CEU "Topics in Combinatorics" course

Some of the topics covered in the February 14, 2017 class

The k-dimensional sphere as the unit sphere in Euclidean $(k+1)$-space: $S^k := \{x \in R^{k+1} : ||x|| = 1\}$. The points x and $-x$ in S^k are called *antipodal*.

Borsuk-Ulam Theorem: If $f : S^k \to R^k$ is continuous, then $\exists x \in S^k$ satisfying $f(x) = f(-x)$.

Lyusternik-Schnirelmann Theorem (1. form): If $H_i \subseteq S^k$ are (relative) open for $i = 1, \ldots, k$ and none of them contains a pair of antipodal points, then there is a pair of antipodal points not covered by any of the sets, that is, $x, -x \in S^k \setminus \cup_{i=1}^k H_i$.

Lyusternik-Schnirelmann Theorem (2. form): If $H_i \subseteq S^k$ are (relative) open for $i = 1, \ldots, k+1$ and none of them contains a pair of antipodal points, then together they do not cover S^k (i.e., $\cup_{i=1}^{k+1} H_i \neq S^k$).

A combinatorial proof through Tucker's lemma (similar to the proof of Brouwer's fixed point theorem through Sperner's lemma studied in class) is possible. A more standard proof is using simple algebraic topology.

Here we prove their equivalence only.

BU implies LS1: Given open sets $H_i \subseteq S^k$ for $i = 1, \ldots, k$ we define $f : S^k \to R^k$ by $f(x) := (d(x, S^k \setminus H_1), \ldots, d(x, S^k \setminus H_k))$, where d stands for the (Euclidean) distance. Note that this is continuous, so by BU we have $x \in S^k$ with $f(x) = f(-x)$. We claim that this antipodal pair satisfies the statement of LS1. Fix i. As H_i is open we have $x \in H_i$ if and only if $d(x, S^k \setminus H_i) > 0$ (and similarly for $-x$). We have $d(x, S^k \setminus H_i) = d(-x, S^k \setminus H_i)$: this value cannot be positive, because then H_i would contain an antipodal pair, so it is 0, and thus H_i contains neither x or $-x$. This holds for all i. Q.E.D.

LS1 implies LS2: We have $k+1$ open subsets in the sphere S^k. Apply LS1 for the first k: there is an antipodal pair x, $-x$ missed by all of them. But the last set can only cover on of this two points, so the other one is not covered. Q.E.D.

LS2 implies BU: Let v_i be the i'th coordinate vector (all zero with a 1 at position i) for $i = 1, \ldots, k$ and $v_{k+1} = (-1, \ldots, -1)$. Consider a continuous function $f : S^k \to R^k$. For $i = 1, \ldots, k+1$ set $H_i = \{x \in S^k : (f(x) - f(-x)) \cdot v_i > 0\}$. This is open and does not contain an antipodal pair. By LS1 there is $x \in S^k$ outside all these sets. This means $(f(x) - f(-x)) \cdot v_i \leq 0$ for all i. This means that all coordinates of $f(x) - f(-x)$ are non-positive, but there sum is non-negative. Thus, we have $f(x) = f(-x)$. Q.E.D.

Borsuk graph B^k_δ: Here $0 < \delta < 2$ and k is a positive integer. Vertex set: S^k (infinite). Vertices x and y are adjacent if and only if $d(x, y) > \delta$ (Euclidean distance). Take δ close to 2, then "almost antipodal" pairs are adjacent.

Finite Borsuk graph: Take a finite set $S \subseteq S^k$ with every point $x \in S^k$ having $y \in S$ with $d(x, y) < \epsilon := 1 - \delta/2$. Form the induced subgraph $B^k_\delta[S]$.

Claim: $\chi(B^k_\delta[S]) \geq k + 2$

Proof using LS2: Let $c : S \to [k+1]$ be a coloring. Let H_i be the ϵ-neighborhood of $c^{-1}(i)$, namely $H_i := \{x \in S^k : \exists y \in S, d(x, y) < \epsilon, c(y) = i\}$. As union of open ϵ-neighborhoods, the sets H_i are open. By the choice of S together they cover S^k. By LS1, one of the sets H_i must contain an antipodal pair, x and $-x$. By the definition of H_i it means that vertices $y, z \in S$ must exist with $c(y) = c(z) = i$, $d(x, y) < \epsilon$ and $d(-x, z) < \epsilon$. Here $d(x, -x) = 2$, so by the triangle inequality we have $d(y, z) > 2 - 2\epsilon = \delta$. Thus, the coloring c is not proper. Q.E.D.

Kneser graph $KG(n, k)$ ($n \geq 2k > 0$ is assumed, otherwise the graph has no edges.) Vertices: $\binom{X}{k}$, for some fixed n-set X, i.e., the k-subsets of X. Vertices x and y are adjacent if and only they are disjoint.

Remark: Both the Borsuk graph B^k_δ and the Kneser graph $KG(n, k)$ are *vertex-transitive* graphs, i.e., its *automorphisms* move every vertex to every vertex. The Kneser graphs are also *edge-transitive*, i.e., every edge is moved to every edge by an automorphism. For the Borsuk graphs automorphisms come from rotations of the sphere, for the Kneser graphs they come from permutations of X. Another similarity: with appropriate choice of the parameters (namely δ close to 2 and n close to $2k$) the graphs B^k_δ and $KG(n, k)$ contains no short odd cycles. To see this for the Borsuk graphs observe that if $d(x, y) > \delta$ for $x, y \in S^k$, then $d(x,-y) < \epsilon'$ for $\epsilon' = \sqrt{4-\delta^2}$ (Pythagoras). Now for an odd cycle $x_0 x_1 \ldots, x_{2l}$ in B^k_δ we have $\delta < d(x_0, x_{2l}) \leq d(x_0, -x_1) + d(-x_1, x_2) + \cdots + d(-x_{2l-1}, x_{2l}) < 2l\epsilon'$. Thus $l > \delta/(2\epsilon')$.

Similarly, if x and y have a common neighbor z in $KG(n,k)$, then $y - x \subseteq X - (x \cup z)$ and therefore $|y - x| \leq |X - (x \cup z)| = n - 2k$. For an odd cycle $x_0 x_1 \ldots, x_{2l}$ in $KG(n,k)$ we have $k = |x_{2l} \setminus x_0| \leq |x_2 \setminus x_0| + |x_4 \setminus x_2| + \cdots + |x_{2l-2} \setminus x_{2l}| \leq l(n-2k)$. Thus $l \geq k/(n-2k)$.

Kneser's conjecture (1955), proved by Lovász (1979), proof simplified by Bárány (1979) and Greene (2002): $\chi(KG(n,k)) = n - 2k + 2$

Easy direction (Kneser): $\chi(KG(n,k) \leq n-2k+2$. Proof by constructing a proper $(m-2k+2)$-coloring. We take an arbitrary subset $X' \subset X$ of size $2k-1$ and let the color of the vertex x be any element of x outside X'. We used $n-2k+1$ colors so far and all color classes are independent sets. We use an extra color for the remaining vertices, namely those contained in X'. This is also an independent set. Q.E.D.

Hard direction (Lovász): $\chi(KG(n,k) \geq n - 2k + 2$.

Proof (Greene, using LS1): Let $u = n - 2k + 1$ and use a set X of n points in S^u in *general position* for the base set in $KG(n,k)$. General position means that each $u+1$ points are linearly independent as vectors in R^{u+1}. So now the vertex set of $KG(n,k)$ is $V = \binom{X}{k}$.

For $x \in S^u$ define the *hemisphere around* x as $HS(x) := \{y \in S^u : x \cdot y > 0\}$. For a coloring $c : V \to [u]$ define the sets $H_i := \{x \in S^u : \exists v \in V, c(v) = i, v \subseteq HS(x)\}$ for $i = 1, \ldots, u$. These sets are open, as $H_i = \cup_{v \in V, c(v)=i} \cap_{x \in v} HS(x)$. Consider an arbitrary $x \in S^u$. The hemispheres $HS(x)$ and $HS(-x)$ are disjoint and together they cover S^u except for the vectors orthogonal to x. By the general position assumption at most u elements of X are orthogonal to x, thus at least $n - u = 2k - 1$ points fall in $HS(x) \cup HS(-x)$. Thus, one of the two hemispheres contain at least k elements of X. Let v be a k-subset of X contained in either $HS(x)$ or $HS(-x)$. The set $H_{c(v)}$ contains x or $-x$. Now LS1 implies that one of the sets H_i must contain an antipodal pair $x, -x$. This means that both $HS(x)$ and $HS(-x)$ contains a color-i vertex. These vertices must be disjoint, thus adjacent, so c is not a proper coloring. Q.E.D.