Short reminder of two bounds on the hypergraph Ramsey numbers $R^{(3)}(k,k)$

Definition: $R^{(3)}(k,k)$ stands for the smallest integer n such that in any 2-coloring of the 3-subsets of a set of n elements one finds a monochromatic subset of size k.

Both of the following arguments establish that $R^{(3)}(k,k)$ exists and both use the graph Ramsey theorem, namely that the Ramsey numbers $R(k,k)$ exist, i.e., they prove the 3-uniform hypergraph Ramsey from the (2-uniform) graph Ramsey. Both easily generalize to prove the 4-uniform version of Ramsey theorem from the 3-uniform version, etc. They also generalize to an arbitrary number of colors, thus eventually proving the Ramsey theorem in full generality.

Both of the arguments can be used to bound the Ramsey numbers $R^{(3)}(k,k)$, but they lead to widely different bounds as we will see. For simplicity we use the bound $R(k,k) < 4^k$ in our calculations.

Argument presented in class: Consider a 2-coloring $c : \binom{V}{3} \to \{1,2\}$ of all 3-subsets of an n element set V. Our goal is to find a monochromatic k-subset of V if n is large enough.

In an iterative process we choose the vertices $x_1, x_2, \ldots, x_m$ of V and along the way we also define the subsets V_i of V as follows.

We start with $V_0 = V$. We stop the process when V_m is empty.

If V_{i-1} is defined and not empty we choose an arbitrary $x_i \in V_{i-1}$. Then we consider the coloring $c_i : \binom{V_{i-1}-\{x_i\}}{2} \to \{1,2\}$ given by $c_i(\{y,z\}) = c_i(\{x_i,y,z\})$. We let V_i be a monochromatic subset of $V_{i-1} - \{x_i\}$ for this coloring of largest possible size. We call the color of this monochromatic subset the *color of* v_i. (Minor point: if $|V_i| \leq 1$, then there is no colored edges in V_i. For simplicity we just define the color of x_i arbitrarily in this case. This will happen for x_m and x_{m-1}.)

The set $\{x_1, x_2, \ldots, x_m\}$ is not quite monochromatic for c yet, but it has the property that the color of $\{x_i, x_j, x_k\}$ is the color of x_i whenever $i < j < k$, since in this case we have $c(\{x_i, x_j, x_k\}) = c_i(\{x_j, x_k\}$ and $x_j, x_k \in V_i$ and V_i is monochromatic in the color of x_i for the coloring c_i.

Clearly, half of the vertices x_i has to have the same color and they form a monochromatic subset of size at least $m/2$.

Rough calculation to bound $R^{(3)}(k,k)$. We find a monochromatic subset of size k if $m = 2k-1$. (In fact, $m = 2k-3$ is enough, because x_m and x_{m-1} do not really have colors, see minor point above.) Using $R(t,t) < 4^t$ we have that $|V_i| \geq \log_4 |V_{i-1}|$, so $|V_i| \geq \log_4^{(i)} n$ (i-times-iterated $\log_4$ of n). We need $|V_{m-1}| \geq 1$, so $n = 4^{4^{\cdot^{\cdot^{4}}}}$ is enough, with a tower of height $m-1 = 2k-2$. This is therefore an upper bound for $R^{(3)}(k,k)$, although a very weak one.

Argument asked for in problem 2, assignment 4: This argument is similar. Again, we consider a 2-coloring $c : \binom{V}{3} \to \{1,2\}$ of all 3-subsets of an n element set V and our goal is to find a monochromatic k-subset of V if n is large enough. We use a similar iterative process to chose vertices $x_1, x_2, \ldots, x_m$ of V and along the way we also define the subsets V_i of V.

We start with $V_0 = V$. We stop the process when V_m is empty.

If V_{i-1} is defined and not empty we choose an arbitrary $x_i \in V_{i-1}$. The main (and only) difference between the two processes, that we 2-color the elements of $V_{i-1} - \{x_i\}$ and not the edges as before. For all $1 \leq j < i$ we define the coloring $c_{ji}(z) = c(\{x_j, x_i, z\})$ and choose V_i as a largest subset of $V_{i-1} - \{x_i\}$ that is monochromatic in each of these colorings. We call the color of any element of V_i in the coloring c_{ji} the *color of the edge* $\{x_j, x_i\}$. (Minor point: in case $V_i = \emptyset$, that is for $i = m$ we can take an arbitrary color as the color of the edges $\{x_j, x_i\}$.)

The set $\{x_1, x_2, \ldots, x_m\}$ is "even less" monochromatic, then before, but it has the property that the color of a triple is determined by the first two elements. For $i < j < k$ we have $c(\{x_i, x_j, x_k\}) = c_{ij}(x_k)$ and this is the color of the pair (edge) $\{x_i, x_j\}$ as $x_k \in V_j$. So any subset of $\{x_1, x_2, \ldots, x_m\}$

that is monochromatic for the edge coloring defined in the process is also monochromatic for the hyperedge coloring c.

Rough calculation to bound $R^{(3)}(k,k)$. We find a monochromatic subset of size k if $m \geq R(k,k)$, so definitely for $m \geq 4^k$. We have $|V_i| \geq (|V_{i-1}|-1)/2^{i-1}$ (we have 2^{i-1} possible combinations of the colors c_{ji} for $1 \leq j < i$). So $|V_i| \geq \lfloor (n-1)/2^{\binom{i}{2}} \rfloor$. To obtain x_m we need $|V_{m-1}| \geq 1$, this is ensured by $n \geq 2^{\binom{m-1}{2}}+1$. If $m \geq 4^k$ this implies the existence of a monochromatic k-subset, so we have $R^{(3)}(k,k) \leq 2^{\binom{4^k}{2}} + 1 < 2^{2^{4k-1}}$. Note that this "only" doubly exponential in k, much better than the previous bound.

Moral: It is better to apply the pigeon hole principle several times and the graph Ramsey once than vice versa because the pigeon hole principle is efficient, but we lose a lot with each application of the graph Ramsey.

Historical notes: Ramsey's original 1930 proof of his theorem was (basically) the former proof above. Erdős and Rado gave the latter proof in 1952. (Using better bounds on the graph Ramsey numbers one can improve the exponent of the exponent from $4k-1$ to around $4k-\log k$.) Conlon, Fox and Sudakov gave the best currently known bound in 2010. Their bound is still doubly exponential: $2^{2^{2k+O(\log k)}}$.

These doubly exponential upper bounds are very far from the best known lower bound that is the $2^{\Omega(k^2)}$ bound you obtained in the solution for problem 1 of assignment 5 (this is usually attributed to Erdős and Hajnal). One of the most pressing problems of Ramsey theory is to decide whether these numbers are simply exponential in k (as the current lower bound) or doubly exponential (as the current upper bound).