CEU "Topics in Combinatorics" course

Some of the topics covered in the February 3, 2015 class

Kneser graph $KG(m,k)$ ($m \geq 2k > 0$ is assumed, otherwise the graph is empty.) Vertices: $\binom{[m]}{k}$, i.e., the size-k subsets of the set $[m]$ (or equivalently of any set of size m). Vertices x and y are adjacent if and only they are disjoint.

Remark: Both the Borsuk graph B^k_δ and $KG(m,n)$ are *vertex-transitive* graphs, i.e., its *automorphisms* move every vertex to every vertex. The Kneser graphs are also *edge-transitive*, i.e., every edge is moved to every edge by an automorphism. Another similarity: with appropriate choice of the parameters (namely $m \approx 2k$) the graph $KG(m,n)$ contains no short odd cycles. Indeed, if x and y have a common neighbor z in $KG(m,k)$, then $x \cup y \subseteq [m] \setminus z$, thus $|x \cup y| \leq m-k$ and consequently $|y \setminus x| \leq m-2k$. For an odd cycle $x_0x_1 \dots, x_{2l}$ in $KG(m,n)$ we have $k = |x_{2l} \setminus x_0| \leq |x_2 \setminus x_0| + |x_4 \setminus x_2| + \dots + |x_{2l-2} \setminus x_{2l}| \leq l(m-2n)$. Thus $l \geq k/(m-2k)$.

Kneser's conjecture (1955), proved by Lovász (1979), proof simplified by Bárány (1979) and Greene (2002): $\chi(KG(m,k)) = m-2k+2$

Easy direction (Kneser): $\chi(KG(n,k) \leq m-2k+2$. Proof by constructing a proper $(m-2k+2)$-coloring: $c(x) = \min(x)$ if $\min(x) \leq m-2k+1$ and we use one additional color for the remaining vertices, i.e., $c(x) = m-2k+2$ if $\min(x) > m-2k+1$. Q.E.D.

Recall from last time one of the forms of the Borsuk theorem:

Lyusternik-Schnirelmann Theorem (2. form): If $H_i \subseteq S^k$ are open for $i = 1, \dots, k$ and none of them contains a pair of antipodal points, then there is a pair of antipodal points not covered by any of the sets, that is, $x, -x \in S^k \setminus \cup_{i=1}^k H_i$.

Hard direction (Lovász): $\chi(KG(m,k) \geq m-2k+2$.

Proof (Greene, using LS2): Let $u = m-2k+1$ and S be a size-m subset of S^u in *general position*, i.e., each $u+1$ vertices are linearly independent as vectors in R^{u+1}. Consider the vertex set of $KG(m,k)$ as $V := \binom{S}{k}$ (edges are still formed by disjoint vertices — this changes the names of vertices not the graph structure). For $x \in S^u$ define the *hemisphere around* x as $HS(x) := \{y \in S^u : x \cdot y > 0\}$. For a coloring $V \to [u]$ define the sets $H_i := \{x \in S^u : \exists v \in V, c(v) = i, v \subseteq HS(x)\}$ for $i = 1, \dots, u$. These sets are open, as $H_i = \cup_{v \in V, c(v)=i} \cap_{x \in v} HS(x)$. Consider an arbitrary $x \in S^u$. The hemispheres $HS(x)$ and $HS(-x)$ are disjoint and together they cover S^u except for a *great circle*, i.e., the vectors orthogonal to x. By the general position assumption at most u elements of S are on this great circle, thus at least $m-u = 2k-1$ points fall in $HS(x) \cup HS(-x)$. Thus, one of the two hemispheres contain at least k elements of S. Let v be a size-k subset of S contained in either $HS(x)$ or $HS(-x)$. The set $H_{c(v)}$ contains x or $-x$. Now LS2 implies that one of the sets H_i must contain an antipodal pair x, $-x$. This means that both $HS(x)$ and $HS(-x)$ contains a color-i vertex. These vertices must be disjoint, thus adjacent, so c is not a proper coloring. Q.E.D.