a loop at $u$. The image of $e$ must separate a pair of vertices, $v, w \in V(H) \setminus \{u\}$ from each other, as otherwise $e$ would be a trivial loop. However, then we could draw another loop $e'$ along or very close to some parts of $e$ with no self-intersection, so that it also separates $v$ and $w$. Therefore, $e'$ is not trivial either. See Fig, 2. (One can find a minimal separating loop in the image of $e$, and then join it with $u$ by a minimal curve in or near the image of $e$.)

Let $H'$ be the topological multigraph obtained from $H$ by replacing $e$ by $e'$. The loops $e$ and $e'$ are not necessarily homotopic, but there is no other edge in $H'$ homotopic to $e'$, because there is no other loop at $u$. Hence, $H'$ is a loose non-homotopic multigraph. This contradicts the minimality of $H$, because $H'$ has the same number of vertices and edges as $H$ does, but its crossing number is smaller. This contradiction proves the lemma. □

**Lemma 8** *In the plane, any loose non-homotopic multigraph with $n \geq 1$ vertices has at most $3n - 3$ edges. This bound can be achieved for every $n$.*

**Proof.** Let $H$ be a loose non-homotopic multigraph in the plane with $n \geq 1$ vertices and $m$ edges. Consider the plane as the sphere $S^2$ with a point $p^*$ removed. Add $p^*$ to $H$ as an isolated vertex, to obtain a topological multigraph $H'$ on the sphere. Then $H'$ is a loose non-homotopic multigraph with $n + 1$ vertices and $m$ edges. If $n > 1$, applying Lemma 7 to $H'$, we obtain that $m \leq 3n - 3$, as required. If $n = 1$, then $H$ is a single-vertex topological multigraph in the plane, so all of its edges must be trivial loops. However, by definition, a non-homotopic multigraph cannot have any trivial loop. This completes the proof of the upper bound.

There are many different constructions for loose non-homotopic multigraphs for which the bound in the lemma is achieved. Such a topological multigraph may have several components and several self-intersecting loops. (However, all self-crossings of non-loop edges must be "homotopically trivial": the removal of the closed curve produced by such a self-crossing does not change the homotopy type of the edge.)

Here, we give a very simple construction. If $n > 2$, we start with a triangulation with $n$ vertices and $3n-6$ edges. Let $uvw$ be the boundary of the unbounded face. Add another non-self-intersecting edge connecting $u$ and $v$ in the unbounded face, which is not homotopic with the arc $uv$ of $uvw$. Finally, we add two further loops at $u$. First, a simple loop $l$ that has all other edges and vertices (except $u$) in its interior, and then another loop $l'$ outside of $l$, which goes twice around $l$. (Of course, $l'$ must be self-intersecting.)

If $n = 1$, the graph with no edge achieves the bound of the lemma. For $n = 2$, draw an edge $e$ connecting the two vertices, $u$ and $v$. Then add two loops at $u$, as above: a simple loop $l$ around $e$ and another loop $l'$ that winds around $l$ twice. □

**Proof of Theorem 1.** Let $G$ be a non-homotopic topological multigraph in the plane with $n > 1$ vertices and $m > 4n$ edges.

Let $D$ denote the *non-crossing graph* of the edges of $G$, that is, let $V(D) = E(G)$ and connect two vertices of $D$ by an edge if and only if the corresponding edges of $G$ do not share an interior point. Any clique in $D$ corresponds to a loose non-homotopic sub-multigraph of $G$. Therefore, by Lemma 8, $D$ has no clique of size $3n-2$. Thus, by Turán's theorem [26],

$$|E(D)| \leq \frac{|V(D)|^2}{2}\left(1 - \frac{1}{3n-3}\right) = \frac{m^2}{2}\left(1 - \frac{1}{3n-3}\right).$$

The crossing number $\mathrm{cr}(G)$ is at least the number of crossing pairs of edges in $G$, which is equal to the number of non-edges of $D$. Since $m > 4n$, we have

$$\mathrm{cr}(G) \geq \binom{m}{2} - \frac{m^2}{2}\left(1 - \frac{1}{3n-3}\right) = \frac{m^2}{6}\left(\frac{1}{n-1} - \frac{3}{m}\right) > \frac{m^2}{6}\left(\frac{1}{n} - \frac{3}{4n}\right) = \frac{1}{24}\frac{m^2}{n},$$

as claimed. □

The proof above gives a lower bound on the number of crossing pairs of edges in $G$, and in this respect it is tight up to a constant factor. To see this, suppose for simplicity that $n$ is even and $m$ is divisible by $n$. Let $G_0$ be a non-homotopic topological multigraph with two vertices and $\frac{2m}{n}$ non-homotopic loops on one of its vertices. Taking $\frac{n}{2}$ disjoint copies of $G_0$, we obtain a non-homotopic topological multigraph with $n$ vertices, $m$ edges, and fewer than $\frac{m^2}{n}$ crossing pairs of edges.

# 3 Two constructions—Proofs of Theorems 6 and 2

The aim of this section is to demonstrate how to construct topological graphs with many edges and families consisting of many loops, without creating many crossings. The constructions are based on the description of the fundamental group of the plane from which a certain number of points have been removed.

**Proof of Theorem 6.** Let $S = \mathbb{R}^2 \setminus \{a_1, \ldots, a_n\}$, where $a_1, \ldots, a_n$ are distinct points in the plane, and let $x \in S$ be also fixed. Assume without loss of generality that $a_i = (i, 0)$, $1 \leq i \leq n$, and $x = (0, -1)$. Recall that an *$x$-loop* is a (possibly self-crossing) oriented path in $S$ from $x$ to $x$, i.e., a continuous function $f\colon [0,1] \to S$ with $f(0) = f(1) = x$.

Note that the homotopy group of $S$ is the free group $F_n$ generated by $g_1, \ldots, g_n$, where $g_i$ can be represented by a triangular $x$-loop around $a_i$, for example the one going from $x$ to $(2i-1, 1)$, from here to $(2i+1, 1)$, and then back to $x$ along three straight-line segments; see [12].

We define an *elementary loop* to be a polygonal $x$-loop with intermediate vertices

$$(1, \epsilon_1),\ (2, \epsilon_2),\ \ldots,\ (n, \epsilon_n),\ (n+1, -1),$$

in this order, where each $\epsilon_i$ ($1 \le i \le n$) is equal either to $1/2$ or to $-1/2$. There are $2^n$ distinct elementary loops, depending on the choice of the $\epsilon_i$. Each of them represents a distinct homotopy class of the form $g_{i_1} \cdots g_{i_t}$, where the indices form a strictly increasing sequence. By making infinitesimal perturbations on the interior vertices of the elementary loops, we can make sure that every pair of them intersect in at most $n-1$ points. Thus, we have $f(n, n) \ge 2^n$. See Fig. 3.

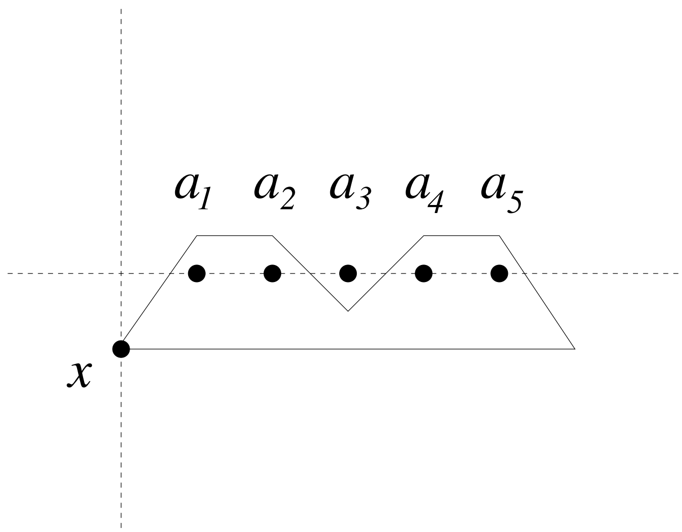


Figure 3: An elementary $x$-loop ($n = 5$).

If $n \le k \le 9n$, we have $f(n, k) \ge f(n, n) \ge 2^n \ge 2^{\sqrt{nk}/3}$, and we are done.

In the case $k > 9n$, we consider all $x$-loops which can be obtained as the product (concatenation) of $j = \left\lfloor \sqrt{\frac{k-1}{n}} \right\rfloor \ge 3$ elementary loops. Unfortunately, some of these concatenated $x$-loops will be homotopic. For example, if the elementary loops $l_1, l_2, l_3$, and $l_4$ represent the homotopy classes $g_1, g_2g_3, g_1g_2$, and $g_3$, respectively, then $l_1l_2$ and $l_3l_4$ are homotopic. To avoid this complication, we only use the $2^{n-1}$ elementary loops that represent homotopy classes involving $g_1$ (that is, the ones with $(1, 1/2)$ as their first intermediate vertex). Concatenating $j$ such elementary loops, we obtain $2^{j(n-1)}$ different $x$-loops, no pair of which are homotopic. By infinitesimal perturbation of the interior vertices of these $x$-loops (including the $j-1$ interior vertices at $x$), we can ensure that they do not pass through $x$, and no two polygonal paths corresponding to a single elementary loop intersect more than $n$ times. Therefore, any pair of perturbed concatenated loops cross at most $j^2 n < k$ times, and the same bound holds for the number of self-intersections of any concatenated loop. This yields that $f(n, k) \ge 2^{j(n-1)} \ge 2^{\sqrt{nk}/3}$, completing the proof of the theorem. □

**Proof of Theorem 2.** We want to construct a non-homotopic topological multigraph $G$ with $n$ vertices, $m$ edges, and few crossings. We distinguish 3 cases.

*Case A:* If $n = 3$, we set $k = \lceil (9/2)\log_2^2(2m) \rceil$. Theorem 6 guarantees that $f(2,k) \geq 2m$. Thus, there are $2m$ pairwise non-homotopic $x$-loops in $S = \mathbb{R}^2 \setminus \{a_1, a_2\}$ such that each of them has fewer than $k$ self-intersections and any pair intersect fewer than $k$ times. Regard this arrangement as a topological multigraph $G$ with $2m$ edges on the vertex set $\{a_1, a_2, x\}$. All edges are $x$-loops. At most one of them is trivial, and for each loop edge there is at most one other loop edge homotopic to it (which must come from an $x$-loop with inverse orientation). Therefore, we can always select $m$ edges that form a non-homotopic multigraph. We have

$$\mathrm{cr}(G) < k\left(m + \binom{m}{2}\right) < \left\lceil \frac{9}{2}\log_2^2(2m) \right\rceil m^2.$$

*Case B:* If $n > 3$, we set $n^* = \lfloor n/3 \rfloor$, $m_0 = \lceil m/n^* \rceil$. Take $n^*$ disjoint copies of the non-homotopic multigraph $G_0$ with 3 vertices and $m_0$ edges constructed in Case A. We add at most 2 isolated vertices and remove a few edges if necessary to obtain a non-homotopic multigraph on $n$ vertices and $m$ edges. We have

$$\mathrm{cr}(G) \leq n^*\mathrm{cr}(G_0) \leq n^* m_0^2 \left\lceil \frac{9}{2}\log_2^2(2m_0) \right\rceil.$$

*Case C:* If $n = 2$, we cannot use Theorem 6 directly. Note that all edges of the non-homotopic multigraphs $G$ constructed in Case A were loops at a vertex $x$, and these $x$-loops were pairwise non-homotopic even in the space obtained from the plane by keeping $x$, but removing every other vertex. Now we cannot afford this luxury since in this case typical edges would go around the other vertex $\Omega(m)$ times and would cross other edges typically $\Omega(m)$ times, creating $\Omega(m^3)$ crossings. However, even for $n = 2$, we can construct a topological multigraph $G$ with many pairwise non-homotopic edges and relatively few crossings, as sketched below.

Let $V(G) = \{a_1, a_2\}$, where $a_1$ and $a_2$ are distinct points in the plane, and set $S = \mathbb{R}^2 \setminus V(G)$. Choose a base point $x \in S$ not on the line $a_1a_2$. Now the homotopy group of $S$ is the free group generated by two elements, $g_1$ and $g_2$, that can be represented by triangular $x$-loops around $a_1$ and $a_2$, respectively. By the proof of Theorem 6, with the notation used there, we can construct $2^j$ pairwise non-homotopic $x$-loops in $S$ with few crossings. Namely, each of these $x$-loops have at most $2j^2$ self-intersections and each pair intersect at most $2j^2$ times. Now, each of these $x$-loops, $l$, can be turned into either a loop edge at the vertex $a_1$ or into an $a_1a_2$ edge, as follows: we start with the straight-line segment $a_1x$, then follow $l$, finally add a straight-line segment from $x$ to either $a_1$ (for a loop edge) or to $a_2$ (to obtain a non-loop edge). After infinitesimally perturbing the resulting edges, one can maintain that each pair of edges cross at most $2(j+1)^2$ times and this also bounds the number of self-intersections of any edge. However, now we face a new complication: there may be a large number of pairwise homotopic edges. In Case A, when we regarded $x$-loops as loop edges in a topological multigraph having $x$ as a vertex, two loop edges could only be homotopic if the corresponding $x$-loops represented the same or inverse homotopy classes. Now the situation is more complicated: a loop edge constructed from an $x$-loop representing an element $g$ in the homotopy group is homotopic to an another edge constructed from another $x$-loop representing $g'$ if and only if we have $g' = g_1^s g g_1^t$ or $g' = g_1^s g^{-1} g_1^t$ for some integers $s$ and $t$. (For non-loop edges the corresponding condition is $g' = g_1^s g g_2^t$.) We may have constructed more than two (even an unbounded number of) homotopic edges, but considering only those of the $2^j$ $x$-loops that start and end with $g_1g_2$, we have a set of $2^{j-2}$ non-homotopic edges. Thus, we choose $j = \lceil \log_2 m \rceil + 2$ and we can select $m$ non-homotopic edges. The crossing number of the non-homotopic graph $G$ so obtained satisfies

$$\mathrm{cr}(G) \leq 2m^2(\lceil \log_2 m \rceil + 3)^2.$$

One can prove by simple calculation that the bounds proved for the crossing numbers of the graphs constructed in all three cases are within the bound stated in the theorem. □

**Remark.** Note that the statement of Theorem 2 does not distinguish between loops and non-loop edges. For $n \geq 3$, all edges of the non-homotopic graphs constructed above are loops, and this can be also attained for $n = 2$.

On the other hand, it is not hard to modify the above constructions so that the resulting graphs have no loops at all. For $n = 2$, one version of the construction described in the last paragraph of the proof uses no loops. For $n > 2$, instead of taking the 3-point construction described in Case A as our base, we can start with a 2-vertex non-homotopic graph that has no loops. The union of $n^* = \lfloor n/2 \rfloor$ pairwise disjoint copies of such a graph, with an additional isolated vertex if necessary, will meet the requirements.

## 4 Even more crossings—Proof of Theorem 3

Let $x \in S^2$ and consider a family $L$ of $x$-loops in $S^2$ that start and end at $x$, but do not pass through $x$. With infinitesimal perturbations of the elements of $L$ and without creating any further intersections, one can attain that all intersections are simple: no point other than $x$ appears more than twice on the same loop or on different members of $L$. This will be assumed for all families of $x$-loops used in the rest of this section. A (possibly self-intersecting) closed curve in $S^2$ is said to be an *$L$-circle* if it is either a segment of a loop $l \in L$ between two appearances of a self-intersection point of $l$, or it consists of two segments of the same loop or two segments belonging to different loops in $L$, connecting the same pair of intersection points. If the two segments belong to the same loop, they are not allowed to overlap. We call a family of $L$-circles *non-overlapping* if no two members of the family share a segment.

**Claim 9** *Let $L$ be a family of $x$-loops consisting of a single loop with at least $k$ self-intersections or consisting of two loops intersecting each other at least $k$ times.*

*Then there is a non-overlapping family of $L$-circles, consisting of at least $k^{1/3} - 1$ members.*

**Proof.** Suppose first that $L$ consists of two $x$-loops, $l_1$ and $l_2$. By the Erdős-Szekeres lemma [6], we can find $k' \geq \sqrt{k}$ intersection points $a_1, a_2, \ldots, a_{k'}$ that appear either in this order or in the reverse order on both $l_1$ and $l_2$. In this case, the segments of $l_1$ and $l_2$ between $a_i$ and $a_{i+1}$ form an $L$-circle, for each $1 \leq i < k'$. The family of these $L$-circles is non-overlapping, as claimed.

Alternatively, suppose that $L = \{l\}$ is a singleton family. The segment of $l$ between the two appearances of a self-intersection point $a$ is an $L$-circle. If there are at least $k^{1/3}$ among these $L$-circles that form a non-overlapping family, then we are done. If this is not the case, then we have a point $p \neq x$ on $l$ which is not an intersection point, but appears in at least $k^{2/3}$ of these single-part $L$-circles. That is, at least $k^{2/3}$ intersection points appear both on the initial segment $l_1$ of $l$ ending at $p$, and on the final segment $l_2$ of $l$, starting at $p$. We can then argue as we did in the case $|L| = 2$. Using the Erdős-Szekeres lemma, we can find $k' \geq k^{1/3}$ intersection points $a_1, a_2, \ldots, a_{k'}$ that appear either in this order or its reverse both on $l_1$ and $l_2$. For every $1 \leq i < k'$, the segments of $l_1$ and $l_2$ between $a_i$ and $a_{i+1}$ form an $L$-circle, and the family of these $L$-circles is non-overlapping. This finishes the proof. □

**Claim 10** *Let $L$ be a family of $x$-loops not passing through a point $p \in S^2$, $p \neq x$. Let $L_1$ and $L_2$ be two disjoint subfamilies of $L$ such that for each $i \in \{1, 2\}$ there is a non-overlapping family $C_i$ consisting of $k$ $L_i$-circles, each of which separates $x$ from $p$.*

*Then the total number of intersections between a loop in $L_1$ and a loop in $L_2$ is at least $k$.*

**Proof.** For a family $H$ of loops in $S^2$, we call a connected component of the part of $S^2$ not covered by $H$ an *$H$-face*. Let $F$ be the $(L_1 \cup L_2)$-face which contains $p$. Let $q$ be an arbitrary non-intersection point on the boundary of $F$. Obviously, $q$ lies on an $x$-loop $l \in L_i$ with $i = 1$ or 2. Thus, $q$ belongs to the $L_{3-i}$-face containing $p$, and all $L_{3-i}$-circles in $C_{3-i}$ separate $x$ from $q$. The loop $l$ connects $x$ to $q$, so it must intersect all of these $L_{3-i}$-circles. As $C_{3-i}$ is a non-overlapping family, these intersections must be distinct intersection points of $l$ and some $x$-loop in $L_{3-i}$. This proves the claim. □

Let us fix $n > 1$ and a set $T$ of $n$ points on the 2-sphere $S^2$. Let $S = S^2 \setminus T$, and fix a point $x \in S$.

**Lemma 11** *The minimal number $a_n(m)$ of crossings among $m$ pairwise non-homotopic $x$-loops in $S$ is super-quadratic in $m$.*

$$a_n(m) = \begin{cases} \Omega\left(m^2 \frac{(\log m/\log\log m)^{1/6}}{n^4}\right) & \text{for } m > 4n, \\ \Omega\left(m^2 \log^{2/3} m\right) & \text{for fixed } n. \end{cases} \tag{2}$$

**Proof.** Let $L$ be a collection of $m$ non-homotopic $x$-loops in $S$ with the minimum overall number, $a_n(m)$, of crossings.

Choose the largest $k$ such that $f(n-1, k) < m/2$. By the definition of $f$ (see Problem 4) this means that in any collection of at least $m/2$ pairwise non-homotopic $x$-loops in $S$, there is one with at least $k$ self-crossings or two that cross each other at least $k$ times. By Theorem 5, we have $k = \Omega(\sqrt{\log m/(n\log(n\log m))})$ in general and $k = \Omega(\log^2 m)$ if $n$ is fixed.

We greedily divide the loops of $L$ into *blocks*, as follows. Each block is either a single loop crossing itself at least $k$ times, or a pair of loops crossing each other at least $k$ times. We do not use the same loop of $L$ twice. Having formed at most $m/4$ blocks, we still have at least $m/2$ unused loops, and so we can form yet another block. Therefore, the greedy procedure yields more than $m/4$ blocks.

By Claim 9, for each block $B$, one can find a non-overlapping collection $C_B$ of at least $k^{1/3} - 1$ $B$-circles. A $B$-circle $\gamma$ is called *trivial* if it does not separate $x$ from any point of $T = S^2 \setminus S$. The existence of a trivial $B$-circle would contradict the minimality of the total number of crossings in the collection $L$ of $x$-loops. Indeed, if a trivial $B$-circle consists of a single segment of an $x$-loop $l$, then deleting this segment does not affect the homotopy type of $l$, but decreases the number of crossings. If a trivial $B$-circle consists of two segments, then interchanging these segments does not affect the homotopy types of the corresponding loops. Now the number of crossings can be reduced by an infinitesimal perturbation of the original family $L$ or of the family obtained by this switch.

As no $B$-cycle in $C_B$ is trivial, for every block $B$, we can find a point $p_B \in T$ such that the number of $B$-cycles in $C_B$ which separate $p_B$ from $x$ is at least $k' = \left\lceil \frac{k^{1/3}-1}{n} \right\rceil$. Clearly $k' = \Omega((\log m/\log\log m)^{1/6}/n^2$ in general and $k' = \Omega(\log^{2/3} m)$ for fixed $n$. There are only $n$ points in $T$, so there exists $p \in T$ such that $p_B = p$ for at least $\lceil m/(4n) \rceil$ blocks. By Claim 10, any two distinct blocks $B$ and $B'$ for which $p_B = p_{B'}$, cross each other at least $k'$ times. This gives a total of at least $\binom{\lceil m/(4n)\rceil}{2} k'$ crossings. Combining this with our bounds on $k'$, the lemma follows. □

Note that the proof of Lemma 11 finds crossings between distinct edges, self-crossings are not even counted.

**Proof of Theorem 3.** Let $G$ be a non-homotopic multigraph with $n$ vertices, $m$ edges, and with the smallest possible crossing number $\mathrm{cr}(n, m)$. As before, we can assume that there is no triple intersection among the edges, because we can get rid of these by infinitesimal perturbations. Obviously, we can find a set $E'$ of $m' \geq m/n^2$ parallel edges in $G$. We fix such a set $E'$, and in the rest of the proof we ignore all other edges of $G$. There are two cases.

*Case A:* $E'$ consists of loops at a vertex $x$. We will use Lemma 11. In the lemma, we have non-homotopic $x$-loops on a surface $S$ with $x \in S$. Therefore, we remove all vertices of $G$ from the plane, except $x$. To maintain that the edges are non-homotopic, we also remove a point $p$, very close to $x$ but not on any of the loops in $E'$. Let $S$ be the set obtained from the plane by deleting $p$ and all vertices of $G$ except $x$. The edges in $E'$ are pairwise non-homotopic $x$-loops in $S$. As $S$ can be obtained from the sphere $S^2$ by deleting $n+1$ points, these loops determine at least $a_{n+1}(m')$ intersections. According to Lemma 11, for a fixed $n$, this quantity is super-quadratic in $m'$ and, hence, also in $m$.

*Case B:* $E'$ consists of edges between two distinct vertices, $x$ and $y$. In this case, we pick two points, $p$ and $q$, very close to $x$ and $y$, respectively, which do not lie on any edge in $E'$. Now choose $S$ to be the set obtained from the plane by deleting all vertices of $G$ except $x$ and $y$, and also deleting $p$ and $q$. Any two edges of $E'$ form an $x$-loop. Moreover, for any $e_1, e_2, e_3 \in E'$ with $e_2 \neq e_3$, the $x$-loop formed by $e_1$ and $e_2$ is not homotopic in $S$ to the $x$-loop formed by $e_1$ and $e_3$.

We build a collection of pairwise non-homotopic $x$-loops by pairing up edges of $E'$ in a greedy way, using every edge at most once. Suppose that the process stops with a collection $L$ of $m''$ $x$-loops. There are $m'-2m''$ unused edges left in $E'$. Fix any one of them, and combine it with each of the remaining ones to obtain $m'-2m''-1$ pairwise non-homotopic $x$-loops. Since we were unable to extend $L$ by another $x$-loop, each of these $x$-loops is homotopic to one of the $m''$ loops we have constructed so far. Therefore, we have $m'' \geq m'-2m''-1$, and $m'' \geq (m'-1)/3$.

All $x$-loops in $L$ pass through $y$. With an infinitesimal perturbation, one can get rid of this multiple intersection without changing the homotopy classes of the $x$-loops or creating any additional intersection. Denote the resulting family of $x$-loops by $L'$. All loops in $L$ intersected at $y$. This may introduce up to $\binom{m''}{2}$ intersections between loops in $L'$ close to $y$. All other intersections among the members of $L'$ correspond to actual intersections between edges in $E'$.

Just like in Case A, $S$ can be obtained from the sphere by removing $n+1$ points. Hence, altogether there are at least $a_{n+1}(m'')$ intersections between the loops in $L'$, and the number of intersections between the edges of $E'$ is at least $a_{n+1}(m'') - \binom{m''}{2}$. In view of Lemma 11, this is super-quadratic in $m''$ and, hence, also in $m$.

Using the estimates in Lemma 11 for $a_n(m)$, we obtain the asymptotic bounds for $\mathrm{cr}(n,m)$ claimed in the theorem. □

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