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On edge-ordered graphs with linear extremal functions

(Extended abstract)

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Abstract

The systematic study of Turán-type extremal problems for edge-ordered graphs was initiated by Gerbner et al. in 2020. Here we characterize connected edge-ordered graphs with linear extremal functions. This characterization is similar in spirit to results of Füredi et al. (2020) about vertex-ordered and convex geometric graphs.

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1 History

Turán-type extremal graph theory asks how many edges an n-vertex simple graph can have if it does not contain a subgraph isomorphic to a *forbidden graph*. We introduce the relevant notation here.

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Definition 1.1. *We say that a simple graph G* avoids *another simple graph H, if no subgraph of G is isomorphic to H. The Turán number* $\mathrm{ex}(n, H)$ *of a* forbidden *finite simple graph H (having at least one edge) is the maximum number of edges in an n-vertex simple graph avoiding H.*

This theory has proved to be useful and applicable in combinatorics, as well as in combinatorial geometry, number theory and other parts of mathematics and theoretical computer science. Turán-type extremal graph theory was later extended in several directions, including hypergraphs, geometric graphs, convex geometric graphs, vertex-ordered graphs, etc. Here we work with *edge-ordered graphs* as introduced by Gerbner, Methuku, Nagy, Pálvölgyi, Tardos and Vizer in [4]. The several extension of extremal graph theory each proved useful and applicable in different parts of mathematics and this also holds for the (still new) edge-ordered version discussed here, see e.g. [2]. Let us recall the basic definitions.

Definition 1.2. *An* edge-ordered graph *is a finite simple graph G together with a linear order on its edge set E. We often give the edge-order with an injective labeling $L : E \to \mathbb{R}$. We denote the edge-ordered graph obtained this way by G^L, in which an edge e precedes another edge f in the edge-order (denoted by $e < f$) if $L(e) < L(f)$. We call G^L the* labeling *or* edge-ordering *of G and call G the simple graph* underlying *G^L.*

An isomorphism between edge-ordered graphs must respect the edge-order. A subgraph of an edge-ordered graph inherits the edge-order and so it is also an edge-ordered graph. We say that the edge-ordered graph G contains *another edge-ordered graph H, if H is isomorphic to a subgraph of G otherwise we say that G* avoids *H.*

For a positive integer n and an edge-ordered graph H, let the Turán number $\mathrm{ex}_<(n, H)$ *be the maximal number of edges in an edge-ordered graph on n vertices that avoids H. Fixing the* forbidden edge-ordered graph *H,* $\mathrm{ex}_<(n, H)$ *is a function of n and we call it the* extremal function *of H. Note that this definition does not make sense if H has no edges, so we insist that H is* non-trivial*, that is, it has at least one edge.*

Braß, Károlyi and Valtr, [1] introduced *convex geometric graphs* while Pach and Tardos, [7] introduced *vertex-ordered graphs* and studied their extremal theories. In both cases a simple graph is given extra structure by specifying an order on their vertices (a cyclic order for convex geometric graphs and a linear order for vertex-ordered graphs). Characterizing the convex geometric or vertex-ordered graphs with a linear extremal function seems to be beyond reach (so far), but Füredi, Kostochka, Mubayi and Verstraëte, [3] found such a characterization for *connected* convex geometric graphs and also for *connected* vertex-ordered graphs. The situation seems to be similar for edge-ordered graphs: while we could not give a general characterization of edge-ordered graphs with linear extremal functions, in Section 2 we characterize when connected edge-ordered graphs have linear extremal functions. This characterization is also a dichotomy result: we show that whenever the extremal function of a connected edge-ordered graph is not linear, it must be $\Omega(n \log n)$.

2 Main result

In classical (unordered) extremal theory the following dichotomy is immediate:

Observation 1. *If H is a forest, then* $\mathrm{ex}(n, H) = O(n)$*, otherwise* $\mathrm{ex}(n, H) = \Omega(n^c)$ *for some $c = c(H) > 1$.*

The analogous statement fails for edge-ordered graphs: the paper [4] exhibits several edge-ordered paths with extremal functions $\Theta(n \log n)$. Therefore, when looking for an analogous result for edge-ordered graphs, we have a choice to make. Either we want to characterize the edge-ordered graphs with linear extremal functions, or the ones with extremal functions that are *almost linear*, i.e., $n^{1+o(1)}$. In the latter direction the authors of [4] formulated a conjecture that we recently verified, see [6]. The former problem seems to be considerably more difficult as there is not even a reasonable conjecture characterizing all edge-ordered graphs with a linear extremal function.

The first result in this direction appeared in the MSc thesis of the first author, [5]: he gave a simple characterization of edge-ordered *paths* with linear extremal functions. In this section we generalize this result and provide a characterization for *connected* edge-ordered graphs with linear extremal functons, see Theorem 1. This is the same restriction considered in [3] with respect to vertex-ordered and convex geometric graphs. Our theorem also states that if the extremal function of a connected edge-ordered graph is not linear, then it is $\Omega(n \log n)$. Such a dichotomy does not hold for edge-ordered graphs in general as [4] exhibits a (necessarily disconnected) edge-ordered graph whose extremal function is $\Theta(n\alpha(n))$, where α is the inverse of the Ackermann function.

In order to formulate the characterization and dichotomy in Theorem 1 we need to introduce some terminology. The *reverse* G^R of an edge-ordered graph G is obtained from G by reversing its edge-order. The *order chromatic number* $\chi_<(G)$ of an edge-ordered graph G is the smallest chromatic number $\chi(H)$ of a simple graph H such that all edge-orderings of H contain G. (If no such H exists we write $\chi_<(G) = \infty$). The order chromatic number was introduced in the paper [4] to play the role of the (ordinary) chromatic number in a version of the Erdős-Stone-Simonovits theorem for edge-ordered graphs, see Theorem 2.3 in [4]. For the purposes of our Theorem 1, one does not even have to apply this definition, it is enough to apply Lemma 2.1 below that gives a simple characterization when the order chromatic number of an edge-ordered forest is two. We call a vertex v of an edge-ordered graph *close* if the edges adjacent to v form an interval in the edge-order.

Lemma 2.1 ([4]). *A non-trivial edge-ordered forest has order chromatic number 2 if and only if it has a proper 2-coloring such that all vertices in one of the color classes are close.*

We call the edges $e_1 < e_2$ *consecutive* in an edge-ordered graph G if no edge e of G satisfies $e_1 < e < e_2$. An edge-ordered graph G is a *semi-caterpillar* if the underlying simple graph is a non-trivial tree and any pair of consecutive edges in G are either adjacent in G or they are directly connected by an edge larger than both of them.

Theorem 1 (Dichotomy). *If G or its reverse G^R is a semi-caterpillar of order chromatic number 2, then $\mathrm{ex}_<(n, G) = O(n)$. For any other non-trivial connected edge-ordered graph G we have $\mathrm{ex}_<(n, G) = \Omega(n \log n)$.*

Neither direction of the above dichotomy seems to follow from earlier results. For lack of space we do not give the full proofs, just sketch the main concepts involved. We start with saying a few words on semi-caterpillars.

Recall that the definition of semi-caterpillars insists that each pair of consecutive edges must either be adjacent or they are connected directly by a single larger edge. If we insist that all pair of consecutive edges are adjacent in an edge-ordered tree, we obtain a sub-class of semi-caterpillars, let us call these *basic caterpillars*. Note that the underlying simple graphs of basic caterpillars are (conventional) caterpillars: each vertex is at distance at most one from a single path. It is easy to prove that the order chromatic number of all basic caterpillars is 2. Neither statement generalizes to all semi-caterpillars, but it is not hard to prove that all vertices of a semi-caterpillar are at distance at most *two* of a single path. See Figure 1 for an example of an order chromatic number 2 semi-caterpillar whose underlying graph is not a caterpillar. The fact that basic caterpillars have linear extremal functions follows easily from the following two observations about a concept we call basic extension: A *basic extension* of a non-trivial edge-ordered graph G is an edge-ordered graph obtained by adding a single new edge to G that connects one end of the smallest edge of G to a new vertex outside G and making this new edge smaller than any edge in G.

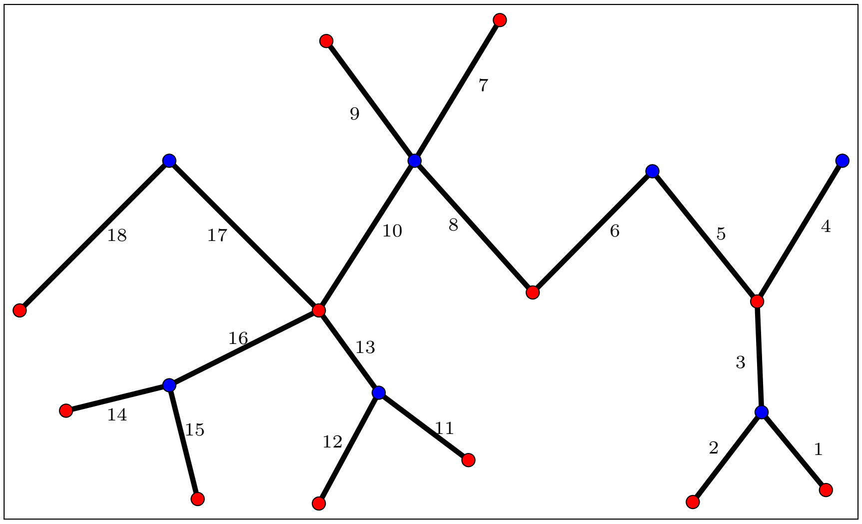

Figure 1: A semi-caterpillar with a linear extremal function

Lemma 2.2. *A non-trivial edge-ordered graph G without isolated vertices is a basic caterpillar if and only if G is obtained (up to isomorphism) from the (only) edge-ordering of the graph K_2 by a sequence of basic extensions.*

Lemma 2.3. *If G' is a basic extension of the edge-ordered graph G, then* $\mathrm{ex}_<(n, G') = \mathrm{ex}_<(n, G) + O(n)$.

Lemma 2.2 is very easy to prove and Lemma 2.3 was already implicit in [4]. We use a similar approach for proving the first statement of Theorem 1, but we will have to resolve several complications on the way. To formulate a version of Lemma 2.2 for semi-caterpillars of order chromatic number 2 we will introduce a generalization of basic extensions we call extensions. We deal with edge-ordered trees, so the underlying simple graphs are bipartite. We will have to break symmetry and distinguish the two sides. This is largly motivated by Lemma 2.1.

An *edge-ordered bigraph* G is an edge-ordered graph G_0 together with a proper 2-coloring to *left* and *right* vertices, so each edge has a left end and a right end. We call G_0 the edge-ordered graph *underlying* G. Note that we use many terms, like edge-ordered forest, edge-ordered tree, edge-ordered path in a simpler sense meaning an edge-ordered graph whose underlying simple graph is a forest, a tree, or a path, respectively. Our use of edge-ordered bigraph as explained above is more than an edge-ordered graph whose underlying simple graph is bipartite. The notions of *isomorphism*, *subgraph*, *contain* and *avoid* naturally extend to edge-ordered bigraphs.

The paper [4] introduced edge-ordered bigraphs in order to break symmetry. Using them one can distinguish the two ways a connected edge-ordered graph may be embedded in another edge-ordered graph if both underlying simple graphs happen to be bipartite: after making them into edge-ordered bigraphs by designating left and right vertices in both graphs either all left vertices map to left vertices and the mapping ensures containment between the edge-ordered bigraphs or all left vertices map to right vertices in which case it does not.[1]

Let G be a non-trivial edge-ordered bigraph and let e be the smallest edge in G. We call the edge-ordered bigraph G' an *extension* of G if G' is obtained from G by adding new edges to it, such that

1. every new edge connects one end of e to a new degree 1 vertex;
2. all new edges are smaller than the edge e;
3. all new edges incident to the left end of e are smaller than any new edge incident to the right end of e.

Let T_0 denote the unique edge-ordered bigraph with a single edge and two vertices. We are now ready to formulate our analogue of Lemma 2.2 for semi-caterpillars of order chromatic number 2.

Lemma 2.4. *An edge-ordered graph is a semi-caterpillar of order chromatic number 2 if and only if it is isomorphic to the underlying edge-ordered graph of an edge-ordered bigraph obtained by a sequence of extensions from T_0.*

The proof of this lemma uses among other things the characterization in Lemma 2.1. If we could complement Lemma 2.4 with an appropriate analogue of Lemma 2.3, that would finish the proof of the first statement of Theorem 1. This analogue should state that if G' is an extension of the edge-ordered bigraph G and their underlying edge-ordered graphs are G_0' and G_0, respectively, then $\mathrm{ex}_<(n, G_0') - \mathrm{ex}_<(n, G_0) = O(n)$ or—at least—that $\mathrm{ex}_<(n, G_0')$ is linear if $\mathrm{ex}_<(n, G_0)$ is linear. Unfortunately, neither statement holds.

This makes our proof of the first statement of Theorem 1 necessarily more involved: instead of being able to concentrate on a single extension step, we have to argue about the entire sequence of extensions that produces a certain edge-ordered bigraph.

And now we say a few words on the proof of the second statement of Theorem 1 which states that the extremal functions of connected edge-ordered graphs not covered by the first statement are $\Omega(n \log n)$. Its main ingredient is the following lemma.

[1] The paper [4] used the terms *edge-ordered bipartite graph* instead of edge-ordered bigraph and the terms *left-contain* and *right-contain* for the two ways an edge-ordered bigraph can contain a path.

We denote the simple path on k vertices by P_k and denote its labeling by listing the labels along the path in the upper index. For example, P_4^{213} mentioned in the lemma below is the edge-ordered 3-edge path whose middle edge is the smallest.

Lemma 2.5. *Let G be a non-trivial edge-ordered tree. G is a semi-caterpillar if and only if it does not contain any of the edge-ordered paths P_4^{213}, P_5^{1342} or P_5^{1432}.*

Using this lemma one can finish the proof of the second statement of Theorem 1 as follows. If the order chromatic number of G is not 2, then by the edge-ordered version of the Erdős-Stone-Simonovits theorem (see [4]) $\mathrm{ex}_<(n,G) = \Theta(n^2)$. Recall that G is assumed to be connected, so if it is not an edge-ordered tree, then its underlying simple graph contains a cycle C_k and therefore $\mathrm{ex}_<(n,G) \geq \mathrm{ex}(n,C_k) = \Omega(n^{1+1/k})$. If G is a non-trivial edge-ordered tree, but not a semi-caterpillar, then it contains one of the three edge-ordered paths listed in Lemma 2.5 and therefore the extremal function $\mathrm{ex}_<(n,G)$ is at least the extremal function of the corresponding edge-ordered path. The extremal functions of these edge-ordered paths were studied in the paper [4] and we know that $\mathrm{ex}_<(n,P_5^{1432}) = \Theta(n \log n)$ and $\mathrm{ex}_<(n,P_5^{1342}) = \Omega(n \log n)$, so we are done in these cases. In the only remaining case G contains P_4^{213}.

The paper [4] calculates the extremal function of P_4^{213} also, but unfortunately it is linear. Now we apply the same argument to the reverse G^R of G, which is also connected. We obtain that if G^R is not a semi-caterpillar of order chromatic number 2, then its extremal function is $\Omega(n \log n)$ or else it contains P_4^{213}. Note that the extremal function of G and G^R coincide, so we are done unless both edge-ordered graphs G and G^R contain P_4^{213} or, in other words, G contains both P_4^{213} and P_4^{132}.

There are edge-ordered graphs with linear extremal functions containing both P_4^{213} and P_4^{132}, for example the disjoint union of P_4^{213} and P_4^{465}. But recall that G is connected. We finish the proof by showing that the extremal function of any *connected* edge-ordered graph G containing both P_4^{213} and P_4^{132} is $\Omega(n \log n)$. The proof uses the construction in Lemma 4.11 of [4].

3 Concluding remarks

Studying the extremal functions of edge-ordered graphs, especially those at the lower end of the spectrum seems very interesting. In particular, Gerbner, Methuku, Nagy, Pálvölgyi, Tardos and Vizer in [4] studied the extremal functions of many edge-ordered graphs, among them all edge-ordered paths consisting of up to four edges. In the journal version of this paper we extend their research to paths of five edges and beyond. Unfortunately, we do not have enough space to include the highlights of this line of research here.

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