CEU "Topics in Combinatorics" course

Topics covered in the February 18, 2014 class

Notation: $[n]$ denotes the positive integers up to n, i.e., $[n] = \{1, 2, \dots, n\}$. An *r-coloring* of a set H is a function $f : H \to [r]$. We call $f(x)$ the *color* of x. We call a subset $A \subseteq H$ *monochromatic* if $f(x) = f(y)$ whenever $x, y \in A$.

An *arithmetic progression* of size k is a set $\{a, a+b, a+2b, \dots, a+(k-1)b\}$, where the increment b is not zero. We write $[k]^d$ for the d-dimensional grid of side k, that is $[k]^d = \{(x_1, \dots, x_d) \mid \forall i \in [d] : x_i \in [k]\}$. A *combinatorial line* in $[k]^d$ is obtained by fixing some, but not all the coordinates and making the remaining coordinates equal. In other words it is the image of an injective function $f : [k] \to [k]^d$ satisfying that each coordinate $i \mapsto (f(i))_j$ is either a constant or the identity function. An example: the following three points form a combinatorial line in $[3]^8$:

$$(1, 1, 2, 1, 1, 3, 1, 1)$$

$$(1, 2, 2, 1, 2, 3, 2, 2)$$

$$(1, 3, 2, 1, 3, 3, 3, 3)$$

Note that each combinatorial line in $[k]^d$ contains k elements.

In the following theorems k, r and n denote positive integers.

Van der Waerden theorem (1927): For all k and r there exists n such that in any r-coloring of $[n]$ there is a monochromatic arithmetic progression of size k.

Erdős-Turán conjecture (1936), Szemerédi theorem (1975) (Roth proved it for $k = 3$ in 1953 and Szemerédi for $k = 4$ in 1969): For all k and r there exists n such that all $H \subseteq [n]$ with $|H| \geq n/r$ contains an arithmetic progression of size k.

Hales Jewett theorem (1963): For all k and r there exists n such that in any r-coloring of $[k]^n$ there is a monochromatic combinatorial line.

Density Hales Jewett theorem by Fürstenberg and Katznelson (1991): For all k and r there exists n such that all $H \subseteq [k]^n$ with $|H| \geq k^n/r$ contains a combinatorial line.

Let us denote the smallest n satisfying the requirements of above theorems $VdW(k, r)$, $Sz(k, r)$, $HJ(k, r)$ and $DHJ(k, r)$, respectively. We can consider the theorems to state that these numbers are finite.

— The density version implies the coloring version, namely $VdW((k, r) \leq Sz(k, r)$ and $HJ(k, r) \leq DHJ(k, r)$. Indeed, the largest color class must contain at least a $1/r$ fraction of all elements, look for a monochromatic structure in that color.

— The combinatorial line version implies the arithmetic progression version, namely $VdW(k, r) \leq k^{HJ(k,r)}$ and $Sz(k, r) \leq k^{DHJ(k,r)}$. To see this let us redefine $[n]$ temporarily to be the numbers from 0 to $n - 1$. This does not change the problems we consider. Now we can make a 1-1 correspondence between $[k^m]$ and $[k]^m$ by writing the numbers in $[k^m]$ as m-digit numbers in base k (possibly with leading zeros). The inequalities follow since all the combinatorial lines in $[k]^n$ correspond to arithmetic progressions. Note, however, that many arithmetic progressions do not correspond to combinatorial lines.

The proofs of the VdW and HJ theorems are tricky inductions, but are relatively simple. The proofs of their density versions are much more involved. By now there are several proofs for both density theorems above, one of the simplest ones is a proof of DHJ that was a result of many mathematicians on-line collaboration in a so called Polymath project in 2009.

In the rest of this note we give a proof of the Hales Jewett theorem through a very tricky induction. We will look for something that is both more and less than what the theorem requires. It is a bigger set but not quite monochromatic.

We start with extending the notion of a combinatorial line. A *d-space* in $[k]^n$ is an injective map $f : [k]^d \to [k]^n$ such that for all $j \in [n]$ its projection $f_j : [k]^d \to [k]$ given by $f_j(x) = (f(x))_j$ is either a constant function or one of the coordinate functions $x \mapsto x_i$. Note that a combinatorial line is the image of a 1-space. Here is an example of a 3-space in $[3]^8$:

$$f(x, y, z) = (1, z, 3, x, x, z, 3, y).$$

We call the following sets A_i^d the *faces* of $[k]^d$ for $i = 0, 1, \ldots, d$:

$$A_i^d = \{x \in [k]^d \mid x_i = k \text{ if } i \leq k \text{ and } x_i < k \text{ if } i > k\}.$$

Note that these are pairwise disjoint subsets in $[k]^d$. Here are the faces of $[3]^2$:

$$A_0^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\},$$

$$A_1^2 = \{(3, 1), (3, 2)\},$$

$$A_2^2 = \{(3, 3)\}.$$

Given a coloring of $[k]^n$ we call a d-subspace f of $[k]^n$ *almost monochromatic* if the image of the faces of $[k]^d$ are monochromatic. The color of the image of different faces may be different.

Let $HJ(k, r, d)$ stand for the smallest n such that in any r-coloring of $[k]^n$ there is an almost monochromatic d-space. At this time we do not know that such an n exists, we let $HJ(k, r, d)$ be infinite if no such n exists. The following recursions will establish that $HJ(k, r)$ and $HJ(k, r, d)$ are finite for all k,r and d.

Lemma 1. For all $k \geq 2$ and r we have $HJ(k, r) \leq HJ(k, r, r)$.

Proof: Let us fix an r-coloring of $[k]^n$ with $n = HJ(k, r, r)$. By the choice of n we have an almost monochromatic r-space $f : [k]^r \to [k]^n$. Each of the images $f(A_i^r)$ for $i = 0, 1, \ldots, d$ are monochromatic. By the pigeon hole principle there are two of them $f(A_i^r)$ and $f(A_j^r)$ for some $0 \leq i < j \leq r$ that have the same color. Consider the combinatorial line $L \subseteq [k]^r$ obtained by fixing the first i coordinates to k, the last $d - j$ coordinates to 1 and making the remaining coordinates equal. One element of L (when the middle coordinates are k) is in A_j^r, all the remaining elements in L are in A_i^r. Thus, the image $f(L)$ of L is a monochromatic line in $[k]^n$ as needed. Q.E.D.

Lemma 2. For all k and r we have $HJ(k, r, 1) \leq HJ(k - 1, r)$.

Proof: Let us fix an r-coloring of $[k]^n$ with $n = HJ(k - 1, r)$. By the choice of n we find a monochromatic combinatorial line L in the restriction of our coloring to $[k-1]^n$. There is a 1-space f in $[k]^n$ such that the image of the face A_0^1 is exactly L, hence monochromatic. This means that f is itself almost monochromatic as there is only one other face to consider, but A_1^1 has a single element, thus its image is automatically monochromatic. Q.E.D.

Lemma 3. For all k, r and d, and for $z = HJ(k, r, d)$ we have $HJ(k, r, d+1) \leq z + HJ(k, r^{k^z}, 1)$.

Proof: We set $r^* = r^{k^z}$, $w = HJ(k, r^*, 1)$, $n = z + w$. For $x \in [k]^z$ and $y \in [k]^w$ we denote their *concatenation* by xy. This is the element of $[k]^n$ whose first z coordinates gives x and last w coordinates give y.

Let $c : [k]^n \to [r]$ be an arbitrary r-coloring of $[k]^n$. For $y \in [k]^w$ we let $c^*(y) : [k]^z \to [r]$ be the r-coloring of $[k]^z$ defined by $c^*(y)(x) = c(xy)$. (This is the restriction of c to the points with the last w coordinates fixed.). Note that there are r^* possible r-colorings of $[k]^z$, thus c^* can be considered an r^*-coloring of $[k]^w$. (In a true r^*-coloring the colors are the elements of $[r^*]$ but the actual values of the colors do not matter, just the number of colors used.)

By the choice of w there is an almost monochromatic 1-space $f : [k] \to [k]^w$ for the coloring c^*. This means that the image $f(A_0^1)$ of $A_0^1 = [k-1]$ is monochromatic in c^*. Let us denote its color by $c_0 : [k]^z \to [r]$. By the choice of z there is an almost monochromatic d-space $g : [k]^d \to [k]^z$ in $[k]^z$ for the r-coloring c_0. We define $h : [k]^{d+1} \to [k]^n$ by $h(x_1, \ldots, x_{d+1}) = g(x_1, \ldots, x_d)f(x_{d+1})$. This is clearly a $(d+1)$-space in $[k]^n$. We claim it is almost monochromatic. If so, that finishes the proof of the lemma.

We need to show that for $i = 0, 1, \ldots, d+1$ the image $h(A_i^{d+1})$ of the face A_i^{d+1} is monochromatic for c. This is trivial for $i = d+1$, as A_{d+1}^{d+1} is a singleton. For $i \le d$ we claim that the color of any point in $f(A_i^{d+1})$ is the same as the color of $g(A_i^d)$ under c_0 (recall that $g(A_i^d)$ is monochromatic in c_0). Indeed, if $x = (x_1, \ldots, x_{d+1}) \in A_i^{d+1}$, then $x' = (x_1, \ldots, x_d) \in A_i^d$ and $x_{d+1} < k$. Thus $c^*(f(x_{d+1})) = c_0$ and $c(h(x)) = c(g(x')f(x_{d+1})) = c^*(f(x_{d+1}))(g(x')) = c_0(g(x'))$. Q.E.D.

Proof of the Hales Jewett theorem: We prove by induction on k that $HJ(k,r)$ is finite for any k and r.

The $k = 1$ base case is trivial with $HJ(1,r) = 1$.

Assume that $k \ge 2$ and $HJ(k-1,r)$ is finite for any r. We need to prove that $HJ(k,r)$ is finite for any r. First we prove by induction on d that $HJ(k,r,d)$ is finite for any d and r.

The base case of this inside induction is $d = 1$. We have $HJ(k,r,1) \le HJ(k-1,r)$ by Lemma 2 and thus it is finite by the (outside) inductive hypothesis.

For the inductive step of the inside induction assume $HJ(k,r,d')$ is finite for every r and $d' \le d$ and we prove the same for $d' = d+1$. Indeed, by the (inside) inductive hypothesis $z = HJ(k,r,d)$ is finite and so is $w = HJ(k, r^{k^z}, 1)$. By Lemma 3 $HJ(k,r,d+1) \le z + w$ is also finite as needed.

Having finished the inside induction we know that $HJ(k,r,r)$ is finite for every r. By Lemma 1 we have $HJ(k,r) \le HJ(k,r,r)$ and so $HJ(k,r)$ is also finite for every r finishing the outside inductive step and the proof of the Hales Jewett theorem. Q.E.D.

Assignment:

1. We saw $VdW(k,r) \le k^{HJ(k,r)}$. Modify the argument to get a better correspondence, i.e., bound $VdW(k,r)$ by a smaller (not exponential) function of $HJ(k,r)$. What about the similar connection between $Sz(k,r)$ and $DHJ(k,r)$?

2. We use a doubly exponential function r^{k^z} of z in the statement of Lemma 3. State and prove a stronger version of this lemma in which this doubly exponential function is replaced by a simple exponential function of k r, d and z.

3. A simple consequence of the Van der Waerden theorem is that if you color *all positive integers* with finitely many colors, then there will be arbitrarily long monochromatic arithmetic progressions. Will there always be monochromatic infinite arithmetic progressions?