Extremal theory of vertex or edge ordered graphs*

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Abstract

We enrich the structure of finite simple graphs with a linear order on either the vertices or the edges. Extending the standard question of Turán-type extremal graph theory we ask for the maximal number of edges in such a vertex or edge ordered graph on n vertices that does not contain a given pattern (or several patterns) as subgraph. The forbidden subgraph itself is also a vertex or edge ordered graph, so we forbid a certain subgraph with a specified ordering, but we allow the same underlying subgraph with a different (vertex or edge) order. This allows us to study a large number of extremal problems that are not expressible in the classical theory. In this survey we report on ongoing research. For easier access we include sketches of proofs of selected results.

1 Definitions

A *vertex ordered graph* or simply an *ordered graph* is a simple graph with a linear order on the vertices. Formally, an ordered graph is a triple $(V, E, <)$, where V is the vertex set, $E \subseteq \binom{V}{2}$ is the edge set (so that (V, E) is a simple graph) and $<$ is a linear order relation on V. Similarly, an *edge ordered graph* is a simple graph with a linear order on its edges, that is $(V, E, <)$, where (V, E) is a simple graph and $<$ is a linear order relation on E. In this

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survey we assume that V is finite. We say that (V, E) is the simple graph *underlying* the vertex or edge ordered graph $(V, E, <)$ and $(V, E, <)$ is a *vertex or edge ordering* of the simple graph (V, E). The notions of isomorphism and subgraph naturally extend to these graphs: An isomorphism between vertex or edge ordered graphs is an isomorphism between the underlying simple graphs that preserves the ordering. A subgraph of a vertex or edge ordered graph is a subgraph of the underlying simple graph with the inherited vertex or edge order.

Armed with these definitions we can extend some classic areas of graph theory to ordered graphs. Here we do this for Turán-type extremal graph theory and for most of this survey we consider vertex orderings. (See the last section for some preliminary results from ongoing research on edge ordered graphs.) Extremal graph theory asks for the maximal number of edges in a simple graph of given size that *avoids* (i.e., does not contain as a subgraph) a specified pattern or any member of a given family of patterns. In particular, we are interested in the maximal number, $\mathrm{ex}(\mathcal{P}, n)$, of edges in an n-vertex simple graph that has no subgraph isomorphic to any member of the family $\mathcal{P}$. Note that we must require that $\mathcal{P}$ does not contain empty graphs (i.e., each member has at least one edge) in order for this definition to make sense. In the case where the forbidden family consists of a single pattern we write $\mathrm{ex}(P, n)$ to denote $\mathrm{ex}(\{P\}, n)$. We call $\mathrm{ex}(\mathcal{P}, n)$ the *extremal function* of the family $\mathcal{P}$ and will concentrate on its asymptotic behavior. Accordingly, all the asymptotic notations like $O(\cdot)$ and $o(\cdot)$ should be interpreted for a fixed family $\mathcal{P}$ and, in particular, the implied constants in $O(\cdot)$ may depend on this family.

For a natural extension of this theory to ordered graphs, we consider a family $\mathcal{P}$ of ordered graphs and we look for the largest number $\mathrm{ex}_<(\mathcal{P}, n)$ of edges in an n-vertex ordered graph with no ordered subgraph isomorphic to any member of $\mathcal{P}$. As before, we require that each member of $\mathcal{P}$ has at least one edge and simplify the notation for singleton families by writing $\mathrm{ex}_<(P, n)$ to denote $\mathrm{ex}_<(\{P\}, n)$. Our remark on the asymptotic notation also applies here.

Let us first observe that the extremal theory of ordered graphs is strictly richer than classical extremal graph theory in the sense that the classical questions can be equivalently asked in this setting, but we can also ask new questions. In particular, for any family $\mathcal{P}$ of simple graphs one can form the family $\mathcal{P}_<$ consisting all orderings of the patterns in $\mathcal{P}$ and then we trivially have:

$$\mathrm{ex}(\mathcal{P}, n) = \mathrm{ex}_<(\mathcal{P}_<, n).$$

On the other hand, if we forbid, say, a single ordered graph P, the corresponding extremal function $\mathrm{ex}_<(P, n)$ has no direct analogue in the classical theory. We naturally have $\mathrm{ex}_<(P, n) \geq \mathrm{ex}(\overline{P}, n)$, where $\overline{P}$ is the simple graph underlying P, but this lower bound is typically very weak, since avoiding $\overline{P}$ in a particular order is often much easier than avoiding it in all possible orders.

In the paper [5] Braß, Károlyi and Valtr establish a very similar theory. Instead of a linear order on the vertices, they consider a *cyclic order*. They have very nice results on the extremal function of certain cyclically ordered graphs. These results have natural translations in the extremal theory of ordered graphs. Let us also give credit to Füredi and Hajnal, who in the last paragraph of their paper [12] explicitly ask for developing both the extremal theory of vertex ordered graphs surveyed here and for that of the cyclically ordered graphs as done later by Braß, Károlyi and Valtr.

Extensions of Ramsey theory to ordered graphs are also studied extensively, see [1, 6].

Most of this survey is about the extremal theory of vertex ordered graphs. In Section 2 the classical Erdős-Stone-Simonovits theorem and its generalization to vertex ordered graphs is presented. This result determines the asymptotics of the extremal function of ordered graphs with interval chromatic number three or higher. We are satisfied with asymptotical results, so we concentrate exclusively on *ordered bipartite* graphs (ordered graphs with interval chromatic number two) in later sections. In Section 3 the close connection between the extremal theory of ordered bipartite graphs and the somewhat older extremal theory of 0-1 matrices is explained. Classical extremal graph theory always gives us a lower bound on the corresponding ordered questions. In Section 4 we explore how far this lower bound can be from the ordered extremal function. The case of forests is treated separately in Section 5. Here the unordered theory gives a linear lower bound, and a prominent open question is to decide if almost linear (say, $n^{1+o(1)}$) bound also hold for all ordered bipartite forests. This is known for a wide class of such forests, but not for all. In Section 6 we present two simple results on the class of ordered graphs with linear extremal functions. Finally, in Section 7 we present results about extremal functions arising from simultaneously forbidding two (or more) ordered graphs. While it is not known in the classical extremal theory of graphs whether forbidding several graphs can result in an extremal function of lower order of magnitude than the ones obtained from just one forbidden graph, we have many such examples in the ordered setting.

In Section 8, the last section of this survey, we summarize recent research on the extremal theory of edge ordered graphs.

2 Basic results

Any survey about extremal graph theory should start with the following classical theorem of Turán from 1941, [30], of which the $r = 2$ special case (the maximal number of edges in a triangle-free graph) was proved by Mantel in 1907, [19]. The result gives the exact extremal function when the forbidden graph is a complete graph. Further, for the $(r + 1)$-vertex complete graph K_{r+1} the theorem states that the unique (up to isomorphism) n-vertex graph with the maximum number of edges avoiding K_{r+1} is the *Turán graph* $T(n, r)$ formed by partitioning the vertices into r almost equal parts and letting a pair of vertices form an edge if and only if they are from distinct parts. Note that the number of edges of the Turán graph $T(n, r)$ is $(1-1/r)n^2/2-O(1)$, where the $O(1)$ error term comes from unequal parts and can go as high as $r/8$. As a consequence, we have:

Theorem 1 (Turán [30]). *For every $r \geq 1$ we have*

$$\mathrm{ex}(K_{r+1}, n) = (1 - 1/r)\frac{n^2}{2} - O(1).$$

A trivial generalization of this result to ordered graphs involves the ordered clique, the unique ordering of the complete graph. Let $K_{r+1,<}$ stand for the $(r + 1)$-vertex ordered clique and we trivially have $\mathrm{ex}_<(K_{r+1,<}, n) = \mathrm{ex}(K_{r+1}, n)$. A more revealing generalization is about the ordered path $P_{r+1,<}$ obtained from the $(r + 1)$-vertex path P_{r+1} with the natural order on the vertices where edges connect neighboring vertices in the order. We have $\mathrm{ex}_<(P_{r+1,<}, n) = \mathrm{ex}(K_{r+1}, n)$. Here the direction $\leq$ follows from the fact that $P_{r+1,<}$ is an ordered subgraph of $K_{r+1,<}$ and $\geq$ follows from the fact that if we order the vertices of $T(n, r)$ in a way that the r parts become intervals in the ordering, then the resulting ordered graph does not contain $P_{r+1,<}$ as an ordered subgraph. Note, however, that in the case r does not divide n, this process yields several non-isomorphic extremal ordered graphs. Note also that the path P_{r+1} has several non-isomorphic orderings for $r > 1$, and by Theorem 3 below, all other orderings have substantially smaller extremal functions.

The most general result in Turán-type extremal graph theory is the following consequence of the Erdős-Stone theorem, [7]. It basically states that the extremal function of *any* simple graph is close to the extremal function of the complete graph with the same *chromatic number*.

Theorem 2 (Erdős-Stone-Simonovits [9, 7])**.** *Let $\mathcal{P}$ be a family of simple graphs and $r + 1 = \min_{P\in\mathcal{P}} \chi(P)$ be the smallest chromatic number of a member of this family. We have*

$$\mathrm{ex}(\mathcal{P}, n) = (1 - 1/r)\frac{n^2}{2} + o(n^2).$$

Pach and Tardos, [24] gave a generalization of this result for ordered graphs. It is based on finding the "correct" version of the chromatic number for ordered graph.

The *interval coloring* of an ordered graph is a proper coloring of the underlying simple graph in which each color class is an interval of the linear order. The *interval chromatic number* of an ordered graph P is the smallest number of colors in an interval coloring of P. We write $\chi_<(P)$ to denote the interval chromatic number of P.

Note that the interval chromatic number is much simpler to compute than the chromatic number because a greedy strategy suffices. Indeed, we can form the first color class by taking longest initial segment of the vertices that form an independent set and proceed similarly for subsequent color classes. The process yields an interval coloring with the fewest possible colors. This is because the first color class of any interval coloring is a subset of first color class found above, so greedily choosing the longest possible interval cannot hurt us later. Using this definition, the generalization of the Erdős-Stone-Simonovits theorem is rather straightforward:

Theorem 3 (Erdős-Stone-Simonovits theorem for ordered graphs [24])**.** *Let $\mathcal{P}$ be a family of ordered graphs and $r + 1 = \min_{P\in\mathcal{P}} \chi_<(P)$ be the smallest interval chromatic number of a member of this family. We have*

$$\mathrm{ex}_<(\mathcal{P}, n) = (1 - 1/r)\frac{n^2}{2} + o(n^2).$$

Proof. Order the vertices of the Turán graph with r-classes such that each class becomes an interval of the ordering. This way we obtain an ordered graph with interval chromatic number r, so it avoids all ordered graphs with higher interval chromatic number, including all members of $\mathcal{P}$. This provides the lower bound for the extremal function $\mathrm{ex}_<(\mathcal{P}, n)$.

Let $P \in \mathcal{P}$ be an ordered graph with interval chromatic number $r+1$. The upper bound comes from the classical Erdős-Stone theorem. Let m be the number of vertices of P. By the theorem, a simple graph on n vertices with at least $(1 - \frac{1}{r} + \varepsilon)n^2$ edges contains the Turán graph T with $r+1$ classes, each containing m vertices if n is large as a function of r, m and ε. We show that any vertex ordering of $T_<$ of T contains P by induction on r. This statement holds trivially for $r = 0$, so we start the induction here (despite the fact that the theorem itself requires $r > 0$ as $\mathcal{P}$ cannot contain an empty graph). For $r > 0$ we explicitly find a monotonic homomorphism from P to $T_<$. For this, identify the first interval I in an optimal interval coloring of P. We map the vertices in I to the first $k = |I|$ vertices of $T_<$ in a single class of the underlying Turán graph. Let x be the last vertex used, so it is the k'th vertex of the chosen class in $T_<$. We choose the class so as to make x appear as early in the vertex order as possible. Let us obtain the induced subgraph T' of $T_<$ by deleting all vertices in the class of x and the first k vertices in all other classes. Note that T' is an ordering of the Turán graph with r classes, each containing $m-k$ vertices, so by the inductive hypothesis it contains $P - I$, an ordered graph of interval chromatic number r on $m-k$ vertices. This gives a mapping from $P - I$ to T' (and so also to $T_<$) and together with our mapping of I provides the required monotonic homomorphism. Indeed, the mapping is monotonic as the image of $P - I$ is inside T', therefore strictly after x, the last vertex in the image of I. It is a homomorphism as $T_<$ contains a complete bipartite graph connecting the image of I (and the whole class containing it) with T'. $\square$

Just as the classic version of this theorem, it gives exact asymptotics for the extremal function of ordered graphs unless the ordered graph is *ordered bipartite* (i.e., has interval chromatic number 2). We will therefore concentrate on ordered bipartite graphs. Containment between ordered bipartite graphs can also be visualized using the language of containment in 0-1 matrices. This connection is explored in the next section.

3 Connection to 0-1 matrices

A 0-1 matrix is simply a matrix with all entries being 0 or 1. The *weight* of such a matrix is the number of its 1-entries. The 0-1 matrix A is said to *dominate* the 0-1 matrix B of the same size if $A_{ij} \geq B_{ij}$ for all i and j, that is, if $B = A$ or B is obtained from A by replacing some 1-entries by 0-entries. A 0-1 matrix A is said to *contain* another 0-1 matrix P, if P is dominated by a submatrix of A. Note that permuting rows or columns is not allowed. If A does not contain P, we say it *avoids* P. The extremal problem for 0-1 matrix containment can be formulated as computing (or estimating) the following extremal function for families $\mathcal{P}$ of 0-1 matrices: $\mathrm{Ex}(\mathcal{P}, n)$ is the maximal weight of an n-by-n 0-1 matrix that avoids all matrices in $\mathcal{P}$. We require that all matrices in $\mathcal{P}$ have positive weights. We write $\mathrm{Ex}(P, n)$ to denote $\mathrm{Ex}(\{P\}, n)$.

For a 0-1 matrix P, let G_P stand for the ordered bipartite graph whose vertices correspond to the rows and columns of P, the order of the vertices agrees with the order of rows and columns in P with all row-vertices preceding all column vertices, and with an edge between a row-vertex and a column-vertex if and only if the corresponding entry in P is 1. This makes P the bipartite adjacency matrix of G_P and turns the weight of P into the number of edges in G_P. The close connection between the extremal theory of ordered bipartite graphs and 0-1 matrices follows from the trivial observation that if a 0-1 matrix A contains another 0-1 matrix P, then the ordered graph G_A also contains G_P. The converse is also true if the homomorphism of G_P to G_A maps row-vertices to row-vertices and column-vertices to column-vertices. This extra condition is automatically satisfied if both the last row and first column of P contain at least one 1-entry, so in this case we have $\mathrm{Ex}(P, n) \leq \mathrm{ex}_<(G_P, 2n)$. There is no equality in general, because $\mathrm{ex}_<(G_P, 2n)$ is the maximum number of edges among all ordered graphs on $2n$ vertices avoiding G_P and the extremal ones may not be ordered bipartite. Still, the two extremal functions are really close to each other as shown by the following observation:

Theorem 4 ([24]). *For a 0-1 matrix P and the corresponding ordered bipartite graph G_P we have*

$$\mathrm{Ex}(P, n) \leq \mathrm{ex}_<(G_P, 2n) = O(\mathrm{Ex}(P, n) \log n).$$

The logarithmic term in the bound above is needed even for some small matrices, e.g., for the matrix

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

For this matrix, we have $\mathrm{Ex}(P, n) = 2n - 1$, but for the corresponding ordered graph G_P one has $\mathrm{ex}_<(G_P, n) = n \log n + O(n)$, where log stands for the binary logarithm. A construction showing the lower bound for this later estimate is an ordered graph whose vertices are adjacent if and only if their distance in the ordering is a power of 2. To see that $\mathrm{Ex}(P, n) \leq 2n - 1$ notice that by removing the first 1 entry in every row and the last 1 entry in every column in an n-by-n 0-1 matrix one removes at most $2n - 1$ 1 entries (any 1 entry in the first column is also the first 1 entry in its row) and if the remaining matrix still contains a 1 entry, then the original matrix contains P. To see the reverse inequality $\mathrm{Ex}(n, P) \geq 2n - 1$ simply consider the n-by-n matrix with ones in the last row and first column and zeros elsewhere and notice that it does not contain P.

The extremal theory of 0-1 matrices predates the related theory of ordered graphs. Zoltán Füredi [11] established the extremal function for a specific 2-by-3 0-1 matrix and used this result for a problem in combinatorial geometry: he bounded the number of diagonals of equal length in a convex n-gon. Independently, Bienstock and Győri [3] found the extremal function of few small 0-1 matrices. Later Füredi and Hajnal [12] started a systematic study of the extremal theory of 0-1 matrices. This latter paper not only contained many nice results, but was also rich in conjectures and had a significant effect on future research.

4 Connections between ordered and unordered extremal functions

Füredi and Hajnal, [12] made the following general conjecture: $\mathrm{Ex}(P, n) = O(\mathrm{ex}(\overline{G}_P, n) \log n)$, where P is any 0-1 matrix with positive weight and $\overline{G}_P$ is the simple graph underlying the ordered graph G_P. We will see that his conjecture is way too strong and fails in general, but it was still influential in subsequent research. It is not hard to see that a reverse of the conjectured inequality, namely $\mathrm{ex}(\overline{G}_P, n) \leq \mathrm{Ex}(P, n)$ always holds, so the conjecture states that the two quantities $\mathrm{Ex}(P, n)$ and $\mathrm{ex}(\overline{G}_P, n)$ are close. We saw in Theorem 4 that $\mathrm{Ex}(P, n)$ and $\mathrm{ex}_<(G_P, n)$ are close, so here we focus on comparing

$\mathrm{ex}_<(G_P, n)$ and $\mathrm{ex}(\overline{G}_P, n)$. In other words, we ask how much more edges can we have in an ordered graph of a given size if we only forbid one particular bipartite ordering of a simple graph G as a subgraph compared to forbidding all orderings of G. Note that it is important to insist that the forbidden ordering of a bipartite graph should have interval chromatic number two, otherwise Theorem 3 provides the easy answer: the two quantities can be very far apart. Indeed, for an ordered tree P of interval chromatic number larger than 2 one has $\mathrm{ex}_<(P, n) = \Theta(n^2)$ but $\mathrm{ex}(\overline{P}, n) = \Theta(n)$.

These considerations lead us to formulate the following question:

Question 1. *How high can the ratio* $\frac{\mathrm{ex}_<(P,n)}{\mathrm{ex}(\overline{P},n)}$ *be for an ordered bipartite graph* P *with more than one edge, where* $\overline{P}$ *stands for its underlying simple graph?*

The paper [24] gives the example with the largest known ratio. Let Q_k be the following ordered graph on $2k$ vertices $v_1, v_2, \ldots, v_{2k}$ (in this order) and the following $2k$ edges: $v_i v_{k+i}$ for $1 \le i \le k$, $v_{i+1} v_{k+i}$ for $1 \le i \le k-1$ and $v_1 v_{2k}$. Note that the interval chromatic number of Q_k is two and its underlying simple graph is the cycle C_{2k}.

Theorem 5 ([24])**.** *There exist ordered bipartite graphs on* n *vertices with* $\Theta(n^{4/3})$ *edges that contain none of the ordered graphs* Q_k*.*

Proof. Consider an arrangement of $n/2$ points and $n/2$ lines in the Euclidean plane with m *incidences*, i.e., point-line pairs with the point on the line. The celebrated Szemerédi-Trotter theorem [27] states that $m = O(n^{4/3})$. This theorem is tight, so we can select the points and the lines in such a way that $m = \Theta(n^{4/3})$. (For example, points of an appropriate square grid and the lines containing the most of these points will work.)

We turn this arrangement into a graph whose vertices are the points and lines in the arrangement and the edges correspond to pairs forming an incident. This graph has n vertices and $m = \Theta(n^{4/3})$ edges. We order the vertices of the constructed graph to turn it into an ordered graph. In our ordering point-vertices preceed line-vertices, this makes the constructed ordered graph ordered bipartite. We fix a coordinate system such that no line in the arrangement is vertical and order the point-vertices according their x-coordinates and we order the line-vertices according their slope (we break ties arbitrarily in both cases).

To finish the proof of the theorem we need to show that the constructed ordered graph does not contain Q_k for any k. Assume for contradiction that it does, so the vertices v_i of Q_k correspond to points and lines in the arrangement. Clearly, a vertex v_i with $i \leq k$ must correspond to a point p_i, while v_i with $i > k$ corresponds to a line l_i. Using the incidences represented by the edges $v_i v_{k+i}$ and $v_{i+1} v_{k+i}$ we conclude that the line segments $p_i p_{i+1}$ (belonging to the line l_{k+i}) form a convex chain for $i = 1, \ldots, k-1$. The contradiction comes from the observation that l_{2k} is the line $p_1 p_k$ connecting the two end points of this chain, thus its slope cannot exceed the slopes of all the segments in the chain (as it should since v_{2k} is the last in the vertex order). $\square$

Theorem 5 implies that $\mathrm{ex}_<(Q_k, n) = \Omega(n^{4/3})$ for every k. On the other hand the simple graph underlying Q_k is the cycle C_{2k} and by the Bondy-Simonovits theorem [4] we have $\mathrm{ex}(C_{2k}, n) = O(n^{1+1/k})$. This gives a lower bound of $\Omega(n^{1/3-1/k})$ for the ratio in Question 1 for the ordered graph Q_k. This also shows that conjecture mentioned in the beginning of this section fails for the bipartite adjacency matrix of Q_k whenever $k > 3$. We do not know if any pattern achieves a ratio of $\Omega(n^{1/3})$ in Question 1. For an upper bound for the same ratio we trivially have $O(n)$, as both the enumerator and the denominator are functions between n and n^2. In fact, they are $O(n^{2-\varepsilon})$ for some $\varepsilon > 0$ depending on the size of P by the Kővári-Sós-Turán theorem [18], so the ratio is always $O(n^{1-\varepsilon})$, but no better upper bound is known in general.

Question 1 asks how far the extremal function of a forbidden ordered bipartite can be from the extremal function of the family of all orderings of the same underlying graph. In similar vein one can ask how far the extremal functions of two distinct bipartite orderings of the same underlying graph might be from each other. I do not know of any results in this very interesting direction, but results of Győri, Korándi, Methuku, Tomon, Tompkins and Vizer on the extremal function of various bipartite orderings of even cycles, [15], might later prove useful to establish such gaps. This question is also related to the problem discussed in Section 7. The gap established in Theorem 5 will also show up between the extremal functions of two different bipartite orderings of the same even cycle or it is the case that forbidding all orderings of an even cycle of length at least 8 results in a substantially smaller extremal function than forbidding just one ordering. The latter would answer a variant of Question 2 in Section 7 (note that for simplicity we ask Question 2 about forbidding a pair of ordered graphs and here we need to consider larger families).

5 Forests

The Füredi-Hajnal paper [12] formulated the special case of their conjecture mentioned in the previous section separately for cycle-free patterns. Here we call a 0-1 matrix P cycle-free if the corresponding simple graph $\overline{G}_P$ is cycle-free, that is a forest. In this case, $\mathrm{ex}(\overline{G}_P, n)$ (the extremal function of an unordered forest) is trivially linear, so their conjecture boils down to stating $\mathrm{Ex}(P, n) = O(n \log n)$ for any cycle-free 0-1 matrix P. The log factor in the conjecture probably came from the the first matrix considered in this context [11, 3]:

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

that happen to be cycle-free and its extremal function is $\Theta(n \log n)$.

Here we formulate a closely related but somewhat weaker conjecture:

Conjecture 1 *For an ordered bipartite forest P and any $c > 1$, we have*

$$\mathrm{ex}_<(P, n) = o(n^c).$$

Note first that if this conjecture is true, then it characterizes the ordered graphs with almost linear extremal functions. Indeed, if P is not ordered bipartite, then $\mathrm{ex}_<(P, n) = \Theta(n^2)$ by Theorem 3, while if the underlying graph $\overline{P}$ contains a cycle, then $\mathrm{ex}_<(P, n) \geq \mathrm{ex}(\overline{P}, n) = \Omega(n^c)$ for some $c > 1$. The latter statement follows from a simple application of the probabilistic method.

Note that stronger conjectures could be formulated by replacing $o(n^c)$ with a bound $O(n \log^c n)$ for a constant $c = c_P$ depending on P, or even with an $O(n \log n)$ bound. Conjecture 1 and the conjecture with the $O(n \log^c n)$ bound are still open and by Theorem 4 are equivalent to the similar conjectures about $\mathrm{Ex}(M, n)$ for cycle-free 0-1 matrices M. The strongest form of the conjecture (an $O(n \log n)$ bound) was also considered for a while and was supported by the fact that it was easy to find an extremal function of order $\Theta(n \log n)$, but there was no known example of an ordered bipartite forest whose extremal function grows faster. If true, it would imply the Füredi-Hajnal conjecture for cycle-free patterns mentioned above. But Seth Pettie, [26], found a cycle-free 0-1 matrix with extremal function slightly higher than $n \log n$: for this matrix M one has $\mathrm{Ex}(M, n) = \Omega(n \log n \log \log n)$. By this, he also disproved the strengthening of Conjecture 1 with the $O(n \log n)$ upper bound, but the conjecture may still hold with the bound $O(n \log^2 n)$. Pettie's result was slightly improved and the current best lower bound is due Park and Shi [25]. They found cycle-free 0-1 matrices M_m with $\mathrm{Ex}(M_m, n) = \Omega(n \log n \log\log n \cdots \log^{(m)} n)$, where $\log^{(m)}$ denotes the m-times-iterated logarithm function.

On the positive side, $\mathrm{ex}_<(P, n) = O(n \log^c n)$ was established in [24] for all ordered bipartite forests with at most 6 vertices. The exponent c in this result can be chosen to be three less than the number of vertices in P. For most of the small ordered bipartite graphs the bound follows from this simple observation.

Lemma 1 ([24])**.** *Let P be a 0-1 matrix. Suppose that the last column of P contains a single* 1 *entry and let us obtain P' from P by deleting this last column. We have*

$$\mathrm{Ex}(P, n) = O(\mathrm{Ex}(P', n) \log n).$$

As the example of the matrix T above shows, the extra log factor is sometimes necessary in Lemma 1. It is reasonable to conjecture the following stronger form of this lemma also holds. If so, it easily implies Conjecture 1, even with the stronger $O(n \log^c n)$ bound.

Conjecture 2. *Let P be 0-1 matrix and let us obtain P' from P by deleting a column that contains a single 1 entry. We have*

$$\mathrm{Ex}(P, n) = O(\mathrm{Ex}(P', n) \log n).$$

The most general positive result toward Conjecture 1 appears in the paper [17] by Korándi, Tardos, Tomon and Weidert. They call a split of a 0-1 matrix P into two matrices P' and P'' an *legal horizontal split* if P is obtained by placing P' atop P'' (so in particular all three matrices have the same number of columns) and at most one of the columns have a one entry in both P' and P''. A a 0-1 matrix P is *vertically degenerate* if it can be partitioned into single line matrices by a sequence of legal horizontal splits. Note that all vertically degenerate 0-1 matrices are cycle-free. All cycle-free 0-1 matrices with at most three rows are vertically degenerate, but there are 4-row cycle-free 0-1 matrices that are not vertically degenerate (see below). Using a density increment argument, they prove the following theorem.

Theorem 6 ([17])**.** *Let P be a vertically degenerate 0-1 matrix with l rows. We have*

$$\mathrm{Ex}(P,n) = n2^{O(\log^{1-1/l} n)}.$$

This result implies that Conjecture 1 holds for all ordered graphs G_P, where P is a vertically degenerate 0-1 matrix. By symmetry, Conjecture 1 is also true for all G_P, where P is *horizontally degenerate*, that is, the transpose of P is vertically degenerate, but it has not been proved for any larger class of ordered bipartite forests. The smallest open case is an ordered path on 8 vertices, specifically G_M for the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Note that M has no legal horizontal or vertical split (where a *legal vertical split* is the legal horizontal split of the transpose). The following matrix N can be split into into trivial (one-by-one) matrices using a sequence of vertical and horizontal splits, but still it is neither vertically nor horizontally degenerate because the splits alternate in direction. Verifying Conjecture 1 for such matrices is probably simpler than for matrices like M above and may be the next logical step in verifying Conjecture 1.

$$N = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

In the rest of this section we give a very rough sketch of the proof of Theorem 6 because it may give some insights as to the limitation of this particular technique. For the technical details (for example the exact settings of the parameters) we refer to the paper [17]. Let us introduce a few notations. Let A be an m-by-kn 0-1 matrix. We consider A to be the union of k vertical blocks, each consisting of n consecutive columns. We say that the 0-1 matrix Q with k columns has a *block respecting embedding* in A if A contains Q in such a way that the submatrix of A dominating Q has all its columns coming from distinct vertical blocks (that is, the column corresponding to the ith column of Q must come from the ith vertical block of A for every i). The advantage of block respecting embeddings is that if Q has legal horizontal split into matrices Q' and Q'', then it is much easier to combine the block respecting embeddings of Q' and Q'' to form a block respecting embedding of Q than it is without the extra condition. Indeed, one only has to check that Q' uses rows of A higher than the ones used by Q'' and the single column that has a 1 entry in both Q' and Q'' use the same column of A. The disadvantage is that simply requiring that A has a lot of 1 entries is not enough to force the existence of a block respecting embedding. Instead we will insist that the 1 entries are evenly distributed with the following definition: we say that A is (k, u)-*complete* if among the n entries in the intersection of any row and any vertical block, one always finds at least u 1-entries.

We say that a 0-1 matrix Q with k columns *easily embeds* if for a certain range of the parameters n, m and u and for every m-by-kn (k, u)-complete 0-1 matrix A either Q has a block respecting embedding in A or one can find a submatrix of A which is significantly denser than A itself. We do not give the precise values of m and u required here, but the reader may think of $u = n^{\varepsilon}$ for some $\varepsilon > 0$ and "significantly denser" may mean an s-by-s submatrix with weight $s^{1+\varepsilon'}$, where $\varepsilon' > \varepsilon$ depends on ε but not of n. The proof of Theorem 6 is based on two lemmas. The first states that if Q has a legal horizontal split to Q' and Q'' and both Q' and Q'' easily embed, then so does Q (with a slight deterioration in the parameters). As single line matrices easily embed, this lemma implies the same for all vertically degenerate matrices. The second lemma takes care of the extra uniformity condition. It states that any 0-1 matrix A has a (k, u)-complete submatrix with comparable size and density to A or a submatrix of significantly larger density.

Consider the 0-1 matrix

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Q is horizontally degenerate, so G_Q satisfies the statement of Conjecture 1, but it is not vertically degenerate, and so we do not know if it easily embeds. It is easy to see that permuting the columns of a 0-1 matrix does not ruin the property that it easily embeds, neither does adding extra columns with a single 1 entry. In this way showing that Q easily embeds would imply the same for both matrices M and N above and would establish the statement of Conjecture 1 for G_M and G_N.

6 Linear extremal functions

Füredi and Hajnal [12] conjectured and later Marcus and Tardos [20] proved that $\text{Ex}(P, n) = O(n)$ for permutation matrices P. It is not hard to see that this result can be restated in the following equivalent form (although Theorem 4 does not directly imply this equivalence).

Theorem 7 ([20])**.** *The extremal function of any ordered bipartite matching P is linear. That is,*

$$\text{ex}_<(P, n) = O(n).$$

Conjecture 1, if true, characterizes all ordered graphs with almost linear extremal functions. It would be nice to find a characterization of ordered graphs or 0-1 matrices with linear extremal functions. One possibility is finding all *minimally nonlinear matrices.* We call a 0-1 matrix P minimally nonlinear, if its extremal function $\text{Ex}(P, n)$ is nonlinear, but $\text{Ex}(P', n) = O(n)$ for all 0-1 matrices $P' \neq P$ contained in P. It might be possible to find such a characterization, but the following theorem indicates that this is a difficult task:

Theorem 8 (Geneson and Keszegh [13, 16])**.** *There are infinitely many minimally nonlinear matrices.*

Note that "minimally nonlinear" simple graphs (for the classic extremal graph theory) are well understood despite the fact that there are infinitely many of them: they are the cycles. Keszegh, [16], foud a sequence of 0-1 matrices $H_0, H_1, \ldots$ (shown below with the zeros omitted for clarity) that show some repetitive behavior. He did not prove that they are minimally nonlinear, instead he showed that they are nonlinear, specifically $\text{Ex}(H_i, n) = \Omega(n \log n)$ for all i, and thus each contains a minimally nonlinear matrix. Then Geneson, [13], showed that no two of them can contain the same nonlinear matrix.

$$H_0 = \begin{pmatrix} & 1 & 1 & \\ & & & 1 \\ & & & 1 \\ 1 & & & \end{pmatrix}$$

$$H_1 = \begin{pmatrix}
 & 1 & 1 & & & & \\
 & & & & & 1 & \\
 & & & & 1 & & \\
1 & & & & & & \\
 & & & & & & 1 \\
 & & & & & & 1 \\
 & & & 1 & & &
\end{pmatrix}$$

$$H_2 = \begin{pmatrix}
 & 1 & 1 & & & & & & & \\
 & & & & & 1 & & & & \\
 & & & & 1 & & & & & \\
1 & & & & & & & & & \\
 & & & & & & & & 1 & \\
 & & & & & & & 1 & & \\
 & & & 1 & & & & & & \\
 & & & & & & & & & 1 \\
 & & & & & & & & & 1 \\
 & & & & & & 1 & & &
\end{pmatrix}$$

7 Interaction between ordered graphs

In this section we compare the extremal functions of families of several forbidden patterns with the extremal function of just a single member of that family. Let us start with the classical extremal theory of graphs. Clearly, we have

$$\mathrm{ex}(\{G, H\}, n) \leq \min(\mathrm{ex}(G, n), \mathrm{ex}(H, n)). \qquad (*)$$

By Theorem 2, the two sides are asymptotically the same for non-bipartite graphs G and H. It is easy to see that they differ by a factor of less than 2 if only one of the graphs is bipartite. Indeed, if G is bipartite and H is not, one can make any simple graph avoiding G itself bipartite by deleting less than half of its edges. For bipartite graphs, the situation is more complicated. We say that G and H *interact* if the two sides of (*) differ more than by a constant factor. It is not known if there exists any interacting pair of graphs. Erdős and Simonovits, [8], conjecture that there exists no interaction between graphs, but more recently Faudree and Simonovits, [10], conjecture the opposite. Specifically, they conjecture that the cycle C_4 and the subdivision of the complete graph K_4, in which each edge is subdivided with a single new vertex, do interact.

Let us emphasize that here we do not care for constant factors in the extremal functions. Finding *weakly interacting* pairs, that is, where $\mathrm{ex}(\{G,H\})$ is a constant factor less than $\min(\mathrm{ex}(G,n),\mathrm{ex}(H,n))$ is considerably simpler. The Erdős-Stone-Simonovits theorem prevents even weak interactions between non-bipartite graphs, but a bipartite and a non-bipartite graph can interact weakly. Specifically, Erdős and Simonovits [8] prove that C_4 and C_5 do interact weakly. Similar questions were also studied in the context of uniform hypergraphs, where answering a question of Mubayi and Rödl, [23], Mubayi and Pikhurko, [22] find weakly interacting pairs of r-uniform hypergraphs with extremal function $\Theta(n^r)$ for all $r>2$. The weakly interacting pairs (or families) with such high extremal functions are called *non-principal* families. Earlier József Balogh, [2], found non-principal families of 3-uniform hypergraphs of larger finite size. Note that for graphs (i.e., $r=2$) non-principal families do not exist.

In contrast to graphs, it is not hard to find a lot of interactions in the extremal theory of ordered graphs and 0-1 matrices. Consider the 3-by-2 matrix $T=\begin{pmatrix}1&0&1\\1&1&0\end{pmatrix}$. Füredi [11] and Bienstock and Győri [3] proved that $\mathrm{Ex}(T_1)=\Theta(n\log n)$. By symmetry, the extremal functions of the matrices $T_2=\begin{pmatrix}1&1&0\\1&0&1\end{pmatrix}$, $T_3=\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}$ and $T_4=\begin{pmatrix}0&1&1\\1&0&1\end{pmatrix}$ are the same. The following theorem implies that each of T_2, T_3 and T_4 interacts with T:

Theorem 9 ([28])**.**

$$\mathrm{Ex}(\{T,T_2\},n)=\Theta(n)$$
$$\mathrm{Ex}(\{T,T_3\},n)=\Theta(n\log n/\log\log n)$$
$$\mathrm{Ex}(\{T,T_4\},n)=\Theta(n\log\log n)$$

The close connection between the extremal functions of 0-1 matrices and ordered graphs makes it easy to turn these interactions into interactions between ordered graphs.

These results represent the first step toward exploring interactions between different patterns. It would be interesting to find stronger interactions, where the ratio between the right and left sides of (*) is larger than logarithmic, ideally a power on n. We remark that the conjectured interaction between bipartite graphs in [10] is of this stronger nature.

Question 2 *Are there ordered graphs G and H such that*

$$\mathrm{ex}_<(\{G,H\},n)=O(\min(\mathrm{ex}_<(G,n),\mathrm{ex}_<(H,n))/n^\epsilon)$$

for some $\epsilon > 0$?

8 Edge ordered graphs

In this final section we survey some preliminary results from ongoing research of the author with Dániel Gerbner, Abhishek Methuku, Dániel T. Nagy, Dömötör Pálvölgyi and Máté Vizer [14] on the extremal theory of edge ordered graphs. Recall that we defined edge ordered graphs in Section 1 analogously to (vertex) ordered graphs, but now the linear order is on the edges of a simple graph. The extremal function for a family $\mathcal{P}$ of edge ordered graphs is also defined analogously: we are looking for the largest number $\mathrm{ex}'_<(\mathcal{P}, n)$ of edges in an n-vertex edge ordered graph with no edge ordered subgraph isomorphic to any member of $\mathcal{P}$. We require that $\mathcal{P}$ does not contain empty graphs and we write $\mathrm{ex}'_<(P, n)$ to denote $\mathrm{ex}'_<(\mathcal{P}, n)$ when $\mathcal{P} = \{P\}$ is a singleton.

As a natural first step we generalize the Erdős-Stone-Simonovits theorem for edge ordered graphs. As for the corresponding theorem for vertex ordered graph, namely Theorem 3, the main thing here is to find the "correct" notion of the chromatic number. Then the result follows easily from the original Erdős-Stone theorem [9]. We say that a simple graph *strongly contains* the edge ordered graph G if every edge ordering of H contains G as an edge ordered subgraph. We define the *order chromatic number* of an edge ordered graph G to be the smallest chromatic number of a simple graph strongly containing G. In case no graph strongly contains G we say that the order chromatic number of G is infinity.

Theorem 10 (Erdős-Stone-Simonovits Theorem for a single edge ordered forbidden graph)**.** *If the order chromatic number of the edge ordered graph G is infinity we have*

$$\mathrm{ex}'_<(G, n) = \binom{n}{2}.$$

If the order chromatic number of G is $r + 1$ we have

$$\mathrm{ex}'_<(G, n) = (1 - 1/r)\frac{n^2}{2} + o(n^2).$$

This theorem determines the extremal function $\mathrm{ex}'_<(G, n)$ exactly if the order chromatic number of G is infinity, it determines the extremal function asymptotically, if the order chromatic number is larger than 2, but tells fairly little in case the order chromatic number is 2.

As an example consider the cycle C_4 and its three different edge orderings: C_4^{1234}, C_4^{1324} and C_4^{1243}. Here we number the edges according the ordering and the upper index represents the numbers along the cycle, so in C_4^{1243} the first and last edges are opposite in the cycle. Any simple graph can be edge ordered by first imposing a linear order on the vertices, then ordering the edges ab with $a < b$ by the lexicographic order on the pairs (a, b). We call this the *lexicographic edge order*. It is easy to see that no graph with a lexicographic edge order contains either C_4^{1234} or C_4^{1324}. This means that these two edge ordered graphs have order chromatic number infinity. The order chromatic number of C_4^{1243} is 2. This can be seen directly or follows from the following result. For the statement of the result we need to define the edge ordered graph $K_{n,n}^{\text{lex}}$. This is the lexicographic edge ordering of the complete bipartite graph $K_{n,n}$ obtained from a vertex order in which one vertex class precedes the other.

Theorem 11. *The non-empty edge ordered graph G on n vertices has order chromatic number 2 if and only if it is contained in $K_{n,n}^{\text{lex}}$.*

Theorem 10 says little about the extremal function of C_4^{1243}. Naturally, the edge ordering can only increase the extremal function, so we have

$$\mathrm{ex}'_<(C_4^{1243}, n) \geq \mathrm{ex}(C_4, n) = \Theta(n^{3/2}).$$

Applying techniques of the paper [21] we prove a nearly matching upper bound:

Theorem 12. $\mathrm{ex}'_<(C_4^{1243}, n) = O(n^{3/2} \log n)$.

We do not know if the logarithmic term is needed in this estimate.

We list here results on a selection of other specific forbidden edge ordered graphs of order chromatic number two. We start with edge ordered graphs whose connected components are (edge ordered) stars. We call them edge ordered star forests. Their extremal functions are obtained through known estimates in generalized Davenport-Schinzel theory.

Theorem 13. *For any edge ordered star forest F we have $\mathrm{ex}'_<(F, n) \leq n2^{\alpha(n)^c}$ for some exponent c depending on F. Here α denotes the inverse of the Ackermann function. For the edge ordered star forest F_0 consisting of two components and five edges with one component consisting of the second and fourth edge we have $\mathrm{ex}'_<(F_0, n) = \Omega(n\alpha(n))$.*

We specify edge orderings of a path P_{k+1} by an upper index listing the ranks of the k edges along the path, so for example P_5^{1342} stands for the edge ordered path where the edges along the path follow as first-third-fourth-second. This is the shortest edge ordered path where we could not find the exact order of magnitude of the extremal function.

Theorem 14. *For an edge ordered path P on three edges we have $\mathrm{ex}'_<(P,n) = \Theta(n)$. For an edge ordered path P on four edges we have either $\mathrm{ex}'_<(P,n) = \Theta(n)$ or $\mathrm{ex}'_<(P,n) = \Theta(n \log n)$ or $\mathrm{ex}'_<(P,n) = \binom{n}{2}$ or P is isomorphic to P_5^{1342} or the equivalent P_5^{4213}. In this last case we have $\mathrm{ex}'_<(P,n) = \Omega(n \log n)$ and $\mathrm{ex}'_<(P,n) = O(n \log^2 n)$.*

In the rest of this section we consider forbidden families of edge ordered graphs and possible weak interaction (see Section 7) between the members of such a family. Note that we formulated Theorem 10 for a single forbidden edge ordered graph. This is because some families of forbidden edge ordered graphs behave differently than any of their members alone, see Theorem 16(ii) below for an example. Note that this contrasts with the situation for simple graphs and vertex ordered graphs, where the Erdős-Stone-Simonovits theorem (Theorem 2) and its variant for vertex ordered graphs (Theorem 3) prevent any such weak interaction. To be more specific, we will see that such weak interaction does not happen between two edge ordered graphs of order chromatic number three but it does happen between certain edge ordered graphs of order chromatic number four and above. To be able to generalize the Erdős-Stone-Simonovits theorem to families of forbidden edge ordered graphs we need to extend the definition of order chromatic number from edge ordered graphs to families of such graphs. Let the *order chromatic number of a family $\mathcal{P}$* be the smallest chromatic number of a simple graph H such every edge ordering of H contains a member of $\mathcal{P}$ as an edge ordered subgraph. Again, if no such H exists, the order chromatic number is infinite. With this definition we have the following generalization of Theorem 10.

Theorem 15 (Erdős-Stone-Simonovits Theorem for a family of forbidden edge ordered graphs)**.** *If the order chromatic number of the family $\mathcal{P}$ of edge ordered graphs is infinity we have*

$$\mathrm{ex}'_<(\mathcal{P},n) = \binom{n}{2}.$$

If the order chromatic number of $\mathcal{P}$ is $r+1$ we have

$$\mathrm{ex}'_<(\mathcal{P},n) = (1-1/r)\frac{n^2}{2} + o(n^2).$$

The following result gives a specific example when a pair of edge ordered graphs behaves differently than either members.

Theorem 16. (i) *The order chromatic number of a family of edge ordered graphs is* 2 *if and only if the family has a member with order chromatic number* 2*.*

(ii) *The order chromatic number of both edge orderings P_5^{1423} and P_5^{2314} of the path P_5 is infinity, but the order chromatic number of the family $\{P_5^{1423}, P_5^{2314}\}$ is* 3*.*

Part (i) of this theorem follows easily from Theorem 11. For part (ii) note that no graph with a lexicographic edge ordering contains P_5^{2314}, so the order chromatic number of P_5^{2314} is infinity. The same statement about P_5^{1423} follows from symmetry. The statement on the pair $\{P_5^{1423}, P_5^{2314}\}$ can be derived directly or follows from an analogue of Theorem 11 for families of edge ordered graphs of order chromatic number 3. While these analogous results for order chromatic number three and above can be easily deduced from Ramsey's theorem, one has to deal with exceedingly many different *homogeneous* edge orderings. Here we state the result for order chromatic number infinity only, where we have to consider only four homogeneous edge orderings of the complete graph K_n as follows: Let $K_n^{(1)}$ be the lexicographic edge ordering of K_n. For $K_n^{(2)}$ let us consider the vertex set of K_n to be $\{1,2,\ldots,n\}$ and order the edges ab with $a<b$ according to the lexicographic order on the pairs $(a,-b)$. Let us obtain $K_n^{(3)}$ and $K_n^{(4)}$ by reversing the edge order in $K_n^{(1)}$ and $K_n^{(2)}$, respectively.

Theorem 17. *The order chromatic number of a family $\mathcal{P}$ of edge ordered graphs is infinity if and only if there exists $1 \le i \le 4$ such that the graphs $K_n^{(i)}$ contain no member of $\mathcal{P}$ for any n.*

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