CEU "Topics in Combinatorics" course

On weak ϵ-nets with respect to convex sets - covered in the March 18, 2014 class

Convex sets on the line are intervals. There is an $k = \lfloor 1/\epsilon \rfloor$ size ϵ-net for any set S on the line with respect to the intervals: If $S = \{x_1, \ldots, x_n\}$ and the points are indexed according their order on the line, then the subset $T = \{x_j, x_{2j}, x_{3j}, \ldots, x_{kj}\}$ is an ϵ-net, where $j = \lceil \epsilon n \rceil$.

The situation is radically different for convex sets in the plane. These sets shatter any point set in convex position, thus their VC-diension is infinite. Accordingly, there is no non-trivial ϵ-net for points in convex position with respect to all convex sets in the plane. (More precisely: any ϵ-net for a n points in convex position must have more than $(1-\epsilon)n$ elements.)

Let $\mathcal{R}$ be a set of ranges and S be a finite set with $|S| = n$, and $0 < \epsilon \leq 1$. We call a set T a *weak ϵ-net for S with respect to $\mathcal{R}$*, if for all $R \in \mathcal{R}$ satisfying $|R \cap S| \geq \epsilon n$ we have $R \cap T \neq \emptyset$.

Note that the only difference between a weak ϵ-net for a set S and an ϵ-net for the same set (with respect to the same ranges), is that the ϵ-net is required to be a subset of S, while the weak ϵ-net is not. This flexibility allows us to find reasonably small weak ϵ-nets in the plane with respect to all convex sets.

Theorem, Alon-Bárány-Füredi-Kleitman, 1992. There is a weak ϵ-net of size $O(1/\epsilon^2)$ for any finite set in the plane with respect to all convex sets.

Proof. Assume the size n of our finite set S in the plane is even and take a line$l$ that separates S into two subsets S_1 and S_2 of size $n/2$. Consider the $n^2/4$ line segments connecting a point of S_1 with a point of S_2. All these segments intersect l. Let V be the set of intersection points. For simplicity we assume all these points are distinct, thus $|V| = n^2/4$.

Let T_1 be a weak $3\epsilon/2$-net for S_1, T_2 a weak $3\epsilon/2$-net for S_2 and T_3 a $\epsilon^2/4$-net for V. All nets in the proof are with respect to the convex sets.

Claim: $T := T_1 \cup T_2 \cup T_3$ is a weak ϵ-net for S.

Indeed, if for a convex set R we have $|R \cap S_1| \geq 3\epsilon n/4$, then $R \cap T_1 \neq \emptyset$..

Similarly, if R is convex and $|R \cap S_2| \geq 3\epsilon n/4$, then $R \cap T_2 \neq \emptyset$.

Finally if $|R \cap S| \geq \epsilon n$ but none of the above two possibilities happen, then $|R \cap S_1| \geq \epsilon n/4$ and $|R \cap S_2| \geq \epsilon n/4$. In this case R contains the endpoints of at least $n^2/16$ of the connecting segments and by convexity we have $|R \cap V| \geq n^2/16$, and therefore $R \cap T_3 \neq \emptyset$ finishing the proof of the claim.

Observe that V is on the line so we can choose $|T_3| \leq 4/\epsilon^2$ as discussed earlier. For choosing T_1 and T_2 we use recursion. This yields the following recursion on the optimal size c_k of a weak $(2/3)^k$-nets in the plane with respect to convex sets:

$$c_k \leq 2c_{k-1} + 4/(2/3)^{2k}.$$

We have $c_0 = 1$ for the starting point of this recursion. Repeated application of the above recursion gives:

$$c_k \leq 4/(2/3)^{2k} + 2 \cdot 4/(2/3)^{2k-2} + 2^2 \cdot 4/(2/3)^{2k-4} + \cdots + 2^{k-1} \cdot 4/(2/3)^2 + 2^k.$$

This is a decreasing geometric sequence, so the sum can be estimated $c_k = O(4/(2/3)^{2k})$. In other words $O(1/\epsilon^2)$ size weak ϵ-nets exist with respect to the convex sets whenever $\epsilon = (2/3)^k$ for some integer k. To obtain a similar result for arbitrary ϵ set k to be the smallest integer with $(2/3)^k \leq \epsilon$ and notice that the $c_k = O(1/\epsilon^2)$ and any weak $(2/3)^k$-nets are also weak ϵ-nets. Q.E.D.

Recursive application of this same proof yields the existence of polynomial size weak ϵ-nets with respect to convex sets in d-space. The simplest proof gives the size bound $O(\epsilon^{-2^{d-1}})$, but a more clever proof gives $O(\epsilon^{-d})$.